Chapter 10

Trefftz Method

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10.1 Introduction

In recent years, by a boundary method, it is usually understood a numerical procedure in which a subregion or the entire region, is left out of the numerical treatment, by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problem leading to considerable economy in the numerical work and constitute a very convenient manner of treating adequately unbounded regions by numerical means. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced [1–2].

There are two main approaches for the formulation of boundary methods; one is based on boundary integral equations and the other one, on the use of complete systems of solutions. In numerical applications, the first one of these methods has received most of the attention [3–4]. This is in spite of the fact that the use of complete systems of solutions presents important numerical advantages; e.g., it avoids the introduction of singular integral equations and it does not require the construction of a fundamental solution. The latter is especially relevant in connection with complicated problems, for which, it may be extremely laborious to build up a fundamental solution. This is illustrated by the fact that there are methods for synthesizing fundamental solutions starting from plane waves, which can be shown to be a complete system [5].

The use of complete systems of solutions is frequently associated with the name of Trefftz [6]. The idea of his method [7] consists in looking for approximate solutions, from among the appropriate class of functions that satisfy exactly the differential equation but do not necessarily satisfy the prescribed boundary conditions. Although Trefftz's original formulation was linked to a variational principle, this is not required. Indeed, complete systems of solutions can be used to treat differential equations which are linear but otherwise arbitrary [8].

The method has been used in many fields. For example, applications to Laplace's equation are given by Mikhlin [9], to the biharmonic equation by Rektorys [10] and to elasticity by Kupradze [11]. Also many scattered contributions to the method can be found in the literature. Special mention is made here of work by Amerio, Fichera, Kupradze, Picone and Vekua [12–16]. Colton [17, 18] has constructed families of solutions which are complete for parabolic equations.

However, systematic analysis applicable to arbitrary linear differential equations were lacking. Motivated by this situation the author started a systematic research of the subject. The aim of the study has been two-fold; firstly, to clarify the theoretical foundations required for using complete systems of solutions in a reliable manner, and secondly, to expand the versatility of such methods, making them applicable to any problem which is governed by partial differential equations which are linear.

Considerable progress has been made [8, 19]. The systematic development includes: a) approximating procedures and conditions for their convergence [5, 8, 20–22]; b) formulation of variational principles [23–28]; and c) development of complete systems of solutions [29–33]. In addition, the algebraic frame-work [34] in which the theory has been constructed has been used for developing biorthogonal systems of solutions [35].

Recently, the theory has been extended to free-boundary problems [36, 37]. These are non-linear even when the governing differential equations are linear. Boundary integral equations have been applied to some of these problems [38]. For seepage problem, numerical experiments using Trefftz method were reported [22], but a theoretical analysis was not included. Many others have been treated using finite element approximations; for example, contact problems in elasticity [39, 40].

Systems of solutions which are complete, frequently preserve this property when the region of definition of the problem is changed. This is the case, for example, of the systems given in Tables 10.1 and 10.2 (see Section 10.6). It is important to develop procedures for constructing such systems and to formulate criteria to elucidate their completeness. This subject is being studied at present and a preliminary survey appeared recently [33].

10.2 Scope

To fix ideas we first consider a simple example. Take Laplace or Poisson's equation in a bounded region Ω, illustrated in Fig. 10.1, and subjected to

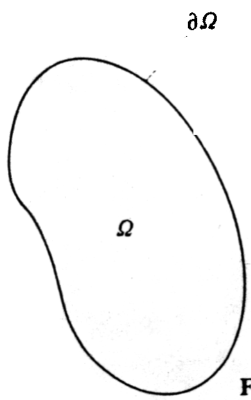

Fig. 10.1.

boundary conditions of Dirichlet type:

$$\Delta u = f_\Omega; \quad \text{in } \Omega \tag{10.1a}$$

and

$$u = f_{\partial\Omega}; \quad \text{on } \partial\Omega \tag{10.1b}$$

where f_Ω and $f_{\partial\Omega}$ are given functions.

In general, the application of boundary methods requires transforming equation (10.1a) into a homogeneous equation. This can be achieved by introducing a particular solution U of equation (10.1a); i.e.

$$\Delta U = f_\Omega; \quad \text{in } \Omega \tag{10.2a}$$

In applications the construction of function U is not difficult, because it is not required to satisfy any prescribed boundary conditions. For example, when a fundamental solution is available, it can be obtained by quadrature.

In addition, let V be a function such that

$$V = f_{\partial\Omega}; \quad \text{on } \partial\Omega \tag{10.2b}$$

Then, Dirichlet problem (10.1) is equivalent to

$$\Delta(u - U) = 0; \quad \text{in } \Omega \tag{10.3a}$$

and

$$u = V; \quad \text{on } \partial\Omega \tag{10.3b}$$

In order to formulate this problem precisely, it is necessary to define a space D of admissible functions. Consider Sobolev space $H^s(\Omega)$, where s is any real number $(-\infty < s < \infty)$. As it is well known the trace operator (i.e. the boundary values) is not defined for some elements of $H^s(\Omega)$ when $s \leqq 1/2$ [41, 42]. However, there is a wide class of functions of $H^s(\Omega)$ for which this trace is defined and belongs to $H^{s-1/2}(\partial\Omega)$. Thus, define

$$D^e = \bigcup h^s(\Omega)$$

and

$$D = \left\{ u \in D^e \,|\, \gamma_0 u \in \bigcup_s H^s(\partial\Omega) \right\} \tag{10.5}$$

where γ_0 stands for the trace of u on $\partial\Omega$. In general, for simplicity the symbol γ_0 will be omitted when it is clear from the context that we refer to the boundary values. It can be noticed that the linear space D defined by (10.5) is not closed. Indeed, a metric is not defined in the whole space.

Let

$$N_P = \{ u \in D \,|\, \Delta u = 0 \quad \text{in } \Omega \} \tag{10.6}$$

and

$$I = \{ u \in D \,|\, u = 0 \quad \text{on } \partial\Omega \} \tag{10.7}$$

Then, Dirichlet problem can be formulated as a problem of linear restrictions. Given any $U \in D$ and $V \in D$ (these functions can be taken as data of the problem), find and element $u \in D$ such that

$$u - U \in N_P \quad \text{and} \quad u - V \in I \tag{10.8}$$

The first of equations (10.8) is equivalent to (10.3a), while the second one to (10.3b).

A first advantage of formulating the problem in this manner is connected with its existence properties. Clearly equation (10.3b) is equivalent to $u - U = V - U$ on $\partial\Omega$. By well known results on the existence of solution [41], this problem possesses a unique solution. Indeed, given $U \in D$ and $V \in D$ there are real numbers r and s such that $U \in H^r(\Omega)$ and the trace $\gamma_0(V - U) \in H^s(\partial\Omega)$. Then, $u - U \in H^{s+1/2}(\Omega)$. Therefore, $u = U + (u - U)$ belongs to $H^t(\Omega)$ where $t = \min\{r, s + 1/2\}$. This shows $u \in D$.

The above discussion also shows that there is no lack of generality by restricting attention to the homogeneous case; i.e.

$$\Delta u = 0; \quad \text{on } \Omega \tag{10.9a}$$

and

$$u = f_{\partial\Omega}; \quad \text{on } \partial\Omega \tag{10.9b}$$

The boundary method to be applied depends on the continuity of the solutions on their boundary values. In principle it can be applied when the space of admissible functions D is given by (10.5). However, this would lead to consider inner products in the space of boundary values $H^s(\partial\Omega)$ with arbitrary s, which may be inconvenient in numerical applications. It is preferable to keep the computations in $\mathscr{L}^2(\partial\Omega) = H^0(\partial\Omega)$, which, as will be seen, leads to least-squares fitting. This, can be achieved if attention is restricted to functions with boundary values belonging to $H^0(\partial\Omega) = \mathscr{L}^2(\partial\Omega)$. When this condition is incorporated in the definition of the space of admissible functions, one gets

$$D = \{u \in D^e \mid \gamma_0 u \in H^0(\partial\Omega)\} \tag{10.10a}$$

This is again a linear space which is not closed.

In addition, in many applications it is necessary to compute the normal derivative $\partial u/\partial n$ on the boundary $\partial\Omega$. Similar considerations lead to require that $\partial u/\partial n$ belong to $H^0(\partial\Omega) = \mathscr{L}^2(\partial\Omega)$. When these two requirements are incorporated in the definition of the space of admissible functions, equation (10.5) becomes

$$D = \{u \in D^e \mid \gamma_0 u \in H^0(\partial\Omega), \gamma_1 u \in H^0(\partial\Omega)\} \tag{10.11a}$$

Here, as it is costumary, $\gamma_1 u$ stands for the trace of the normal derivative on $\partial\Omega$. This is again a linear space.

General results on the existence and continuity properties of solutions of elliptic equations [41], imply that any harmonic function u whose trace $\gamma_0 u$ belongs to $H^0(\partial\Omega)$, necessarily is a member of $H^{1/2}(\Omega)$. Therefore $N_P \subset H^{1/2}(\Omega)$ in this case. Even more, due to the continuity properties just mentioned, N_P is a closed subspace of $H^{1/2}(\Omega)$. This will be represented by $N^{1/2}(\Omega)$. Thus

$$N_P = N^{1/2}(\Omega) \tag{10.11b}$$

when D is defined by equation (10.10a). Similarly, when equation (10.11a) holds corresponding properties imply that

$$N_P = N^{3/2}(\Omega) \tag{10.11c}$$

where $N^{3/2}(\Omega)$ is the subspace of harmonic functions belonging to $H^{3/2}(\Omega)$ which can also be shown to be closed.

by its boundary values. instead. The coefficients b_n^N will be chosen so that

$$\left\| \frac{\partial u}{\partial n} - \sum_{n=1}^{N} b_n^N w_n \right\|^2 \tag{10.24}$$

is minimized.

This leads to take the projection of $\partial u/\partial n$ on the space spanned by $\{w_1, \ldots, w_N\} \subset H^0(\partial\Omega)$. This requires the orthogonality condition

$$\left(\frac{\partial u}{\partial n} - \sum_{n=1}^{N} b_n^N w_n, w_m \right) = 0\,, \qquad m = 1, \ldots, N \tag{10.25}$$

to be satisfied. Expanding (10.25), one gets

$$\sum_{n=1}^{N} K_{nm} b_n^N = d_m \tag{10.26}$$

where

$$K_{nm} = \int_{\partial\Omega} w_n w_m \, dx; \qquad n, m = 1, \ldots, N \tag{10.27a}$$

and

$$d_m = \int_{\partial\Omega} \frac{\partial u}{\partial n} w_m \, dx = \int_{\partial\Omega} f_{\partial\Omega} \frac{\partial w_m}{\partial n} dx; \qquad m = \quad , N \tag{10.27b}$$

Observe that the use of the reciprocity relation (10.20) has permitted to express d_m in terms of boundary data only.

An additional point must be mentioned. In order for the approximating sequence $\sum_{n=1}^{N} b_n^N w_n$ to be convergent, it is necessary that the solution $\frac{\partial u}{\partial n} \in H^0(\partial\Omega)$. This is granted if $f_{\partial\Omega} \in H^1(\partial\Omega)$. Alternatively, this condition can be expressed in matrix form. Let $\mathbf{K}^N$ be the $N \times N$ square matrix whose elements are given by (10.27a). Similarly $\mathbf{d}^N$ is the $1 \times N$ vector defined by (10.27b). Assume, the system of traces $\{w_1, \ldots, w_N\} \subset H^1(\partial\Omega)$ is linearly independent, which is required in order for the system (10.26) to be invertible, and denote by $(\mathbf{K}^N)^{-1}$ the inverse of $\mathbf{K}^N$. Then, the sequence of real numbers

$$\left\| \sum_{n=1}^{N} b_n^N w_n \right\|^2 = \mathbf{d}^N \cdot (\mathbf{K}^N)^{-1} \cdot \mathbf{d}^N \geqq 0\,, \qquad N = 1, 2, \ldots \tag{10.28}$$

is non-negative and increasing. Convergence, of the approximating sequence is granted when the sequence (10.28) is bounded. The meaning of this condition is more easily understood by observing that when the system of traces $\{w_1, w_2, \ldots\}$ is orthonormal (i.e. $K_{nm} = \delta_{nm}$), in which case the coefficients d_n are independent of N, it becomes

$$\sum_{n=1}^{\infty} d_n^2 < \infty \tag{10.29}$$

The treatment of Neuman problem is similar. Let the space of admissible functions be given again by equations (10.11a). Then equation (10.9b) is replaced by

$$\frac{\partial u}{\partial n} = g_{\partial\Omega}\,; \qquad \text{on } \partial\Omega \tag{10.30}$$

where the boundary values $g_{\partial\Omega} \in \{1\}^{\perp} \subset H^0(\partial\Omega)$. The previous argument still holds if (10.17) is replaced by

$$M_{nm} = \int_{\partial\Omega} \frac{\partial w_n}{\partial n} \frac{\partial w_m}{\partial n} dx \tag{10.31 a}$$

$$c_m = \int_{\partial\Omega} g_{\partial\Omega} \frac{\partial w_m}{\partial n} dx \tag{10.31 b}$$

In this case $u^N \to u$ in $H^{3/2}(\Omega)$; therefore, also $u^N \to u$ in $H^{1/2}(\Omega)$. It must be observed that this assertion is not strictly true because the solution of Neuman's problem contains an undetermined constant. To remove it one can take $\mathscr{B} = \{1, w_1, w_2, \ldots\} \subset N^{3/2}(\Omega)$ and require

$$\int_{\partial\Omega} w_j \, dx = 0 \, ; \quad j = 1, 2, \ldots \tag{10.32}$$

$$\int_{\partial\Omega} u \, dx = 0 \tag{10.33}$$

In general, if the normal derivative $\dfrac{\partial u^N}{\partial n} \to g_{\partial\Omega}$ in $H^0(\partial\Omega)$, then $u^N \to u$ in $H^{3/2}(\Omega)$; hence, on the boundary $u^N \to u$ in $H^1(\partial\Omega)$, which implies $u^N \to u$ in $H^0(\partial\Omega)$. Thus, the boundary values (i.e. $\gamma_0 u$ on $\partial\Omega$), which in case of Neuman problem are not known beforehand, can be derived from the approximating sequence directly. However, the use of the reciprocity relation (10.20) offers an alternative for computing them. Indeed, one simply has to replace equations (10.21) and (10.27), by

$$u^N = \sum_{n=1}^{N} b_n^N \frac{\partial w_n}{\partial n} \to u \, , \quad \text{in } H^0(\partial\Omega) \tag{10.34}$$

$$K_{nm} = \int_{\partial\Omega} \frac{\partial w_n}{\partial n} \frac{\partial w_m}{\partial n} dx \tag{10.35 a}$$

and

$$d_m = \int_{\partial\Omega} u \frac{\partial w_m}{\partial n} dx = \int_{\partial\Omega} g_{\partial\Omega} w_m \, dx \tag{10.35 b}$$

Again, equations (10.26) have to be satisfied. When this is the case the solution u in (10.34) fulfills (10.33). This method can be used to accelerate the convergence of the approximating sequence on the boundary. As a matter of fact, when the system of equations (10.12), (10.16) and (10.31) is applied, the norm $\left\| \dfrac{\partial u^N}{\partial n} - g_{\partial R} \right\|$ in the $\mathscr{L}^2(\partial\Omega)$ sense, is minimal; however, $\| u^N - u \|$ in $\mathscr{L}^2(\partial\Omega)$ in general is not minimal. When equations (10.26), (10.34) and (10.35) are applied, on the contrary, $\| u^N - u \|$ in the $\mathscr{L}^2(\partial\Omega)$ sense, is minimal; i.e. in the first case, the approximation of the boundary data is optimal, while by the second method, the approximation of the unknown boundary values is optimal. In applications, generally, the latter would be preferable.

Generally, when dealing with partial differential equations only some boundary values of the functions and their derivatives are relevant in the discussion of the problems. For example, for Laplace equation these are the function u and its normal derivative $\partial u/\partial n$. For Elasticity the displacements $\mathbf{u}$ and tractions $\mathbf{T}(\mathbf{u})$. When a boundary value problem is formulated, only one part of this boundary information is prescribed and the other part must be derived after the solution has been obtained. For Dirichlet problem, for example, u is prescribed and $\partial u/\partial n$ is derived. The converse has to be done in the case of Neuman problem. Approximating sequences for the complementary boundary values which depend on reciprocity relations, such as (10.20) can be derived for very general classes of differential equations. The reciprocity relations can be obtained from corresponding Green's formulas. For example, from

$$\int_{\Omega} \{v\, \Delta u - u\, \Delta v\}\, dx = \int_{\partial\Omega} \left\{v\, \frac{\partial u}{\partial n} - u\, \frac{\partial v}{\partial n}\right\} dx \tag{10.36}$$

one obtains

$$\int_{\partial\Omega} v\, \frac{\partial u}{\partial n}\, dx = \int_{\partial\Omega} u\, \frac{\partial v}{\partial n}\, dx \tag{10.37}$$

when u and v are harmonic in Ω. Equation (10.37) can be recognized as (10.20). The procedure used to derive approximations (10.21) and (10.34) can be traced back to a group of italian mathematicians [12–14] and was discussed extensively by Kupradze [15]. The author has introduced an abstract formulation which permits extending this procedure to problems with prescribed jumps [34] (applications to elasticity are given in [5]). This is linked to a systematic classification of boundary values and will be explained in Section 3.

The possibility of applying the boundary method here explained depends on the availability of a system of solutions $\mathscr{B} = \{w_1, w_2, \ldots\} \subset N^{3/2}(\Omega)$ of Laplace equation which spans $N^{3/2}(\Omega)$. In this connection, there are two general categories of theoretical questions which must be analyzed in order to increase the flexibility and versatility of the procedure. These are: criteria for deciding when a system $\hat{\mathscr{B}}$ is complete and methods for constructing complete systems which can be applied to many problems.

Regarding the first one, we have seen that what is required is that the system $\mathscr{B} = \{w_1, w_2 \ldots\}$ spans $N^{3/2}(\Omega)$. However, in applications it is frequently difficult to verify this in a direct manner and it is necessary to use alternative criteria; these can be established by analyzing the spaces spanned by the boundary values. For example, for Laplace equation, given a system of functions $\mathscr{B} = \{w_1, w_2, \ldots\}$ defined in Ω, let us denote by $\hat{w}_\alpha = [w_{\alpha 1}, w_{\alpha 2}]$ the system of traces $w_{\alpha 1} = \gamma_0 w_\alpha$ and $w_{\alpha 2} = \gamma_1 w_\alpha$. In addition $\hat{\mathscr{B}} = \{\hat{w}_1, \hat{w}_2, \ldots\}$, $\mathscr{B}_1 = \{w_{11}, w_{21}, \ldots\}$ and $\mathscr{B}_2 = \{w_{12}, w_{22}, \ldots\}$. For example, when the region Ω is a circle (the unit circle for definiteness), by separation of variables one obtains (in polar coordinates)

$$\mathscr{B} = \{1; r^n \cos n\,\theta, r^n \sin n\,\theta;\ n = 1, 2, \tag{10.38}$$

This system is made of harmonic polynomials

$$\mathscr{B} = \{1, x^2 - y^2, x\,y, \ldots\} \tag{10.39}$$

which can be recognized as $\operatorname{Re} z^n$ and $\operatorname{Im} z^n$ ($n = 0, 1,$).

Setting $r = 1$ in (10.38), one obtains the system of traces

$$\mathscr{B}_1 = \{\cos n\,\theta, \sin n\,\theta;\ n = 0, 1, \ldots \tag{10.40 a}$$

and

$$\mathscr{B}_2 = \{-n \sin n\,\theta, n \cos n\,\theta;\ n = 0, 1, \ldots\} \tag{10.40 b}$$

Denote by N_1 and N_2 the spaces spanned in the $\mathscr{L}^2(\partial\Omega)$ metric by the traces $\gamma_0 u$ and $\gamma_1 u$, respectively, when u ranges over $N^{3/2}(\Omega)$. Clearly, $N_1 = \mathscr{L}^2(\partial\Omega) = H^0(\partial\Omega)$ while $N_2 = \{1\}^\perp \subset \mathscr{L}^2(\partial\Omega)$, Here, the orthogonal complement $\{1\}^\perp$ is taken in the $\mathscr{L}^2(\partial\Omega)$ inner product.

Let $\mathscr{B} \subset N^{3/2}(\Omega)$ be a system such that

$$\operatorname{span} \mathscr{B}_1 = N_1 = \mathscr{L}^2(\partial\Omega) \quad \text{and} \quad \operatorname{span} \mathscr{B}_2 = N_2 = \{1\}^\perp \tag{10.41}$$

where the spans are taken in the $\mathscr{L}^2(\partial\Omega)$ sense.

For simplicity, assume that the constant function $w_0 = 1$ is a member of $\mathscr{B}$, so that

$$\mathscr{B} = \{1\} \cup \mathscr{B}' \tag{10.42}$$

where $\mathscr{B}' = \{w_1, w_2, \quad\}$. It will also be assumed that

$$\int_{\partial\Omega} w_\alpha\, dx = 0\,; \qquad \alpha = 1, 2, \tag{10.43}$$

Any harmonic function $u \in N^{3/2}(\Omega)$ can be written uniquely as

$$u = a_0 + u' \tag{10.44}$$

where a_0 is the constant

$$a_0 = \int_{\partial\Omega} u\, dx\,, \quad \text{while} \int_{\partial\Omega} u'\, dx = 0 \tag{10.45}$$

In view of (10.41) and $\gamma_1 w_0 = 0$, it is clear that

$$\operatorname{span} \mathscr{B}_2' = \{1\}^\perp \tag{10.46}$$

Also $\gamma_1 u' \in \{1\}^\perp$, since u' is harmonic in Ω, so that $\gamma_1 u'$ is in the $\mathscr{L}^2(\Omega)$ – span of $\mathscr{B}_2'$. This shows that there is a sequence v^N of linear combination of $\mathscr{B}'$ such that

$$\gamma_0 v^N \xrightarrow[N \to \infty]{} \gamma_0 u'\,, \qquad \text{in } \mathscr{L}^2(\partial\Omega) \tag{10.47}$$

In view of (10.43), the second of conditions (10.45) and continuity properties [41] of solutions of elliptic equations, it is clear that $v^N \to u'$ in the metric of $H^{3/2}(\Omega)$. Therefore, the linear combination $u^N = a_0 + v^N$ of elements of $\mathscr{B} \subset N^{3/2}(\Omega)$, is such that $u^N \to u$ in $H^{3/2}(\Omega)$. This shows that

$$\operatorname{span} \mathscr{B} = N^{3/2}(\Omega) \tag{10.48}$$

where the span is taken in the $H^{3/2}(\Omega)$ metric. Thus, in this case we have derived the completeness of the system $\mathscr{B} \subset N^{3/2}(\Omega)$ from the fact that the system of traces $\mathscr{B}_1$, spans the same space as the traces of harmonic functions (i.e. solutions of the homogeneous equation) in $N^{3/2}(\Omega)$. Similar results hold in a more general context.

Let $\hat{\mathscr{H}} = H^0(\partial\Omega) \oplus H^0(\partial\Omega)$ be the space of pairs $\hat{u} = [u_1, u_2]$ with $u_1 \in H^0(\partial\Omega)$ and $u_2 \in H^0(\partial\Omega)$, provided with the usual inner product

$$((\hat{u}, \hat{v})) = (u_1, v_1)_\partial + (u_2, v_2)_\partial \tag{10.49}$$

Denote by $\hat{\mathcal{N}} \subset \hat{\mathcal{H}}$ the image of $N^{3/2}(\Omega)$ under the mapping $u \to \hat{u} = [\gamma_0 u, \gamma_1 u] \in \hat{\mathcal{H}}$. It can be shown that $\hat{\mathcal{N}} \subset \hat{\mathcal{H}}$ is closed in the metric of $\hat{\mathcal{H}}$. Notice that the reciprocity relation (10.20) becomes (we assume Hilbert-spaces are being taken with real coefficients):

$$(v_1, u_2) = (v_2, u_1) \; \forall \; \hat{u} \in \hat{\mathcal{N}} \; \& \; \hat{v} \in \hat{\mathcal{N}} \tag{10.50}$$

A system of function $\mathcal{B} \subset N^{3/2}(\Omega)$ will be said to be T-complete* if, for every $\hat{u} \in \hat{\mathcal{H}}$, one has

$$(w_1, u_2) = (w_2, u_1) \; \forall \; w \in \mathcal{B} \Rightarrow \hat{u} \in \hat{\mathcal{N}} \tag{10.51}$$

Using this notation the following characterization of complete systems holds [19, 34].

Theorem 10.1. *Let $\mathcal{B} \subset N^{3/2}(\Omega)$. Then the following assertions are equivalent:*

(i) *$\mathcal{B} \subset N^{3/2}(\Omega)$ spans $N^{3/2}(\Omega)$ in the metric of $H^{3/2}(\Omega)$;*
(ii) *$\hat{\mathcal{B}} \subset \hat{\mathcal{H}}$ spans $\hat{\mathcal{N}}$ in the metric of $\hat{\mathcal{H}}$;*
(iii) *$\hat{\mathcal{B}} \subset \hat{\mathcal{N}}$ is a T-complete system;*
(iv) *Equations (10.41) are satisfied when the spans are taken in the $\mathcal{L}^2(\partial\Omega)$ sense.*

An advantage of having a system which satisfies any of the criteria (i) to (iv), is that the same system can be used for both a Dirichlet and a Neuman problem. Indeed, the same T-complete system can be used for any linear boundary condition which is prescribed point-wise. Such condition can be written as

$$a_1 u + a_2 \partial u / \partial n = f_{\partial R}, \quad \text{on } \partial\Omega \tag{10.52}$$

The arguments presented previously, can be extended to this case by introduction of more general Green's formulas. This will be discussed in Section 3.

It has interest to observe that it is possible to develop systems which are complete in regions which are, to a large extent, arbitrary. For example, the system of harmonic polynomials given by (10.38) and (10.39), is T-complete in any bounded and simply connected region [29]. Also, the system

$$\{\operatorname{Log} r, \operatorname{Re} z^{-n}, \operatorname{Im} z^{-n}; \; n = 1, 2, \ldots\} \tag{10.53}$$

is T-complete in the exterior of any simply connected and bounded region which contains the origin.

To develop general criteria establishing conditions under which a system which is complete in a region is also complete in another one, is quite valuable. Especially if such criteria are applicable to a wide class of partial differential equations. For this purpose the notion of T-completeness is useful.

10.3 Green's Formulas

The development of Green's formulas for general classes of partial differential equations is a classical topic of the theory of partial differential equations [41]. A

* Trefftz-complete. Previously, such systems had been called c-complete by the author.

theory which permits obtaining such formulas systematically and which in some respects enlarges the kind of problems that can be treated in this manner, has been developed by the author [19, 28, 34]. The fundamental notions are closely related with simplectic geometry [44].

Basically, what is done is to characterize the space of boundary values which are relevant for each differential equation or system of such equations. Then such space is decomposed into two subspaces. With every Green's formula there is associated such a decomposition and conversely with every decomposition there is a unique Green's formula. A procedure for reconstructing the Green's formula when the decomposition is known, is established [34].

We consider a bilinear functional P defined on an arbitrary linear space D; it will be denoted by $P: D \to D^*$ because it can be thought as an operator defined on the linear space D and taking values on its algebraic dual D^* (this is the space of linear functionals defined on D) [45]. The value of such bilinear functional at elements $u \in D$ and $v \in D$, will be denoted by $\langle P u, v \rangle$. The transposed bilinear functional of $P: D \to D^*$, will be $P^*: D \to D^*$; thus

$$\langle P^* u, v \rangle = \langle P v, u \rangle \tag{10.54}$$

The theory is applicable to general non-symmetric linear operators, although its application to formally symmetric ones is simpler, because it does not require the introduction of a formal adjoint. Here, attention is restricted to such operators. Given an operator $P: D \to D^*$, we define the antisymmetric bilinear form

$$A = P - P^* \tag{10.55}$$

The operator A, given by (10.55) plays a central role in the theory. Firstly, we are going to use it, to define the relevant boundary values. For this purpose, we consider the null subspace N_A of A; i.e.

$$N_A = \{u \in D \mid A u = 0\} \tag{10.56}$$

With reference to the reduced wave equation

$$\Delta u + k^2 u = 0, \quad \text{on } R \quad . \tag{10.57}$$

as an example (recall that Laplace equation corresponds to the case $k = 0$), consider the bilinear functional $P: D \to D^*$, given by

$$\langle P u, v \rangle = \int_R v\, (\Delta u + k^2 u)\, dx \tag{10.58}$$

Then $A = P - P^*$ is

$$\langle A u, v \rangle = \int_{\partial R} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} dx \tag{10.59}$$

The null subspace N_P, is the linear subspace of functions which satisfy (10.57).

There are many ways of taking the linear space D. A convenient one is by means of equation (10.11a). This defines a linear subspace, but we do not introduce a topology in it. We notice that the null subspace N_P is well defined, if (10.57) is interpreted in the sense of distributions [41]. Also the bilinear form $A: D \to D^*$, given by (10.59); however, the operator $P: D \to D^*$, given by (10.58) is not. Many technical difficulties are avoided by leaving the operator P out of the discussion.

It is easy to see that

$$N_A = \left\{ u \in D \mid u = \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega \right\} \tag{10.60}$$

Due to (10.60), the relevant boundary values for Laplace and reduced wave equations (10.60), will be u and $\frac{\partial u}{\partial n}$, on $\partial\Omega$. We notice that given $u \in D$ and $v \in D$, one has that

$$u = v; \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n}, \qquad \text{on } \partial\Omega \tag{10.61}$$

if and only if $u - v \in N_A$; i.e. two functions $u \in D$ and $v \in D$ have the same relevant boundary values, if and only if, $u - v \in N_A$.

Similar notions can be applied to any linear differential equation. Let us consider the biharmonic equation:

$$\Delta^2 u = 0; \qquad \text{on } \Omega \tag{10.62}$$

which occurs, for example, in connection with incompressible flows at low Reynolds numbers. Define

$$\langle P\, u, v \rangle = \int_\Omega v\, \Delta^2 u \, dx \tag{10.63}$$

Then

$$\langle A\, u, v \rangle = \int_{\partial\Omega} \left\{ v \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial v}{\partial n} + \Delta v \frac{\partial u}{\partial n} - u \frac{\partial \Delta v}{\partial n} \right\} dx \tag{10.64}$$

Again, a convenient definition of the space D is (see equation (10.4)):

$$D = \left\{ u \in D^e \mid u, \frac{\partial u}{\partial n}, \Delta u \text{ and } \frac{\partial \Delta u}{\partial n} \text{ belong to } H^0(\partial\Omega) \right\} \tag{10.65}$$

Then, A as given by (10.64) is well defined, and N_P can be taken as the linear subspace of D which satisfies (10.62) in the sense of distributions. The operator $P: D \to D^*$, given (10.63), is not defined for this space D, and we leave it out from our discussion.

The null subspace N_A, is

$$N_A = \left\{ u \in D \mid u = \frac{\partial u}{\partial n} = \Delta u = \frac{\partial \Delta u}{\partial n} = 0, \text{ on } \partial\Omega \right\} \tag{10.66}$$

The classification of boundary values induced by (10.66), is characterized by quadruplets of functions u, $\frac{\partial u}{\partial n}$, Δu, $\frac{\partial \Delta u}{\partial n}$; recall that these functions yield enough information to have u and its derivatives up to order 3 determined

The homogeneous stationary Stokes equations are

$$\nu \Delta \mathbf{u} - \nabla p = 0 \tag{10.67 a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{10.67 b}$$

where ν is the viscosity. In this case, it is convenient to define the bilinear form $P: D \to D^*$, by

$$\langle P\, \hat{u}, \hat{v} \rangle = \int_\Omega \{ \mathbf{v} \cdot (\nu \Delta \mathbf{u} - \nabla p) + q \nabla \cdot \mathbf{u} \} \, dx \tag{10.68}$$

Here, $\hat{u}$ stands for a pair of functions; $\mathbf{u}$ which is vector valued and defined in Ω, and p scalar valued and also defined in Ω. With $\hat{v}$, we have associated the pair $\mathbf{v}, q$. Then

$$\langle A\,\hat{u}, \hat{v}\rangle = \int_{\partial\Omega} \left\{ \mathbf{v} \cdot \left(\nu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} \right) - \mathbf{u} \cdot \left(\nu \frac{\partial \mathbf{v}}{\partial n} - q\,\mathbf{n} \right) \right\} dx\,. \tag{10.69}$$

Elements of the linear space D will be pairs $\hat{u} = [\mathbf{u}, p]$ such that the traces $\mathbf{u}$ and $\nu \dfrac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n}$ are well defined and span $\mathbf{H}^0(\partial\Omega)$. One must also require that the set of functions $N_P \subset D$ which satisfy Stokes equations (10.67) in the sense of distributions be well defined. In general, $P: D \to D^*$ may not be defined in this space. The null subspace

$$N_A = \left\{ \hat{u} \in D \mid \mathbf{u} = \nu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} = \mathbf{0}, \text{ on } \partial\Omega \right\}. \tag{10.70}$$

The classification of boundary values induced by (10.70) is characterized by the values of $\mathbf{u}$ and $\nu \dfrac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n}$ on the boundary $\partial\Omega$.

As it has been seen in the specific examples given thus far, in general, it is not necessary to define on operator $P: D \to D^*$ in order for the theory to be applicable. Thus, in what follows, it will simply be assumed that there is available an antisymmetric bilinear form $A: D \to D^*$.

A subspace $I \subset D$ is said to be regular for A, when

(i) For every $u \in I$ and $v \in I$,

$$\langle A\,u, v\rangle = 0 \tag{10.71}$$

i.e. I is a commutative subspace for A.

(ii)

$$I \supset N_A \tag{10.72}$$

We have seen that the null subspace N_A, induces a classification of D which defines what could be properly called, the boundary values which are relevant for the differential equation considered. In the light of this fact, condition (ii) implies that a regular subspace is characterized by boundary values, only.

To illustrate this fact, assume, $I \subset D$ is a regular subspace. In connection with the examples given previously, let $u \in D$ and $v \in D$, be such that

$$u = v\,; \ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n}, \quad \text{on } \partial\Omega \tag{10.73}$$

when the reduced wave equation is considered; or

$$u = v\,; \ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n}\,; \ \Delta u = \Delta v\,; \ \frac{\partial \Delta u}{\partial n} = \frac{\partial \Delta v}{\partial n}\,; \quad \text{on } \partial\Omega \tag{10.74}$$

for the biharmonic equation. Then, there are only two mutually exclusive possibilities

a) u and v belong to I, or
b) neither u, nor v belongs to I.

A corresponding proposition holds for $\hat{u} \in D$ and $\hat{v} \in D$, in connection with Stokes equations, when it is assumed that

$$\mathbf{u} = \mathbf{v}; \quad \nu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} = \nu \frac{\partial \mathbf{v}}{\partial n} - q\,\mathbf{n}; \quad \text{on } \partial\Omega \tag{10.75}$$

The following statement summarizes this discussion. A regular subspace, is a commutative subspace which is defined through boundary values only.

Examples of regular subspaces for the reduced wave equation are

$$I_1 = \{u \in D \mid u = 0, \text{ on } \partial\Omega\} \tag{10.76 a}$$

$$I_2 = \left\{u \in D \,\middle|\, \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega\right\} \tag{10.76 b}$$

and

$$I_3 = \left\{u \in D \quad \alpha \frac{\partial u}{\partial n} + \beta u = 0, \text{ on } \partial\Omega\right\} \tag{10.76 c}$$

where $\alpha^2 + \beta^2 \neq 0$.

Many examples of regular subspaces can be given for the biharmonic equation; an interesting set of such subspaces is

$$I_1 = \left\{u \in D \,\middle|\, u = \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega\right\}, \tag{10.77 a}$$

$$I_2 = \{u \in D \mid u = \Delta u = 0, \text{ on } \partial\Omega\} \tag{10.77 b}$$

and

$$I_3 = \left\{u \in D \quad \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \text{ on } \partial\Omega\right\} \tag{10.77 c}$$

More general examples were given previously [8].

For Stokes problem we have the following regular suspaces

$$I_1 = \{\hat{u} \in D \mid \mathbf{u} = 0, \text{ on } \partial\Omega\} \tag{10.78 a}$$

$$I_2 = \left\{\hat{u} \in D \,\middle|\, \nu \frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} \in H^0(\partial\Omega)\right\} \tag{10.78 b}$$

Of course, many more can be given.

Of special interest is the case when a regular subspace $I \subset D$, has the following additional property

(iii) For every $u \in D$

$$\langle A u, v\rangle = 0 \;\forall\, v \in I \Rightarrow u \in I \tag{10.79}$$

A regular subspace, which enjoys (iii) is called completely regular.

It is not difficult to verify that in all the examples given in equations (10.76) through (10.78) the subspaces are, actually, completely regular.

Given an antisymmetric bilinear from $A : D \to D^*$, a pair of subspaces $\{I_1, I_2\}$ is said to be a canonical decomposition of D for A, when

(i) I_1 and I_2 are regular subspaces; and

(ii) $D = I_1 + I_2$. (10.80)

It has been shown [28, 34] that when $\{I_1, I_2\}$ is a canonical decomposition of D, then I_1 and I_2 are necessarily, completely regular and

$$N_A = I_1 \cap I_2 \tag{10.81}$$

Now, condition (10.80) is equivalent to the requirement that given any $u \in D$, one can find elements $u_1 \in I_1$ and $u_2 \in I_2$ such that

$$u = u_1 + u_2 \tag{10.82}$$

In the presence of equation (10.81), this representation of u, is unique except for elements of subspace N_A; more precisely, if $u_1' \in I_1$ and $u_2' \in I_2$ are such that

$$u = u_1' + u_2' \tag{10.83}$$

then $u_1 - u_1' \in N_A$ and $u_2 - u_2' \in N_A$. Taking into account that N_A is the set of functions with vanishing boundary values, it is seen that the boundary values of u_1 and u_2 are uniquely defined. Thus, when a canonical decomposition $\{I_1, I_2\}$ is available, representation (10.82) supplies a convenient manner of dividing the information on the boundary values of the function u into two parts, $u_1 \in I_1$ and $u_2 \in I_2$, which is useful in the formulation of many boundary value problems.

For the reduced wave equation, the pair $\{I_1, I_2\}$, defined by (10.76 a) and (10.76 b), constitutes a canonical decomposition of the space D, with respect to A, as defined by (10.59). In this case, the representation (10.82), breaks the boundary information in the following manner

$$u = u_2; \quad \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n}; \quad \text{on } \partial\Omega \tag{10.84}$$

The pair $\{I_1, I_3\}$, given by (10.76 a) and (10.76 c), is also a canonical decomposition, whenever $\alpha \neq 0$. In this case, if $u = u_1 + u_3$, with $u_1 \in I_1$ and $u_3 \in I_3$, then the boundary values are given by

$$u = u_1 + u_3; \quad \frac{\partial u}{\partial n} = \frac{\partial u_1}{\partial n} + \frac{\partial u_3}{\partial n}; \quad \text{on } \partial\Omega \tag{10.85}$$

If we define

$$I_4 = \left\{ u \in D \quad \gamma \frac{\partial u}{\partial n} + \delta u = 0, \text{ on } \partial\Omega \right\} \tag{10.86}$$

it is easy to see that $\{I_3, I_4\}$ is a canonical decomposition, whenever $\alpha\delta - \beta\gamma \neq 0$. Clearly, the previous ones are particular cases of this more general canonical decomposition.

For the biharmonic equation, the following pair is a canonical decomposition

$$I_1 = \left\{ u \in D \;\middle|\; u = \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega \right\} \tag{10.87 a}$$

$$I_2 = \left\{ u \in D \;\middle|\; \Delta u = \frac{\partial \Delta u}{\partial n} = 0, \text{ on } \partial\Omega \right\} \tag{10.87 b}$$

Also

$$I_1 = \{ u \in D \mid u = \Delta u = 0, \text{ on } \partial\Omega \} \tag{10.88 a}$$

$$I_2 = \left\{ u \in D \;\middle|\; \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \text{ on } \partial\Omega \right\} \tag{10.88 b}$$

Small

Finally, for Stokes problems one has

$$I_1 = \{\hat{u} \in D \mid \mathbf{u} = \mathbf{0}, \text{ on } \partial\Omega\} \tag{10.89 a}$$

$$I_2 = \left\{\hat{u} \in D \;\middle|\; v\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} = \mathbf{0}, \text{ on } \partial\Omega\right\} \tag{10.89 b}$$

Of course many more can be constructed.

In many boundary value problems the prescribed boundary data is given by means of one of the elements in (10.82), for example u_1, and the complementary boundary information u_2, can only be obtained after the boundary value problem has been solved. In Dirichlet problem for example, u is prescribed on $\partial\Omega$ and the derived boundary information $\frac{\partial u}{\partial n}$ on $\partial\Omega$, is obtained, only after the problem has been solved.

The notion of Green's formula is closely related with that of canonical decomposition. Some auxiliary notions are required in order to introduce abstract Green's formulas.

Given the bilinear form $B : D \to D^*$, let

$$N_B = \{u \in D \mid B\,u = 0\} \tag{10.90}$$

be the null subspace of B. Then, if

$$D = N_B + N_{B^*} \tag{10.91}$$

where $B^* : D \to D^*$ is the transposed bilinear form of B, one says that B and B^* can be varied independently. When B and B^* can be varied independently and

$$A = B - B^* \tag{10.92}$$

equation (10.92) is called a Green's formula. It can be shown [34] that in this case $B : D \to D^*$ is necessarily a boundary operator.

There is a general result of the theory according to which there is a one-to-one correspondence between canonical decompositions $\{I_1, I_2\}$ and Green's formulas. This is established as follows:

(i) Given a Green's formula, define

$$I_1 = N_{B^*}; \qquad I_2 = N_B \tag{10.93}$$

then $\{I_1, I_2\}$ is a canonical decomposition.

(ii) Given a canonical decomposition $\{I_1, I_2\}$, let $B : D \to D^*$ be defined by

$$\langle B\,u, v\rangle = \langle A\,u_1, v_2\rangle \tag{10.94}$$

Here, the representation (10.82) of every element $u \in D$ of the space, in terms of its components $u_1 \in I_1$ and $u_2 \in I_2$, has been used.

To illustrate these notions, in the case of Laplace and reduced wave equation, we notice that if we define

$$\langle B\,u, v\rangle = \int_{\partial R} v\,\frac{\partial u}{\partial n}\,dx \tag{10.95}$$

then (10.91) and (10.92) are fulfilled. Also, the canonical decomposition $\{I_1, I_2\}$, given by (10.76 a, b) satisfies (10.93).

In the case of the biharmonic equation, the canonical decomposition (10.87), is associated with

$$\langle B\,u, v\rangle = \int_{\partial R} \left\{ v\,\frac{\partial \Delta u}{\partial n} - \Delta u\,\frac{\partial v}{\partial n} \right\} dx \tag{10.96}$$

The canonical decomposition (10.88), on the other hand, yields

$$\langle B\,u, v\rangle = \int_{\partial R} \left\{ v\,\frac{\partial \Delta u}{\partial n} + \Delta v\,\frac{\partial u}{\partial n} \right\} dx \tag{10.97}$$

Finally, for Stokes equations, the canonical decomposition (10.89), is associated with

$$\langle B\,\hat{u}, \hat{v}\rangle = \int_{\partial R} \mathbf{v}\cdot\left(\nu\,\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} \right) dx \tag{10.98}$$

10.4 Illustration of Green's Formulas

In this section general examples of Green's formulas are presented. Many of the operators listed are formally symmetric in the classical sense; others can be included due to the extension of this concept introduced in the algebraic theory of boundary value problems [19, 34] which supplies the basic frame-work for this chapter.

Elliptic Equations

This subject is classical. The reader is referred to the book by Lions and Magenes [41]. The extension of such formulas to problems with prescribed jumps can be done along the lines presented in Section 5. A general discussion of Green's formulas from the point of view of the algebraic theory will appear soon [19].

Time Dependent Problems

For a discussion of the spaces which are suitable for the formulation of this class of problems, the reader is referred to the second volume of the treatise by Lions and Magenes [41]. In this Section we simply assume that the linear space of functions D is such that the operators to be considered are well defined.

Two examples will be given: the heat and the wave equations. These can be associated with formally symmetric operators, in the sense of the algebraic theory [19, 34], using Gurtin's convolutions [46, 47]. The basic ideas can be applied to more general problems. For each one of these operators we give only one Green's formula; of course, many more can be constructed.

(i) The Heat Equation. Consider the cylinder $\Omega \times [0, T]$ (Fig. 10.2). Let the linear space D, be made of functions defined on $\Omega \times [0, T]$. The operator $P : D \to D^*$, is defined by

$$\langle P\,u, v\rangle = \int_{\Omega} v * \left(\frac{\partial u}{\partial t} - \Delta u \right) dx \tag{10.99}$$

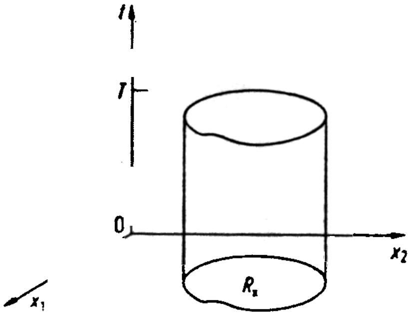

Fig. 10.2.

where the notation

$$u * v = \int_0^T u\,(T-t)\,v\,(t)\,dt \tag{10.100}$$

is used. Let $A = P - P^*$, then

$$A = B + B^* \tag{10.101}$$

where

$$\langle B\,u, v\rangle = \int_{\partial\Omega} u * \frac{\partial v}{\partial n}\,dx - \int_{\Omega} v\,(T)\,u\,(0)\,dx \tag{10.102}$$

(ii) The Wave Equation. The incorporation of this equation in the frame-work here presented is similar.

Taking the region $\Omega \times [0, T]$ and the linear space of functions D, as explained before, define $P: D \rightarrow D^*$ by

$$\langle P\,u, v\rangle = \int_{\Omega} v * \left(\frac{\partial^2 u}{\partial t^2} - \Delta u\right) dx \tag{10.103}$$

with the convention (10.100). A Green's formula for this operator is obtained taking $B: D \rightarrow D^*$ as

$$\langle B\,u, v\rangle = \int_{\partial\Omega} u * \frac{\partial v}{\partial n}\,dx - \int_{\Omega} \left\{ v\,(T)\,\frac{\partial u}{\partial t}\,(0) + \frac{\partial v}{\partial t}\,(T)\,u\,(0) \right\} dx \tag{10.104}$$

Formulas (10.102) and (10.104) are suitable for application to initial value problems when the function u is prescribed on the lateral boundary of the space-time cylinder (Fig. 10.2). More general boundary conditions can be treated by using the Green's formula of the Laplace operator, associated with the canonical decomposition defined by (10.76 c) and (10.86).

Elasticity

Let the elastic tensor C_{ijpq} be $C^{\infty}(\Omega)$, satisfy the usual symmetry conditions [48]

$$C_{ijpq} = C_{pqij} = C_{jipq} \tag{10.105}$$

and be strongly elliptic; i.e.

$$C_{ijpq}\,\xi_i\,\eta_j\,\xi_p\,\eta_q > 0 \quad \text{whenever} \quad \xi \neq 0,\ \|\eta\| \neq 0 \tag{10.106}$$

(i) Static and Periodic Motions. Let $D = H^s(\Omega) = H^s(\Omega) \oplus H^s(\Omega) \oplus H^s(\Omega)$, $s \geq 2$. Define

$$\tau_{ij}(u) = C_{ijpq}\frac{\partial u_p}{\partial x_q}, \quad \text{on } \Omega \tag{10.107}$$

$$\mathscr{S}_i(u) = \frac{\partial \tau_{ij}}{\partial x_j} + \varrho\,\omega^2 u_i, \quad \text{on } \Omega \tag{10.108}$$

where summation convention is understood. Here the density ϱ, is a function of position belonging to $C^\infty(\Omega)$ while ω is a constant. The case $\omega = 0$, is associated with elastostatics.

Let $P : D \to D^*$ be

$$\langle P\,u, v\rangle = \int_\Omega v_i\,\mathscr{S}_i(u)\,dx \tag{10.109}$$

Then, $A = P - P^*$ is given by

$$\langle A\,u, v\rangle = \int_{\partial\Omega} \{v_i\,T_i(u) - u_i\,T_i(u)\}\,dx \tag{10.110}$$

where

$$T_i(u) = \tau_{ij}(u)\,n_j. \tag{10.111}$$

An operator $B : D \to D^*$ that decomposes A is

$$\langle B\,u, v\rangle = -\int_{\partial\Omega} u_i\,T_i(v)\,dx \tag{10.112}$$

There are many more.

(ii) Dynamics. Let D be a suitable linear space of functions defined on $\Omega \times [0, T]$. Define

$$\langle P\,u, v\rangle = \int_\Omega v_i * \left(\varrho\frac{\partial^2 u_i}{\partial t^2} - \mathscr{S}_i\,u\right)dx \tag{10.113}$$

where the conventions (10.100) and (10.108) (with $\omega = 0$) are used. $A = P - P^*$ is given by

$$\langle A\,u, v\rangle = \int_{\partial\Omega} \{u_i * T_i(v) - v_i * T_i(u)\}\,dx \tag{10.114}$$

$$+ \int_\Omega \varrho\left\{v_i(0)\frac{\partial u_i}{\partial t}(T) + \frac{\partial v_i}{\partial t}(0)\,u_i(T) - u_i(0)\frac{\partial v_i}{\partial t}(T) - \frac{\partial u_i}{\partial t}(0)\,v_i(T)\right\}$$

Many operators that decompose A can be constructed. One such operator is

$$\langle B\,u, v\rangle = \int_{\partial\Omega} u_i * T_i(v)\,dx - \int_\Omega \left\{u_i(0)\frac{\partial v_i}{\partial t}(T) - \frac{\partial u_i}{\partial t}(0)\,v_i(T)\right\}dx \tag{10.115}$$

10.5 Green's Formulas in Discontinuous Fields

An advantage of introducing abstract boundary operators is the large class of problems that can be formulated using them; a very general example is the problem of connecting or matching [28]. This is an abstract version of problems formulated in discontinuous fields with prescribed jump conditions.

Such problems occur in many applications. In potential theory, for example, the jumps of the function and its normal derivative are usually prescribed, while in Elasticity the prescribed functions are the jumps of the displacements and the tractions. Variational principles for some of these problems were developed by Prager [49] and Nemat-Nasser [50, 51] presented more recent surveys. Here, general Green's formulas for such problems are developed which are applicable irrespectively of the specific operators.

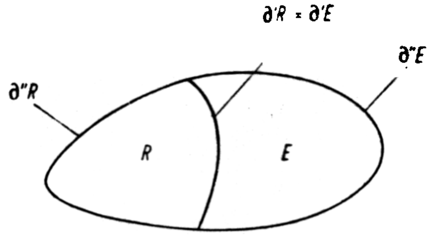

Fig. 10.3.

Consider two neighboring regions R and E (Fig. 10.3), let $\partial' R = \partial' E$ be the common boundary separating them; in addition, $\partial'' R$ and $\partial'' E$ will be the remaining parts of the boundaries of R and E, respectively. Let D_R and D_E be two linear spaces; in the applications to be made their elements will be functions defined on R and E, respectively. Consider the product space $\hat{D} = D_R \oplus D_E$; elements $\hat{u} \in \hat{D}$ are pairs $\hat{u} = \{u_R, u_E\}$ where $u_R \in D_R$ while $u_E \in D_E$. Given operators $P_R \colon D_R \to D_R^*$ and $P_E \colon D_E \to D_E^*$, define $\hat{P} \colon \hat{D} \to \hat{D}^*$ by

$$\langle \hat{P}\hat{u}, \hat{v} \rangle = \langle P_R u_R, v_R \rangle + \langle P_E u_E, v_E \rangle \tag{10.166}$$

This additive property is usually satisfied when $\hat{P} \colon \hat{D} \to \hat{D}^*$ is defined by means of an integral on the region $R \cup E$. From (10.116) it follows that

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle \tag{10.117}$$

The symbol $\hat{N}_A$ will be used for the null subspace of $\hat{A} \colon \hat{D} \to \hat{D} \to \hat{D}^*$. A linear subspace $\hat{S} \subset \hat{D}$ will be considered. Elements $\hat{u} = \{u_R, u_E\} \in \hat{S}$ will be said to be smooth. When $\hat{u} = \{u_R, u_E\}$ is smooth, $u_R \in D_R$ and $u_E \in D_E$ will be said to be smooth extension of each other.

Let $\hat{S} \subset \hat{D} = D_R \oplus D_E$ be a linear subspace. Then $\hat{S}$ is said to be a smoothness relation if every $u_R \in D_R$ possesses at least one smooth extension $u_E \in D_E$ and conversely. A smoothness relation $\hat{S}$ is said to be regular or completely regular for $\hat{P}$, when as a subspace, it is regular or completely regular for $\hat{P}$, respectively. Therefore, a smoothness relation $\hat{S}$ is regular when

a) $$\hat{S} \supset \hat{N}_A \tag{10.118a}$$

and

b) $$\langle \hat{A}\hat{u}, \hat{v} \rangle = 0 \;\forall\; \hat{u} \in \hat{S} \text{ and } \hat{v} \in \hat{S} \tag{10.118b}$$

Similarly, it is completely regular when

$$\langle \hat{A}\hat{u}, \hat{v} \rangle = \langle A_R u_R, v_R \rangle + \langle A_E u_E, v_E \rangle = 0 \quad \forall\; \hat{v} \in \hat{S} \Leftrightarrow \hat{u} \in \hat{S} \tag{10.119}$$

The mapping $\tau: \hat{D} \to \hat{D}$ defined by $\tau\,\hat{u} = \{u_R, -u_E\}$, for every $u = \{u_R, u_E\} \in \hat{D}$ will be used in the following discussion. Given $\hat{S}$, let $\tau\,\hat{S}$ be the image of $\hat{S}$ under this mapping; i.e.

$$\tau\,\hat{S} = \{\hat{u} = \tau\,\hat{v} \in \hat{D} \mid \hat{v} \in \hat{S}\} \tag{10.120}$$

Given $\hat{u} = \{u_R, u_E\}$, let be $\hat{u}' = \{u'_R, u'_E\}$, where u'_R and u'_E are smooth extensions of u_E and u_R, respectively. Define

$$\bar{u} = \tfrac{1}{2}(\hat{u} + \hat{u}') = \tfrac{1}{2}\{u_R + u'_R, u_E + u'_E\} \tag{10.121 a}$$

and

$$[\hat{u}] = \hat{u}' - \hat{u} = \{u'_R - u_R, u'_E - u_E\} \tag{10.121 b}$$

Then

$$\hat{u} = \bar{u} - \tfrac{1}{2}[\hat{u}] \tag{10.122}$$

and it can be seen that $\hat{u} \in \hat{S}$ while $[\hat{u}] \in \tau\,\hat{S}$. Therefore

$$\hat{D} = \hat{S} + \tau\,\hat{S} \tag{10.123}$$

From (10.123), it follows that when $\hat{S} \subset \hat{D}$ is a regular smoothness condition, then the pair $\tau\,\hat{S}$, $\hat{S}$ is a canonical decomposition of $\hat{D}$ with respect to $\hat{P}: \hat{D} \to \hat{D}^*$.

Application of formula (10.96), shows that the relation

$$\hat{P} - \hat{P}^* = \hat{A} = \hat{J} - \hat{J}^* \tag{10.124}$$

is a Green's formula when $\hat{J}: \hat{D} \to \hat{D}^*$ is defined by

$$\langle \hat{J}\,\hat{u}, \hat{v} \rangle = -\tfrac{1}{2}\langle \hat{A}\,[\hat{u}], \hat{v} \rangle \tag{10.125}$$

/ v

This is called jump operator [19, 34] because it characterizes the jumps since

$$\hat{J}\,\hat{u} = \hat{J}\,\hat{v} \Leftrightarrow \hat{u} - \hat{v} \in \hat{S} \tag{10.126}$$

by virtue of the second of equations (10.93).

To apply these results to potential theory and reduced wave equation, given ϱ and non-zero functions k_R and k_E, define (Fig. 10.3)

$$\mathscr{L} u = \nabla \cdot k\,\nabla u + \varrho\, u \tag{10.127}$$

$$\langle P_R u_R, v_R \rangle = \int_R v \mathscr{L} u\, dx + \int_{\partial_1 R} u\, k \frac{\partial v}{\partial n}\, dx - \int_{\partial_1 R} v\, k \frac{\partial u}{\partial n}\, dx \tag{10.128}$$

/ 2

and $P_E: D_E \to D_E^*$ replacing R by E. Integrating by parts it is seen that

$$\langle \hat{A}\,\hat{u}, \hat{v} \rangle = \int_{\partial' R} k_R \left(v_R \frac{\partial u_R}{\partial n} - u_R \frac{\partial v_R}{\partial n} \right) dx + \int_{\partial' E} k_E \left(v_E \frac{\partial u_E}{\partial n} - u_E \frac{\partial v_E}{\partial n} \right) dx \tag{10.129}$$

Observe that the unit normal vector $\mathbf{n}$ is taken pointing outwards from the region of integration. Equation (10.129) implies that

$$\hat{N}_A = \{\{u_R, u_E\} \in \hat{D} \mid u_R = u_E = \partial u_E/\partial n = \partial u_R/\partial n = 0, \quad \text{on} \quad \partial_3 R\} \tag{10.130}$$

Smoothness conditions can be defined in many alternative manners. One which is suitable in many applications (in flow through porous media, for example) is

$$\hat{S} = \{\hat{u} \in \hat{D} \mid u_R = u_E,\, k_R\, \partial u_R/\partial n = k_E\, \partial u_E/\partial n, \quad \text{on} \quad \partial' R\} \tag{10.131}$$

Given $\hat{u} = \{u_R, u_E\} \in \hat{D}$, let be $\hat{u}' = \{u'_R, u'_E\}$, where u'_R and u'_E are smooth extensions of u_E and u_R, respectively. Write

$$[\hat{u}] = \{[\hat{u}]_R, [\hat{u}]_E\}; \quad \bar{u} = \{\bar{u}_R, \bar{u}_E\} \tag{10.132}$$

Applying definitions (10.121) and (10.131), it is seen that

$$[\hat{u}]_R = u'_R - u_R = u_E - u_R, \quad \text{on } \partial' R \tag{10.133a}$$

$$\frac{\partial[\hat{u}]_R}{\partial n} = \frac{\partial u'_R}{\partial n} - \frac{\partial u_R}{\partial n} = \frac{k_E}{k_R}\frac{\partial u_E}{\partial n} - \frac{\partial u_R}{\partial n}, \quad \text{on } \partial' R \tag{10.133b}$$

where the normal derivative is taken pointing outwards from R. When $\hat{v} \in \hat{S}$, equation (10.129) reduces to

$$\langle \hat{A}\,\hat{u}, \hat{v} = \int_{\partial' R} \left\{ k\,[\hat{u}]\frac{\partial v}{\partial n} - v\left[k\frac{\partial \hat{u}}{\partial n}\right]\right\} dx \tag{10.134}$$

Here, as in what follows, the components (R or E) to be used when carrying out the integration are indicated by the subindex under the integral sign. Also the notation

$$\left[k\frac{\partial \hat{u}}{\partial n}\right]_R = k_R\frac{\partial}{\partial n}[\hat{u}]_R = k_E\frac{\partial u_E}{\partial n} - k_R\frac{\partial u_R}{\partial n} \tag{10.135}$$

was introduced. Application of (10.134) in (10.125) yields

$$2\langle J\,\hat{u}, \hat{v}\rangle = \int_{\partial' R} \left\{\bar{v}\left[k\frac{\partial \hat{u}}{\partial n}\right] - k\overline{\frac{\partial v}{\partial n}}[\hat{u}]\right\} dx \tag{10.136}$$

Observe that equations (10.121) imply

$$[\hat{u}]_R = u_E - u_R, \quad 2\bar{v} = v_R \cdot \;\; _E \quad \text{and} \quad 2k_R\overline{\frac{\partial v}{\partial n}} = k_E\frac{\partial v_E}{\partial n} + k_R\frac{\partial v_R}{\partial n} \tag{10.137}$$

In view of equations (10.128) and (10.137), the Green's formula

$$\int_R \{v\mathcal{L}u - u\mathcal{L}v\}\,dx = \langle(\hat{B} + J)\,\hat{u}, \hat{v}\rangle - \langle(\hat{B} + J)\,\hat{v}, \hat{u}\rangle \tag{10.138}$$

is clear. Here

$$\langle \hat{B}\,\hat{u}, \hat{v}\rangle = \int_{\partial_1(R\cup E)} u\,k\frac{\partial v}{\partial n}\,dx - \int_{\partial_2(R\cup E)} v\,k\frac{\partial u}{\partial n}\,dx \tag{10.139}$$

In a similar fashion for static and quasi-static elasticity, when the smoothness criterium consists of continuity of displacements and tractions, one obtains for the jump operator [28]

$$2\langle J\,\hat{u}, \hat{v}\rangle = \int_{\partial' R} \{\bar{v}_i[T_i(\mathbf{u})] - [\hat{u}_i]\overline{T_i(\mathbf{v})}\}\,dx \tag{10.140}$$

This yields corresponding Green's formulas.

The formulation of Greens's formulas in discontinuous fields here presented is applicable to arbitrary formally symmetric operators which are linear. Thus, for example, the biharmonic equation or Stoke's problem are included. Green's formulas for two phases systems have also been derived in this manner [28].

10.6 T-Complete Systems

With every operator $P: D \to D^*$, we can associate a linear subspace $I_P \subset D$, defined by

$$I_P = N_P + N_A \tag{10.141}$$

this equation implies that every element $u \in I_P$, can be written as

$$u = u_P + u_A \tag{10.142}$$

with $u_P \in N_P$ while $u_A \in N_A$; since u_A vanishes on the boundary, we see that a function u belongs to I_P, if and only if, there is a solution u_P of the homogeneous partial differential equation such that the boundary values of u and u_P coincide. As illustration, in the example given previously of the reduced wave equation, a function $v \in I_P$, if and only if, there is a solution $u \in D$ of the homogeneous equation such that $v = u$ and $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n}$, on the boundary $\partial\Omega$.

It can be shown [28, 34] that I_P, as defined by (6.1), is always regular. Due to this fact the concept of T-complete system will be useful. Let $I_P \subset D$, be regular, and $\mathscr{B}$ be a subset of I_P, then we say that $\mathscr{B} \subset I_P$ is T-complete for I_P, when for every $u \in D$

$$\langle A\,u, w\rangle = 0 \quad \forall\, w \in \mathscr{B} \Rightarrow u \in I_P \tag{10.143}$$

Under very general conditions $N_P \subset I_P$ is T-complete for I_P [19, 28, 34]. For the representation of solutions it is, however, of greater interest to have denumerable subsets $\mathscr{B} \subset N_P$ which are T-complete. Examples of such systems are given in Tables 10.1 and 10.2. It has interest to mention that for the reduced wave equations the author has shown that a system of plane waves, which have a very simple structure, is T-complete in any bounded and simply connected region [5].

In these tables $J_n(r)$ and $H_n^{(1)}(r)$ are Bessel and Hankel functions of the first class [52. 53]. P_n^q is the associated Legendre function, while j_n and h_n^1 are the spherical Bessel and Hankel functions [52]. We recall, in addition, that the T-complete systems given in Tables 1 and 2 for Laplace equation in a bounded region are harmonic polynomials expressed in polar and spherical coordinates. Observe that the detailed shape of Ω is arbitrary.

Table 10.1 T-complete systems in two dimensions

Bounded Ω	Ω = exterior of a bounded region
Laplace Equation	
$\{1, r^n \cos n\,\theta, r^n \sin n\,\theta\}$	$\{\mathrm{Ln}\, r^{-n} \cos n\,\theta, r^{-n} \sin n\,\theta\}$
Reduced Wave Equation $\Delta u + u = 0$	
$\{J_0(r), J_n(r) \cos n\,\theta, J_n(r) \sin n\,\theta\}$	$\{H_0^{(1)}(r), H_n^{(1)}(r) \cos n\,\theta, H^{(1)}(r) \sin n\,\theta\}$
$n = 1, 2, .$	

Table 10.2 T-complete systems in three dimensions

Bounded Ω	Ω = exterior of a bounded region
Laplace Equation	
$\{r^n P_n^q(\cos\theta)\, e^{iq\varphi}\}$	$\{r^{-n-1} P_n^q(\cos\theta)\, e^{iq\varphi}\}$
Reduced Wave Equation	
$\{j_n(r)\, P_n^q(\cos\theta)\, e^{iq\varphi}\}$	$\{h_n^{(1)}(r)\, P_n^q(\cos\theta)\, e^{iq\varphi}\}$
$n = 0, 1, 2, \quad -n \leq q \leq n$	

10.7 Hilbert-Space Formulation

Associated with every Green's formula or equivalently, with every canonical decomposition, there is a Hilbert-space formulation.

For this purpose, we focus our attention in boundary values; i.e. we identify functions possessing the same boundary values. More precisely, two functions u and v of D, are identified whenever $u - v \in N_A$. The resulting space $\mathscr{G}$ is called the quotient space; i.e.

$$\mathscr{G} = D/N_A \tag{10.144}$$

Thus, for example

(i) For Laplace and reduced wave equation, $\mathscr{G}$ is made of pairs of functions $u, \dfrac{\partial u}{\partial n}$, defined on the boundary ∂R and square integrable there. Indeed

$$\mathscr{G} = \left\{\left[u, \frac{\partial u}{\partial n}\right] \middle| \ u \in H^0(\partial R), \frac{\partial u}{\partial n} \in H^0(\partial R)\right\} \tag{10.145}$$

(ii) Biharmonic equation

$$\mathscr{G} = \left\{\left[u, \frac{\partial u}{\partial n}, \Delta u, \frac{\partial \Delta u}{\partial n}\right] \middle| \ \text{Each one of } u, \frac{\partial u}{\partial n}, \Delta u, \frac{\partial \Delta u}{\partial n} \in H^0(\partial R)\right\}$$

(iii) Stokes equation

$$\mathscr{G} = \left\{\left[\mathbf{u}, \nu\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n}\right] \middle| \ \mathbf{u} \in \mathbf{H}^0(\partial R), \nu\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n} \in H^0(\partial R)\right\} \tag{10.146}$$

In each of these examples, one can give to $\mathscr{G}$, the structure of a Hilbert space. Possible choices for the corresponding inner products are

$$\text{(i)} \int_{\partial\Omega} \left\{u v + \frac{\partial u}{\partial n}\frac{\partial v}{\partial n}\right\} dx \tag{10.147a}$$

$$\text{(ii)} \int_{\partial\Omega} \left\{u v + \frac{\partial u}{\partial n}\frac{\partial v}{\partial n} + \Delta u\, \Delta v + \frac{\partial \Delta u}{\partial n}\frac{\partial \Delta v}{\partial n}\right\} dx \tag{10.147b}$$

$$\text{(iii)} \int_{\partial\Omega} \left\{\mathbf{u}\cdot\mathbf{v} + \left(\nu\frac{\partial \mathbf{u}}{\partial n} - p\,\mathbf{n}\right)\left(\nu\frac{\partial \mathbf{v}}{\partial n} - q\,\mathbf{n}\right)\right\} dx \tag{10.147c}$$

With these inner products, the linear space $\mathscr{S}$ is isomorphic to the following Hilbert spaces:

(i) $H^0(\partial\Omega) \oplus H^0(\partial\Omega)$ (10.148a)

(ii) $H^0(\partial\Omega) \oplus H^0(\partial\Omega) \oplus H^0(\partial\Omega) \oplus H^0(\partial\Omega)$ (10.148b)

(iii) $H^0(\partial\Omega) \oplus H^0(\partial\Omega)$ (10.148c)

Now, given any canonical decomposition $\{I_1, I_2\}$ it is possible to chose the Hilbert-space structure so that the associated operator $B: D \to D^*$ (equation 10.94) is given by

$$\langle B\,u, v\rangle = (u_1, v_2) \tag{10.149}$$

Thus, for example, when the inner product (10.147a) is used, equation (10.149) yields the operator B associated with the canonical decomposition given by (10.76a and b). The same happens if this decomposition is replaced by (10.76a and c). When one uses the inner product (10.147b), equation (10.149) supplies the operator $B: D \to D^*$ associated with any canonical decomposition corresponding to the biharmonic equation; for example, those given by equations (10.87) or (10.88). For Stokes problem the inner product can be (10.147c) and a possible canonical decomposition is defined by (10.89).

10.8 Representation of Solutions

For the formulation of the general boundary value problem to be considered here, we assume there is a canonical decomposition $\{I_1, I_2\}$, and an operator $B: D \to D^*$ such that (10.92) is a Green's formula. Using the representation (10.82), we formulate the problem as follows; find $u \in N_P$, such that

$$u_1 = U_1 \tag{10.150}$$

where U_1 is a given element of I_1.

Let $\mathscr{N}_P = N_P/N_A \subset \mathscr{S} = \hat{\mathscr{H}}$, be the linear space generated by the boundary values of solutions of the homogeneous equation. Then every $u \in \mathscr{N}_P$ can be written as

$$u = u_1 + u_2; \tag{10.151}$$

where $u_1 \in \mathscr{I}_1 = I_1/N_A$ while $u_2 \in \mathscr{I}_2 = I_2/N_A$. Let $\mathscr{N}_1 \subset \mathscr{I}_1$ be the range of values taken by u_1, in (10.151), when u ranges over $\mathscr{N}_P$. Similarly, let $\mathscr{N}_2 \subset \mathscr{I}_2$, be the range of values taken by u_2, in (10.151), when u ranges over $\mathscr{N}_P$.

Given a system of functions $\mathscr{B} = \{w_1, w_2, \ldots\} \subset N_P$, write

$$w_\alpha = w_{\alpha 1} + w_{\alpha 2} \tag{10.152}$$

We denote

$$\mathscr{B}_1 = \{w_{11}, w_{21}, w_{31}, \ldots\} \subset \mathscr{I}_1; \quad \mathscr{B}_2 = \{w_{12}, w_{22}, w_{32}, \ldots\} \subset \mathscr{I}_2 \tag{10.153}$$

Clearly, we will be able to approximate the boundary values of every solution of (10.150), if and only if

$$\operatorname{span} \mathscr{B}_1 = \bar{\mathscr{N}}_1 \tag{10.154}$$

Here, the bar refers to the closure of $\mathscr{N}_1$.

A result similar to Theorem 10.1, holds in this more general context. Assume $I_P = N_P + N_A$ is completely regular and $\mathscr{B} \subset N_P$. Then the following statements are equivalent:

(i) $\hat{\mathscr{B}} \subset \hat{\mathscr{N}}_P \subset \hat{\mathscr{H}}$ spans $\hat{\mathscr{N}}_P$ in the metric of $\hat{\mathscr{H}}$;
(ii) $\hat{\mathscr{B}} \subset \hat{\mathscr{N}}_P$ is T-complete; and
(iii) span $\mathscr{B}_1 = \bar{\mathscr{N}}_1$ while span $\mathscr{B}_2 = \bar{\mathscr{N}}_2$ (10.155)

Therefore, when $\hat{\mathscr{B}}$ is T-complete, it is possible to construct approximating sequences

$$u^N = \sum_{n=1} a_n^N w_n; \quad N = 1, 2, \ldots \tag{10.156}$$

such that $u_1^N \to U_1$, whenever $U_1 \in \bar{\mathscr{N}}_1$; therefore, if the problem (10.150) has a solution u, then

$$u^N \to u \tag{10.157}$$

The convergence in (10.157), is in any metric in which the solution of the problem, depends continuously on the boundary data U_1.

When a Green's formula is available, the results of Section 7, yield an efficient procedure to compute the complementary boundary data. In this case, for every $w_\alpha \in \mathscr{B}$, we have

$$(w_{\alpha 2}, U_1) = (w_{\alpha 2}, u_1) = (w_{\alpha 1}, u_2) \tag{10.158}$$

which gives $(w_{\alpha 1}, u_2)$ in terms of the boundary data U_1. This gives the approximating sequence

$$u_2^N = \sum_{n=1}^{N} b_n^N w_{n1} \tag{10.159}$$

where the coefficients b_n^N satisfy. for every fixed N, the system of equations.

$$(w_{m2}, U_1) = \sum_{n=1}^{N} b_n^N (w_{m1}, w_{n1}) \tag{10.160}$$

This generalizes the results of Section 2.

Observe that the values of u_2 are approximated by linear combinations of w_{n1}. This implies, for example, that in applications to problems formulated in discontinuous fields, with precribed jump conditions, the averages of the functions across discontinuities are approximated by the jumps of the basic systems (a specific application of this kind is given in [5]).

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