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Hybrid methods from a new perspective

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INTRODUCTION

Usually, by a hybrid method it is understood one in which continuity requirements are reduced or eliminated altogether[1,2,3,4] by introducing auxiliary dependent variables. Finite element methods for elliptic equations of order $2m$, are said to be nonconforming when in the evaluation of the energy, which involves derivatives of order m, the approximations to derivatives of order $m-1$ may have simple discontinuities[5,6].

In this paper a systematic formulation of hybrid and nonconforming element methods is briefly explained and illustrated by applying it to some simple elliptic operators. The approach here presented is quite general, since it is applicable to any linear operator. It is based on the Algebraic Theory of Boundary Value Problems developed by the author[7,8,9,10] and which has just been published in book form[11].

The approach here presented apparently supplies a useful tool of analysis which can be used not only to evaluate the error of approximate methods but also gives insight and orientation to develop more efficient ones. In the case of ordinary differential equations it yields systems which permit computing the exact values of the sought solutions and its derivatives at a finite number of nodes. For special choices of the test functions some results obtained by Rose[12,13] are also derived in this manner. However, the new approach goes beyond, since it supplies algorithms which permit computing the exact values of the derivatives at prescribed nodes without computing the values of the function there.

FINITE ELEMENT FORMULATION

Generally, boundary value problems can be formulated as follows[11]. Let D be a linear space of functions and D^* its dual (space of linear functionals defined on D). Let P: $D \to D^*$ and B: $D \to D^*$ be operators, such that B is a boundary operator for P. Given $U \in D$ and $V \in D$, let $f = PU$ and $g = BV$. Then, $u \in D$ is a solution of the boundary value problem, if and only if

$$(P+B)u = f+g \tag{1}$$

This equation is equivalent to

$$\langle Pu, v\rangle + \langle Bu, v\rangle = \langle f, v\rangle + \langle g, v\rangle \forall\ v \in D \tag{2}$$

To be specific, let

$$\mathscr{L}u = \Delta u \pm u, \quad \text{in } \Omega \tag{3}$$

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or alternatively

$$\mathscr{L}u = \Delta u, \quad \text{in } \Omega \tag{3'}$$

where Ω is an open bounded domain of a two-dimensional Euclidean space satisfying the same restrictions as in Ref. 1. Define P: $D \to D^*$, B: $D \to D^*$, $f \in D^*$ and $g \in D^*$ by

$$\langle Pu, v\rangle = \int_\Omega v\mathscr{L}u\,\mathrm{d}x, \qquad \langle Bu, v\rangle = \int_{\partial\Omega} \frac{\partial v}{\partial n}\gamma_0 u\,\mathrm{d}x \tag{4}$$

$$\langle f, v\rangle = \int_\Omega v f_\Omega\,\mathrm{d}x \quad \text{and} \quad \langle g, v\rangle = \int_{\partial\Omega} \frac{\partial v}{\partial n} g_{\partial\Omega}\,\mathrm{d}x \tag{5}$$

trace operator and the Laplacian is understood in the sense of distributions. Then (2), or equivalently (1), is satisfied if and only if

$$\mathscr{L}u = f_\Omega \quad \text{and} \quad \gamma_0 u = g_{\partial\Omega} \tag{6}$$

Let π be a partition of Ω into a finite collection of E subdomains Ω_e, $1, 2, \ldots, E(\pi)$[1,2]. We define D_e and $P_e: D_e \to D_e^*$, replacing Ω by Ω_e in previous definitions. In addition, B_e: $D_e \to D_e^*$ is defined replacing $\partial\Omega$ by $\partial\Omega \cap \partial\Omega_e$. Let $\hat{D}$ be the product space $D_1 \times D_2 \times \ldots \times D_E$. Define $\hat{P}$: $\hat{D} \to \hat{D}^*$ and $\hat{B}$: $\hat{D} \to \hat{D}^*$ by

$$\langle \hat{P}\hat{u}, \hat{v}\rangle = \sum_{e=1}^{E} \langle P_e u_e, v_e\rangle, \quad \langle \hat{B}\hat{u}, \hat{v}\rangle = \sum_{e=1}^{E} \langle B_e u_e, v_e\rangle \tag{7}$$

Then equations (4) are satisfied by $\hat{P}$ and $\hat{B}$. The operator $\hat{P}+\hat{B}$ is nonsymmetric. However, the restriction of the bilinear functional $\langle(\hat{P}+\hat{B})\hat{u}, \hat{v}\rangle$ to smooth functions is symmetric. Using this fact, it can be seen that $\hat{P}+\hat{B}-\hat{J}$ is symmetric, where

$$\langle \hat{J}\hat{u}, \hat{v}\rangle = \int_\Gamma \left\{[u]\frac{\partial \dot{v}}{\partial n} - \dot{v}\left[\frac{\partial u}{\partial n}\right]\right\}\mathrm{d}x \tag{8}$$

Here, Γ is the union of all the interelement boundaries More formally:

$$\Gamma = \bigcup_{e \neq f} (\partial\Omega_e \cap \partial\Omega_f) \tag{9}$$

In addition

$$[u] = u_f - u_e, \qquad \left[\frac{\partial u}{\partial n}\right] = \frac{\partial u_f}{\partial n} - \frac{\partial u_e}{\partial n} \tag{10a}$$

$$\dot{u} = \tfrac{1}{2}(u_f + u_e), \qquad \frac{\partial \dot{u}}{\partial n} = \frac{1}{2}\left(\frac{\partial u_f}{\partial n} + \frac{\partial u_e}{\partial n}\right) \tag{10b}$$

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where given any point $\mathbf{x} \in \Gamma$, it is assumed that Ω_e and Ω_f are such that $\mathbf{x} \in \partial\Omega_e \cap \partial\Omega_f$. Also, the unit normal vector in (8), must be taken pointing outwards from Ω_e into Ω_f. Then, equation

$$\hat{P}\hat{u} + \hat{B}\hat{u} - \hat{J}\hat{u} = \hat{f} + \hat{g} \tag{11}$$

or equivalently

$$\langle \hat{P}\hat{u}, \hat{v}\rangle + \langle \hat{B}\hat{u}, \hat{v}\rangle - \langle \hat{J}\hat{u}, \hat{v}\rangle = \langle \hat{f}, \hat{v}\rangle + \langle \hat{g}, \hat{v}\rangle \; \forall\, \hat{v} \in \hat{D} \tag{12}$$

if and only if $\hat{u} = \{u_1, u_2, \ldots, u_E\}$, where $u_e \in H^1(\Omega_e)$ are the restrictions to Ω_e of the solution $u \in H^1(\Omega)$ of problem (6).

Observe

$$\langle \hat{J}^*\hat{u}, \hat{v}\rangle = \int_\Gamma \left\{ [v]\frac{\partial \dot{u}}{\partial n} - \dot{u}\left[\frac{\partial v}{\partial n}\right]\right\} \mathrm{d}x \tag{13}$$

by virtue of (8). Given $\hat{u} \in \hat{D}$, the functionals $\hat{J}\hat{u}$ and $\hat{J}^*\hat{u}$ are called the jump and the average values of $\hat{u}$, respectively. Equations (8) and (13) show that this nomenclature is appropriate, since $\hat{J}\hat{u}$ defines uniquely the functions $[u]$ and $[\partial u/\partial n]$ on Γ, while $\hat{J}^*\hat{u}$ defines uniquely the functions $\dot{u}$ and $\partial \dot{u}/\partial n$. Observe that $\hat{J} = \hat{J}_1 + \hat{J}_2$, where

$$\langle \hat{J}_1\hat{u}, \hat{v}\rangle = \int_\Gamma [u]\frac{\partial \dot{v}}{\partial n}\mathrm{d}x, \quad \langle \hat{J}_2\hat{u}, \hat{v}\rangle = -\int_\Gamma \dot{v}\left[\frac{\partial u}{\partial n}\right]\mathrm{d}x \tag{14}$$

Clearly $\hat{J}_1\hat{u}$, $\hat{J}_2\hat{u}$, $\hat{J}_2^*$ and $\hat{J}_1^*$, characterize $[u]$, $[\partial u/\partial n]$, $\dot{u}$ and $\partial\dot{u}/\partial n$, respectively.

To obtain an approximate solution $\hat{u}' \in \hat{D}$, in the finite element method one replaces (12) by

$$\langle \hat{P}\hat{u}', \hat{\varphi}_\alpha\rangle + \langle \hat{B}\hat{u}', \hat{\varphi}_\alpha\rangle - \langle \hat{J}\hat{u}', \hat{\varphi}_\alpha\rangle = \langle \hat{f}, \hat{\varphi}_\alpha\rangle + \langle \hat{g}, \hat{\varphi}_\alpha\rangle \qquad \alpha = 1, \ldots, N \tag{15}$$

where $\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\} \subset \hat{D}$ is a family of 'test functions'. Usually, one imposes the additional *constraint* that $\hat{u}' = \sum_{\beta=1}^{N} a_B \hat{\Phi}_B$ where $\{\hat{\Phi}_1, \ldots, \hat{\Phi}_N\}$ are the basis functions. However, this latter constraint is alien to the problem, while equation (15) is necessarily satisfied by the solution of the problem. Therefore, it has interest to analyse the restrictions implied by (15), when no assumption is made on the representation of the solution.

To this end, observe that (15) can also be written as

$$\langle \hat{P}\hat{\varphi}_\alpha, \hat{u}'\rangle + \langle \hat{B}^*\hat{u}', \hat{\varphi}_\alpha\rangle - \langle \hat{J}^*\hat{u}', \hat{\varphi}_\alpha\rangle = \langle \hat{f}, \hat{\varphi}_\alpha\rangle + \langle \hat{g}, \hat{\varphi}_\alpha\rangle \qquad \alpha = 1, \ldots, N \tag{16}$$

Taking into account (4) and (13), this imposes conditions on the possible values taken by

$$\int_\Omega u'\mathscr{L}\varphi_\alpha\,\mathrm{d}x, \qquad \int_{\partial\Omega} \varphi_\alpha \frac{\partial u'}{\partial n}\mathrm{d}x$$

$$\int_\Gamma \dot{u}\left[\frac{\partial \varphi_\alpha}{\partial n}\right]\mathrm{d}x \quad \text{and} \quad \int_\Gamma [\varphi_\alpha]\frac{\partial \dot{u}}{\partial n}\mathrm{d}x$$

These in turn, can be interpreted as restrictions on the possible values of u' in the region Ω, $\partial u'/\partial n$ on $\partial\Omega$, u' and $\partial u'/\partial n$ on Γ. The kind of such conditions depends of course on the particular choice of the family of test functions. Thus, for example, in standard finite element formulations $\hat{\varphi}_\alpha$ is required to be continuous; i.e. $[\hat{\varphi}_\alpha] = 0$. Then

$$\langle \hat{J}_1^*\hat{u}', \hat{\varphi}_\alpha\rangle = \int_\Gamma [\varphi_\alpha]\frac{\partial \dot{u}'}{\partial n}\mathrm{d}x = 0 \tag{17}$$

and no condition is imposed on $\partial\dot{u}'/\partial n$. Similarly, in some hybrid element methods[2], it is assumed that $\hat{P}\hat{\varphi}_\alpha = 0$. Hence $\int_\Omega u'\mathscr{L}\varphi_\alpha\,\mathrm{d}x = 0$ identically and no restrictions are imposed on the possible values of u' in the region Ω.

Observe that equation (16) is also satisfied by the exact solution $\hat{u} \in \hat{D}$. Therefore, by substraction, one gets

$$\langle \hat{P}\hat{\varphi}_\alpha, \hat{u} - \hat{u}'\rangle + \langle B^*(\hat{u} - \hat{u}'), \hat{\varphi}_\alpha\rangle - \langle \hat{J}^*(\hat{u} - \hat{u}'), \hat{\varphi}_\alpha\rangle = 0 \tag{18}$$

In view of (18), it is clear that the system of equations (16) supplies information about the exact solution $\hat{u}$. Again the kind of information depends on the specific family $\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\}$, of test functions chosen. In particular, if $\hat{\varphi}_\alpha$ are required to be continuous, no information about $\partial u/\partial n = \partial\dot{u}/\partial n$, is gotten. Correspondingly, if $\hat{\varphi}_\alpha$ satisfies the differential equation locally, then no information about the function u on Ω is obtained. It has interest to observe, because this may be useful in some applications, that when the normal derivatives are continuous across interelement boundaries (i.e. $[\partial\varphi_\alpha/\partial n] = 0$), then no information is obtained about $u = \dot{u}$. These facts may be used to focus the information in some aspects of the solution, depending on the applications. Such procedures are illustrated in the following Section.

EXAMPLES

Let us apply the foregoing discussion for the one dimensional case, taking Ω as the open interval (0, 1). Thus, the problems to be considered are

$$\mathscr{L}u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f_\Omega \qquad \text{in } \Omega = (0, 1) \tag{19a}$$

or alternatively

$$\mathscr{L}u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \pm u = f_\Omega \qquad \text{in } \Omega = (0, 1) \tag{19'a}$$

subject to

$$u(0) = g_0 \quad \text{and} \quad u(1) = g_1 \tag{19b}$$

Let the partition contain E subintervals of equal length (x_{k-1}, x_k), with $k = 1, 2, \ldots, E$. Thus $h = |x_k - x_{k-1}| = 1/E$. Assume the system of test functions $\{\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_N\}$ satisfies $\mathscr{L}\alpha_{\alpha k} = 0$; $k = 1, \ldots, E$. Then equations (16) and (18) are

$$\varphi_\alpha \frac{\mathrm{d}u'}{\mathrm{d}x}\bigg|_0^1 + \sum_{k=1}^{E-1}\left\{\dot{u}'\left[\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}\right] - [\varphi_\alpha]\frac{\mathrm{d}\dot{u}'}{\mathrm{d}x}\right\}_k = \int_0^1 f_\Omega\varphi_\alpha\,\mathrm{d}x + g_1\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}(1) - g_0\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}(0) \tag{20}$$

and

$$\varphi_\alpha\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}u'}{\mathrm{d}x}\right)\bigg|_0^1 + \sum_{k=1}^{E-1}\left\{(u - \dot{u}')\left[\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}\right] - [\varphi_\alpha]\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}\dot{u}'}{\mathrm{d}x}\right)\right\}_k = 0 \tag{21}$$

Here $\alpha = 1, \ldots, N$. Also, the dots have been deleted from $\dot{u}_k$ and $(\mathrm{d}\dot{u}/\mathrm{d}x)_k$, since the exact solution u is smooth. Equations (20) can be interpreted as a system of N equations for the $2E$ unknowns $\dot{u}'_k$, $(\mathrm{d}\dot{u}'/\mathrm{d}x)_k$, $k = 1, \ldots, E-1$, $(\mathrm{d}u'/\mathrm{d}x)_0$ and $(\mathrm{d}u'/\mathrm{d}x)_E$. If $N = 2E$, there is a possibility this system had a unique solution; in such case (21) shows that

$$u_k = \dot{u}'_k, \qquad (\mathrm{d}u/\mathrm{d}x)_k = (\mathrm{d}\dot{u}'/\mathrm{d}x)_k \tag{22}$$

where $k=1,\ldots,E-1$, for the first equation while $k=0,1,\ldots,E$ for the second one. Also, when $k=0$ or 1, the dot must be deleted.

A first possibility is to take $N=2E$ with $\{\varphi_1,\ldots,\varphi_{2E}\}$ in such a manner that $\{\varphi_1,\varphi_2\}$ is a system of two linear independent solutions of the homogeneous equation $\mathscr{L}\varphi=0$ in the interval (x_0,x_1) and zero outside. More generally, the system $\{\varphi_{2j-1},\varphi_{2j}\}$ is a system of two linear independent solutions of the homogeneous equation in the interval (x_{j-1},x_j). Using general results of the Algebraic Theory[11], it can be shown that such a system is T-complete and the system of equations (20) has a unique solution. Thus, the solution of (20) yields the exact values of u and its derivatives at all the nodes.

If we are not interested on the derivative, it may be convenient to eliminate it from the system of equations. This is achieved by imposing the additional conditions on the test functions $\hat{\varphi}_\alpha$; namely that $\hat{\varphi}_\alpha$ be continuous and $\hat{\varphi}_\alpha(0)=\hat{\varphi}_\alpha(1)=0$. A complete system satisfying these conditions can be chosen so that, for every $\alpha=1,\ldots,N$, $\hat{\varphi}_\alpha$ has support on the interval $(x_{\alpha-1},x_{\alpha+1})$ and $N=E-1$. Then the system (20) reduces to

$$\sum_{k=\alpha-1}^{\alpha+1} u_k\left[\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}\right]_k = \int_0^1 f_\Omega\varphi_\Omega\,\mathrm{d}x + g_1\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}(1) - g_0\frac{\mathrm{d}\varphi_\alpha}{\mathrm{d}x}(0) \tag{23}$$

Taking the test functions as indicated in Table 1, one obtains the following systems of difference equations for the *exact* values u_k. When $\mathscr{L}=\Delta$, then

$$\frac{u_{\alpha+} \quad u_{\alpha-} \quad 2u_\alpha}{h^2} = \frac{}{h^2}\int_{x_{\alpha-1}}^{x_{\alpha+1}} f_\Omega\varphi_\alpha\,\mathrm{d}x$$

$$\alpha=2, \quad ., N-1=E-2 \tag{24a}$$

$$\frac{u_2-2u_1}{h^2}=\frac{1}{h^2}\int_0^h f_\Omega\varphi_1\,\mathrm{d}x - g_0/h^2$$

$$\frac{u_N-2u_{N+}}{h^2} = \frac{}{h^2}\int_{1-}^{1} f_\Omega\varphi_N\,\mathrm{d}x - g_1/h^2 \tag{24b}$$

Notice that these are central differences. When $\mathscr{L}=\Delta-$

$$\frac{u_{\alpha+1}u_{\alpha-1}-2u_\alpha\cosh b}{b^2}=\frac{1}{b^2}\int_{x_{\alpha-}}^{x_{\alpha+1}} f_\Omega\varphi_\alpha\,\mathrm{d}x$$

$$\alpha=2,\ldots,N \quad =E-2 \tag{25a}$$

$$\frac{u_2-2u_1\cosh b}{b}=\frac{1}{b^2}\int_0^b f_\Omega\varphi_1\,\mathrm{d}x - g_0/b^2$$

$$\frac{u_N-2u_N}{b^2} = \frac{}{b^2}\int_{1-b}^{1} f_\Omega\varphi_N\,\mathrm{d}x - g_1/b^2 \tag{25b}$$

Table 1. Test functions when u_k is sought ($b=h$)

$\mathscr{L}$	$x_{\alpha-1}<x<x_\alpha$	$x_\alpha<x<x_{\alpha+1}$
Δu	$\varphi_\alpha=x-x_{\alpha-1}$	$\varphi_\alpha=(x_{\alpha+1}-x)$
$\Delta u-u$	$\varphi_\alpha=\sinh(x-x_{\alpha-1})$	$\varphi_\alpha=\sinh(x_{\alpha+1}-x)$
$\Delta u+u$	$\varphi_\alpha=\sin(x-x_{\alpha-1})$	$\varphi_\alpha=\sin(x_{\alpha+1}-x)$

where we have written $b=h$, for clarity. Finally, when $\mathscr{L}=\Delta+1$

$$\frac{u_{\alpha+1}+u_{\alpha-1}-2u_\alpha\cos b}{b^2}=\frac{1}{b^2}\int_{x_{\alpha-1}}^{x_{\alpha+1}} f_\Omega\varphi_\alpha\,\mathrm{d}x$$

$$\alpha=2,\ldots,N-1=E-2 \tag{26a}$$

$$\frac{u_2-2u_1\cos b}{h^2}=\frac{1}{b^2}\int_0^b f_\Omega\varphi_1\,\mathrm{d}x - g_0/b^2$$

$$\frac{u_N-2u_{N+}}{b^2}=\frac{1}{b^2}\int_{1-b}^{1} f_\Omega\varphi_N\,\mathrm{d}x - g_1/b^2 \tag{26b}$$

The results just presented, can also be derived applying a method proposed by Rose[12,13] for positive definite self-adjoint operators on the basis of energy considerations. However, the methodology here discussed, is applicable to arbitrary linear operators even if they are non-symmetric, as will be shown elsewhere.

If we were interested in the values $(\mathrm{d}u/\mathrm{d}x)_k$ of the derivative at all the nodes ($k=0,1,\ldots,E$), one can impose the condition that the derivative of the test functions be continuous (i.e. $[\mathrm{d}\varphi/\mathrm{d}x]=0$). Such families of test functions are given in Table 2. Substitution of them in equation (20) yields the following systems of equations. When $\mathscr{L}=\Delta-1$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{\alpha+1} + \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{\alpha-} - 2\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_{\alpha}\cosh b = \quad f_\Omega\varphi_\alpha\,\mathrm{d}x$$

$$\alpha= \quad E \tag{27a}$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right) \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_0\cosh b = \int_0^h f_\Omega\varphi_1\,\mathrm{d}x + g_0\sinh b$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_E \quad \left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)_E \cosh b = \int^1 \quad f_\Omega\varphi_E\,\mathrm{d}x - g_1\sinh b \tag{27b}$$

When $\mathscr{L}=\Delta+1$ and $2h\neq\pi, 3\pi,\ldots$, equations (27) remain valid, if cosh and sinh are replaced by cos and $-\sin$, respectively. When $\mathscr{L}=\Delta u$, the system of test functions given in Table 2, is not complete because it contains only E functions. It is necessary to take $N=E+1$, with $\varphi_{E+1}=x$, $0<x<1$, which clearly has continuous derivatives and is linearly independent of the family given in Table 2. Although the system of difference equations that is obtained for this case is rather trivial, the example has some theoretical interest because it has the peculiarity that it is not possible to construct a function of local support, with continuous derivative and linearly independent of the system $\{\hat{\varphi}_0,\hat{\varphi}_1,\ldots,\hat{\varphi}_E\}$.

It may be of greater interest to mention that by combining the previous constructions, it is possible to construct finite difference schemes which yields the exact

Table 2. Test functions when $(\mathrm{d}u/\mathrm{d}x)_k$ is sought ($b=h$)

$\mathscr{L}$	$x_{\alpha-1}<x<x_\alpha$	$x_\alpha<x<x_{\alpha+1}$
Δu	$\varphi_\alpha=1;\ \alpha=1,\ldots,E$	$\varphi_\alpha=0$
$\Delta u-u$	$\varphi_\alpha=-\cosh(x-x_{\alpha-1}),\ \alpha\neq0$ $\varphi_0=0$	$\varphi_\alpha=\cosh(x_{\alpha+1}-x);\ \alpha\neq E$ $\varphi_E=0$
$\Delta u+u$	$\varphi_\alpha=-\cos(x-x_{\alpha-1}),\ \alpha\neq0$ $\varphi_0=0$	$\varphi_\alpha=\cos(x_{\alpha+1}-x);\ \alpha\neq E$ $\varphi_E=0$

values of the solution at some nodes and the exact values of the derivatives at other ones.

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