Unified Formulation of Numerical Methods. I. Green's Formulas for Operators in Discontinuous Fields

Ismael Herrera
Instituto de Geofisica, Universidad Nacional Autonoma de Mexico, Apdo. Postal 21-524, 04000 Mexico, D.F.

Applying the method of weighted residuals and then interpreting the resulting equations by means of Green's formulas for discontinuous functions, a direct method of analysis is developed. The scheme includes finite differences, finite elements, and boundary methods. This is the first of a sequence of articles in which the methodology is presented. A fundamental ingredient of the procedure are general Green's formulas for operators defined in discontinuous fields. They are developed in this first article.

I. INTRODUCTION

Three of the most powerful numerical methods for partial differential equations are finite elements, finite differences, and boundary element methods. The foundations of each one of these methodologies, as originally formulated, appeared to be unrelated. More recently, however, it has been recognized that it is desirable to develop foundations common to all of them.

This article is the first of a sequence of articles devoted to presenting a unifying theory recently developed by the author. The approach is quite general, since it is applicable to any linear operator, symmetric or nonsymmetric, regardless of its type. Thus, for example, the theory is applicable to steady-state and time-dependent problems.

The starting point for the theory is a rather simple and, as a matter of fact, old idea [see, e.g., 1]. Let $\mathcal{L}$ be a differential operator defined in a region such as Ω in Figure 1, and let $\mathcal{L}^*$ be its formal adjoint. Then, when u and v satisfy suitable boundary conditions, Green's formula

$$\int_{l} v\mathcal{L}u\,dx = \quad u\mathcal{L}^* v\,dx \qquad 1)$$

holds. Equation (1) allows a convenient interpretation of the method of weighted residuals. Consider the equation

$$\mathcal{L}u = f_{\Omega}, \qquad \text{in } \Omega \qquad (2)$$

subjected to homogeneous boundary conditions for which Green's formula (1) applies. As is usual in the method of weighted residuals [2], one says that a function u' is an approximate solution of this problem when

Numerical Methods for Partial Differential Equations, 1, 25–44 (1985)
 CCC 0749-159X/85/010025-20$04.00

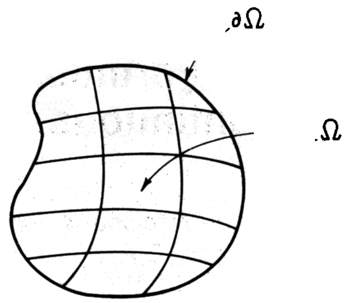


FIG.

$$\int_\Omega \varphi_\alpha(\mathcal{L}u' - f_\Omega)\,dx = 0, \qquad \alpha = \qquad N \tag{3}$$

Here, $\{\varphi_1, \ldots, \varphi_N\}$ is a family of "weighting functions." Usually, the system (3), which is made of N equations, has many solutions. In order to obtain a system possessing a unique solution, it is customary to introduce the representation $u' = \Sigma a_\alpha \Phi_\alpha$ of u' in terms of the system $\{\Phi_1, \ldots, \Phi_N\}$ of base functions. However, this representation is an artifice that bears no relation to the exact solution u. On the other hand, the system (3) is also satisfied by the exact solution. Thus,

$$\int_\Omega \varphi_\alpha(\mathcal{L}u - f_\Omega)\,dx = 0 \qquad \alpha = \qquad N \tag{4}$$

One may inquire what is the actual information about the exact solution contained in an approximate one. To answer this question, compare Eqs. (3) and (4), to obtain

$$\int_\Omega \varphi_\alpha \mathcal{L}u'\,dx = \int_\Omega \varphi_\alpha \mathcal{L}u\,dx, \qquad \alpha = 1, \ldots, N. \tag{5}$$

In this form, Eq. (5) is not informative. A more informative form is obtained by applying Green's formula (1); this yields

$$\int_\Omega u'\mathcal{L}^*\varphi_\alpha\,dx = \int_\Omega u\mathcal{L}^*\varphi_\alpha\,dx, \qquad \alpha = 1, \ldots, N. \tag{6}$$

The system (6) can be interpreted in terms of projections on a Hilbert space for which the inner product of the two functions u and v is given by $\int_\Omega uv\,dx$. Thus, the answer to our questions is:

> An approximate solution u' is any function whose projection on the subspace spanned by the system of functions $\{\mathcal{L}^*\varphi_1, \ldots, \mathcal{L}^*\varphi_N\}$ coincides with that of the exact solution u. Indeed, this projection is "all the information" about the exact solution contained in an approximate solution. In this light, the representation $u' = \Sigma a_\alpha \phi_\alpha$ can be interpreted as a procedure for extrapolating the actual information contained in the approximate solution.

The very simple and precise result just presented clarifies much of the nature of approximate solutions, but up to now it has not been possible to apply it, in a systematic manner, to analyze discrete methods. This is due to the fact that Eqs. (6) hold only when the "admissible functions" are sufficiently smooth and satisfy homogeneous boundary conditions. However, in the formulation of finite-element methods, for example, nonhomogeneous boundary conditions are considered and the admissible functions are discontinuous across the "interelement boundaries" which separate the finite elements from each other (Fig. 1). In many cases the base functions as well as the weighting functions are required to possess certain degree of smoothness, but in some others such as in Petrov–Galerkin methods [3], fully discontinuous weighting functions are considered. Even more, the development of a theory in which the analysis can be carried out when both the base functions as well as the weighting functions are fully discontinuous, would be useful any way, since satisfying the required continuity restrictions frequently complicates the numerical treatment of the problem.

When trying to extend Green's formula (1) to discontinuous functions, it is natural to resort to the theory of "distributions" or generalized functions. This I did, in my first attempts to tackle this problem, but it soon became apparent that the incorporation of the Hilbert-space structure or even the topological structure from the beginning gave rise to inconvenient rigidity. In order to avoid this, a purely algebraic formulation was preferred [4]. This was done by means of bilinear forms such as $\langle Pu, v\rangle$, which can also be thought of as functional-valued operators $P: D \rightarrow D^*$, where D is the linear space of admissible functions, while D^* is the space of linear functionals defined on D (i.e., D^* is the algebraic dual of D). Also, focusing attention on the algebraic structure is useful to obtain Green's formulas of general validity.

Using this formulation, one is led to replace Eq. (1) by the "general Green's formula for operators defined in discontinuous fields:"

$$\langle Pu, v\rangle - \langle Bu, v\rangle - \langle Ju, v\rangle = \langle Q^*u, v\rangle - \langle C^*u, v\rangle - \langle K^*u, v\rangle. \tag{7}$$

Here, the asterisk stands for the transpose of the corresponding bilinear form. Thus, for example $\langle Q^*u, v\rangle = \langle Qv, u\rangle$. In Eq. (7), $\langle Pu, v\rangle$ and $\langle Qv, u\rangle$ are defined in terms of the differential operators $\mathcal{L}$ and $\mathcal{L}^*$, respectively, by means of

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\,dx; \qquad \langle Q^*u, v\rangle = \langle Qv, u\rangle = \int_\Omega u\mathcal{L}^*v\,dx. \tag{8}$$

Then Q is a "formal adjoint" of P in an abstract sense introduced in the theory. Also $\langle Bu, v\rangle$ and $\langle C^*u, v\rangle$ are "boundary operators" in an abstract sense, while $\langle Ju, v\rangle$ and $\langle K^*u, v\rangle$ are the "jump" and "average" operators, respectively. Generally, when formulating a boundary-value problem one prescribes Pu, Bu, and Ju; it will be assumed that $f \in D^*$, $g \in D^*$, and $j \in D^*$ are the corresponding prescribed values. Throughout this section it will also be assumed that the sought solution u is smooth and therefore $j = 0$.

It must be emphasized that the integrals in (8) are understood in an elementary sense (surfaces of discontinuity, on which the operators $\mathcal{L}$ and $\mathcal{L}^*$ are not defined, are left out), so that generalized functions or the theory of distributions are not used. In view of (8), it is clear that Q^*u is characterized by the values of the solution u in the interior of the finite elements. The meaning of the "complementary boundary values" C^*u is illustrated by means of some examples: For the Dirichlet problem of the Laplace equation, in which u is prescribed on the boundary, the complementary boundary values are the normal derivatives $\partial u/\partial n$; for problems of elasticity, when the displacement is prescribed on the boundary, the complementary boundary values are the tractions. Generally, while the boundary values Bu are prescribed, the complementary boundary values C^*u can be computed only after the exact solution u has been obtained. Similarly, the "average values" K^*u of the exact solution (which coincide with the actual values for smooth solutions), on the interelement boundaries of the finite elements, also can only be computed after the exact solution has been obtained. Thus, while Pu, Bu, and Ju constitute the "prescribed data" of the problem Q^*u, C^*u, and K^*u will be called the "sought information."

The "general Green's formula" (7) yields *two variational principles* for any linear boundary-value problem. The first one is

$$\langle Pu, v\rangle - \langle Bu, v\rangle - \langle Ju, v\rangle = \langle f, v\rangle - \langle g, v\rangle, \qquad \forall\, v \in D. \tag{9}$$

This equation, in the presence of (7), is equivalent to the second one:

$$\langle Q^*u, v\rangle - \langle C^*u, v\rangle - \langle K^*u, v\rangle = \langle f, v\rangle - \langle g, v\rangle, \qquad \forall\, v \in D. \tag{10}$$

The variational principles (9) and (10) will be called "direct" and "indirect" or "derived" variational formulations of the original boundary-value problem, respectively.

According to the method of weighted residuals, an approximate solution $u' \in D$ will be any one that satisfies the direct variational formulation (9), and therefore also the indirect one, for every weighting function of the family $\{\varphi_1, \ldots, \varphi_N\} \subset D$. Since the exact solution $u \in D$ necessarily satisfies (10), it is clear that

$$\begin{aligned}\langle Q^*u', \varphi_\alpha\rangle &- \langle C^*u', \varphi_\alpha\rangle - \langle K^*u', \varphi_\alpha\rangle \\ &= \langle Q^*u, \varphi_\alpha\rangle - \langle C^*u, \varphi_\alpha\rangle - \langle K^*u, \varphi_\alpha\rangle, \qquad \alpha = 1, \ldots, N.\end{aligned} \tag{11}$$

This is the equation we are looking for; it replaces (6) when the problem is formulated in general discontinuous fields. Clearly, the functionals $\langle Q^*u, \varphi_\alpha\rangle - \langle C^*u, \varphi_\alpha\rangle - \langle K^*u, \varphi_\alpha\rangle$, $(\alpha = 1, \ldots, N)$, which are correctly supplied by any approximate solution, are part of the "sought information." This is indeed "all the information" that one can extract from an approximate solution u'. The representation $u' = \Sigma a_\alpha \Phi_\alpha$ supplies a procedure for extrapolating such information, but the actual information contained in an approximate solution is independent of such an extrapolation process and only depends on the system of weighting functions $\{\varphi_1, \ldots, \varphi_N\}$ chosen. These observations constitute the conceptual basis of the methodology presented in this sequence of articles.

It must be mentioned that the theory presented here is the outcome of a line of research initiated by the author more than ten years ago and that the genesis of the basic ideas can be found in many previous publications [4–19]. Most of the previous work was done for formally symmetric operators and attention had been focused on boundary methods; more specifically, the Trefftz method. An integrated presentation of those results has just appeared in book form [20]. However, it was only after the draft of the book had been completed that I extended the theory to nonsymmetric operators. This was essential for developing a unified formulation of discrete methods.

This article contains the systematic development of Green's formulas for nonsymmetric operators. The forthcoming articles will present the general formulation of boundary value problems and its application to the analysis of finite differences, finite elements, and boundary methods.

II. PRELIMINARY NOTIONS AND NOTATIONS

Denote by $\mathcal{F}$ the field of real or complex numbers. Let D be a linear space over the field $\mathcal{F}$, whose elements will be called scalars [21]. Elements of D will be denoted by u, v, . . . , and will be said to be admissible functions. Write D^* for the linear space of linear functionals defined on D; i.e., D^* is the algebraic dual of D. Hence, any element $\alpha \in D^*$ is a function $\alpha : D \to \mathcal{F}$, which is linear. Given $v \in D$, the value of the function α at v will be denoted by

$$\alpha(v) = \langle \alpha, v \rangle \in \mathcal{F}. \tag{12}$$

In this work functional-valued operators $P:D \to D^*$ will be used extensively. Given $u \in D$, the value $P(u) \in D^*$ is itself a linear functional. According with Eq. (12), given any $v \in D$, $\langle P(u), v \rangle \in \mathcal{F}$ will be the value of this linear functional at v. When the operator P is itself linear, $\langle P(u), v \rangle$ is linear in u when v is kept fixed. Therefore, as it is customary, we write

$$\langle Pu, v \rangle = \langle P(u), v \rangle \in \mathcal{F} \tag{13}$$

for this value. We shall be concerned, exclusively, with functional-valued operators that are linear.

On the other hand, let $D^2 = D \oplus D$ be the space of pairs $\{u, v\}$ with $u \in D$ and $v \in D$. We may consider functions $\beta : D^2 \to \mathcal{F}$. The value of such a function on a pair $\{u, v\} \in D^2$ will be written as $\beta(u, v)$. Such a function is said to be a bilinear functional if it is linear in u when $v \in D$ is kept fixed, and, conversely, it is linear in v when u is kept fixed. There is a one-to-one correspondence between bilinear functionals and functional-valued operators which are linear. This is given by

$$= \langle Pu, v \tag{14}$$

In what follows, operators $P:D \to D^*$ will be defined by giving the corresponding bilinear functional.

Given $P:D \to D^*$, let Eq. (14) be satisfied, the notation $P^*:D \to D^*$ will be used for the operator associated with the transposed bilinear functional; thus

$$\langle P^*u, v \rangle = \beta^*(u, v) = \beta(v, u) = \langle Pv, u \rangle . \tag{15}$$

Here, β^* is the transpose of β and $P^*:D \to D^*$ will be called the transpose of $P:D \to D^*$. Observe that the transpose operator is well-defined whenever $P:D \to D^*$ is given.

Example 2.1. In applications of the theory, bilinear forms such as $\langle Pu, v\rangle$ will be used to characterize boundary-value problems variationally. Thus, if D is a linear space of functions on a region Ω and $\mathcal{L}$ is a linear differential operator defined in functions $u \in D$, then one can define $P:D \to D^*$ by

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\, dx \tag{16}$$

associating in this manner a functional-valued operator to the original differential operator $\mathcal{L}$.

For any operator $P:D \to D^*$ the null subspace of P will be denoted by N_P. Hence,

$$N_P = \{u \in D \,|\, Pu = 0\}$$

Some relationships between the null subspaces of functional-valued operators will be used in the sequel.

Definition 2.1. One says that the operators $P:D \to D^*$ and $Q:D \to D^*$ can be varied independently, if for every $U \in D$ and $V \in D$ there exists $u \in D$ such that

$$Pu = PU \quad \text{while } Qu = QV. \tag{18}$$

The notion we have just introduced can be defined in several alternative—but equivalent—ways. For later use, they are listed now.

Theorem 2.1. Let $P:D \to D^*$ and $Q:D \to D^*$ be given operators. Then, the following assertions are equivalent:

(i) P and Q can be varied independently.

(ii) For every $U \in D$, there exists $u \in D$ such that

$$Pu = PU \quad \text{while } Qu = 0. \tag{19}$$

(iii) For every $V \in D$, there exists $u \in D$ such that

$$Pu = 0 \quad \text{while } Qu = QV. \tag{20}$$

(iv)
$$D = N_P + N_Q. \tag{21}$$

(v) For every $u \in D$ there exist $u_1 \in D$ and $u_2 \in D$ such that

(a)
$$u = u_1 + u_2 \tag{22a}$$

and

(b)
$$Pu = Pu_1 \quad \text{while } Qu = Qu_2 \tag{22b}$$

Proof. We omit the proof of this result. A slightly different version of this theorem is shown in [20].

If $R = P + Q$, it is easy to see that $N_R \supset N_P \cap N_Q$. When, in addition, P^* and Q^* can be varied independently, one has

$$N_R = N_P \cap N_Q \,. \tag{23}$$

Proposition 2.1. Assume P^* and Q^* can be varied independently. Let $R = P + Q$, then Eq. (23) is satisfied.

Proof. This result is also shown in [20].

III. DECOMPOSITION OF OPERATORS: A REPRESENTATION THEOREM

The unified formulation of discrete methods is based on abstract Green's formulas which are special cases of the kind of decompositions of operators discussed in this section.

Definition 3.1. Let $R : D \to D^*$ be given. The operators R_1 and R_2 are said to decompose R when

(i) $$R = R_1 + R_2 \,. \tag{24}$$

(ii) R_1 and R_2 can be varied independently.
(iii) R_1^* and R_2^* can be varied independently.

Proposition 3.1. Let R_1 and R_2 decompose R. Then

$$N_R = N_{R1} \cap N_{R2} \quad \text{while } N_{R^*} = N_{R^*1} \cap N_{R^*2} \,. \tag{25}$$

Proof. By virtue of Proposition 2.1.

As mentioned before, the general class of Green's formulas to be used in this work are special cases of the kind of decompositions introduced in Definition 3.1. For applications, great flexibility is achieved by characterizing such decompositions abstractly. This depends on properties of the subspaces N_{R1}, N_{R2}, N_{R^*1}, and N_{R^*2}.

Remark 3.1. Assume R_1 and R_2 decompose R. Then

(a) $$\langle Ru, v\rangle = 0 \quad \forall\, u \in N_{R1} \quad \text{and} \quad v \in N_{R^*2} \tag{26}$$

(b) $$N_{R1} \supset N_R \quad \text{while } N_{R^*2} \supset N_{R^*} \,. \tag{27}$$

(c) $$D = N_{R1} + N_{R2} = N_{R^*1} + N_{R^*2} \,. \tag{28}$$

Proof. Equation (26) is clear because

$$\langle Ru, v\rangle = \langle R_1 u, v\rangle + \langle R_2^* v, u\rangle \,.$$

Relations (27) follow from Proposition 3.1, while Eqs. (28) are mere restatements of properties (ii) and (iii) in Definition 3.1.

In a similar fashion, it is clear that

(a′) $$\langle Ru, v\rangle = 0 \quad \forall\, u \in N_{R2} \quad \text{and} \quad v \in N_{R^*1} \,, \tag{26′}$$

(b′) $$N_{R2} \supset N_R \quad \text{while } N_{R^*1} \supset N_{R^*} \,. \tag{27′}$$

The following definitions are motivated by these remarks.

Definition 3.2. Let $I_1 \subset D$ and $I_2 \subset D$ be two linear subspaces of D. Then, the pair $\{I_1, I_2\}$ is said to be a pair of conjugate subspaces for R, when

$$\text{(i)} \qquad \langle Ru, v\rangle = 0 \quad \forall\, u \in I_1 \quad \text{and} \quad v \in I_2 . \tag{29}$$

A pair $\{I_1, I_2\}$ of conjugate subspaces for R is said to be regular when in addition

$$\text{(ii)} \qquad I_1 \supset N_R \quad \text{and} \quad I_2 \supset N_{R^*} , \tag{30}$$

In view of these definitions and previous remarks, it is seen that $\{N_{R1}, N_{R^*2}\}$ is a pair of conjugate subspaces that is regular for R. Another pair with the same property is $\{N_{R2}, N_{R^*1}\}$.

Definition 3.3. Let[†] $\{I_{11}, I_{22}\}$ and $\{I_{12}, I_{21}\}$ be two pairs of regular conjugate subspaces for R. The pairs together are said to be a canonical decomposition of D with respect to R, if

$$D = I_{11} + I_{12} = I_{21} + I_{22} . \tag{31}$$

Clearly, when R_1 and R_2 decompose R, the pair $\{N_{R2}, N_{R^*1}\}$ and $\{N_{R1}, N_{R^*2}\}$ constitute a canonical decomposition of D with respect to R. Indeed, when taking

$$I_{11} = N_{R2}, \qquad I_{22} = N_{R^*1} , \tag{32a}$$

$$I_{12} = N_{R1}, \qquad I_{21} = N_{R^*2} , \tag{32b}$$

Eqs. (31) are satisfied by virtue of (28). In such a case, there exist elements $u_{11} \in I_{11}$, $u_{12} \in I_{12}$, $u_{21} \in I_{21}$ and $u_{22} \in I_{22}$ such that

$$u = u_{11} + u_{12} = u_{21} + u_{22} \tag{33}$$

by virtue of Definitions 3.1 and Theorem 2.1. With this choice

$$\langle R_1 u, v\rangle = \langle Ru_{11}, v_{21}\rangle; \qquad \langle R_2 u, v\rangle = \langle Ru_{12}, v_{22}\rangle , \tag{34}$$

where the decomposition (33) has also been applied to $v \in D$.

In the remaining part of this section, we prove that Eqs. (34) supply the desired abstract characterization of operator decompositions.

Theorem 3.1. Let $\{I_{11}, I_{22}\}$ and $\{I_{12}, I_{21}\}$ be a canonical decomposition of D with respect to R. Then, there exists a unique pair of operators $R_1 : D \to D^*$ and $R_2 : D \to D^*$ which decompose R and satisfy Eqs. (32). Such operators are defined by Eqs. (34), where representation (33) is used for u and v.

Proof. It is given in the Appendix.

Definition 3.4. Let $\{I_1, I_2\}$ be a pair of conjugate subspaces, regular for R. Such a pair is said to be completely regular for R if, in addition, the implications[‡]

[†] The manner in which the subindexes are introduced in this definition yields notational advantages, as will be seen in the sequel.

[‡] In what follows the arrow $\Rightarrow$ must be read as implies and the double arrow $\Leftrightarrow$ as if and only if.

$$\langle Ru, v\rangle = 0 \quad \forall\, v \in I_2 \Rightarrow u \in I \tag{35a}$$

and

$$\langle Ru, v\rangle = 0 \quad \forall\, u \in I_1 \Rightarrow v \in I$$

hold.

Remark 3.2. It is easy to see that a pair of linear subspaces $\{I_1, I_2\}$ is completely regular for R if and only if the following equivalence relations hold:

$$\langle Ru, v\rangle = 0 \quad \forall\, v \in I_2 \Leftrightarrow u \in I_1\,, \tag{36a}$$

$$\langle Ru, v\rangle = 0 \quad \forall\, u \in I_1 \Leftrightarrow v \in I_2\,. \tag{36b}$$

Proposition 3.2. When $\{I_{11}, I_{22}\}$ and $\{I_{12}, I_{21}\}$ is a canonical decomposition of D with respect to R, one has:

(i) Each one of the pairs $\{I_{11}, I_{22}\}$ and $\{I_{12}, I_{21}\}$ is completely regular for R.

(ii) $$N_R = I_{11} \cap I_{12} \quad \text{while } N_{R^*} = I_{21} \cap I_{22}\,. \tag{37}$$

Proof. This is also given in the Appendix.

Remark 3.3. Equations (37) imply that in representation (33) the vectors $u_{11} \in D$ and $u_{12} \in D$ are unique except for elements belonging to N_R. Similarly, $u_{21} \in D$ and $u_{22} \in D$ are unique except for elements belonging to N_{R^*}.

IV. BOUNDARY OPERATORS AND GREEN'S FORMULAS

The notions of boundary operators, formal adjoints, and Green's formulas are introduced in this section.† An abstract characterization of Green's formulas is also given.

Definition 4.1. $B:D \to D^*$ is a boundary operator for $P:D \to D^*$ if

$$\langle Pu, v\rangle = 0 \quad \forall\, v \in N_{B^*} \Rightarrow Pu = 0\,. \tag{38}$$

Example 4.1. Let Ω be an open region of an n-dimensional Euclidean space ($n \geq 1$) and $\partial\Omega$ be its boundary (Fig. 1). Assume the admissible functions (i.e., the elements of D) possess second-order continuous derivatives on the closure of Ω. Define $P:D \to D^*$ and $B:D \to D^*$ by

$$\langle Pu, v\rangle = \int_\Omega v\, \Delta u\, dx \quad \text{and} \quad \langle Bu, v\rangle = \int_{\partial\Omega} uv\, dx$$

In this case it can be shown [see, e.g., 20] that

$$N_{B^*} = \{u \in D \mid u = 0 \quad \text{on} \quad \partial\Omega\}\,. \tag{40}$$

Using arguments that are standard in calculus of variations, it can be seen that implication (38) is fulfilled. Thus, B is a boundary operator for P.

Definition 4.2. The operators $P:D \to D^*$ and $Q:D \to D^*$ are said to be formal adjoints when $R = P - Q^*$ is a boundary operator for P, while R^* is a boundary operator for Q.

†Definitions 4.1 and 4.2 modify slightly those presented in [20].

Remark 4.1. The condition of being formal adjoints defines a symmetric relation among functional-valued operators.

Definition 4.3. An operator $P:D \to D^*$ is said to be formally symmetric when P is formal adjoint of itself.

Formally symmetric operators were discussed in [20]. The following result was shown there.

Proposition 4.1. Given $P:D \to D^*$, define $A = P - P^*$. Then P is formally symmetric if and only if

$$\langle Pu, v\rangle = 0 \quad \forall\, v \in N_A \Rightarrow Pu = 0. \tag{41}$$

Proof. See [20].

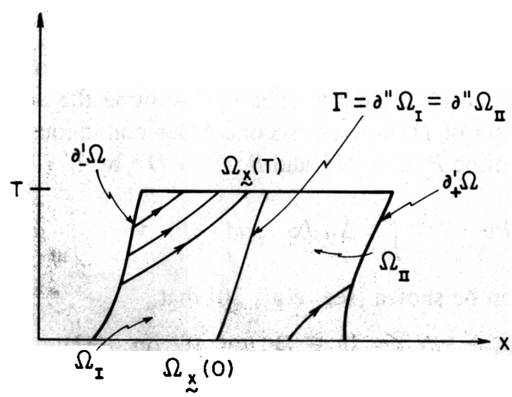
The class of Green's formulas to be considered in this theory is given next.

Definition 4.4. Let $P:D \to D^*$ and $Q:D \to D^*$ be formal adjoints. The equation

$$P - B = Q^* - C^* \tag{42}$$

is Green's formula when C^* is a boundary operator for B, C is a boundary operator for B^*, and, conversely, B is a boundary operator for C^*, while B^* is a boundary operator for C. Equation (42) is Green's formula in the strong sense when B and $-C^*$ decompose $R = P - Q^*$.

Example 4.2. Let the linear space of admissible functions $D = C^1(\overline{\Omega})$, where Ω is the region (space–time) illustrated in Figure 2. Consider the differential operator $\mathscr{L}$, relevant in transport problems, and its formal adjoint (as a differential operator) $\mathscr{L}^*$ given by



$\partial'_+\Omega$ if $n_t + Vn_x > 0$

$\partial'_-\Omega$ if $n_t + Vn_x < 0$

FIG. 2.

$$\mathscr{L}u = \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x}; \qquad \mathscr{L}^*u = -\frac{\partial u}{\partial t} - \frac{\partial (Vu)}{\partial x} \tag{43}$$

Define

$$\langle Pu, v\rangle = \int_\Omega v\mathscr{L}u\,d\mathbf{x}; \qquad \langle Q^*u, v\rangle = \int_\Omega u\mathscr{L}^*v\,d\mathbf{x}, \tag{44}$$

where $d\mathbf{x} = dx\,dt$. Then $P:D \to D^*$ and $Q:D \to D^*$ are formal adjoints Even more, if

$$\langle Bu, v\rangle = \int_{\Omega_x(0)} uv\,d\mathbf{x} + \int_{\partial'_-\Omega} uv(n_t + Vn_x)\,d\mathbf{x}$$

and

$$\langle C^*u, v\rangle = -\int_{\Omega_x(T)} uv\,d\mathbf{x} - \int_{\partial'_+\Omega} uv(n_t + Vn_x)\,d\mathbf{x}, \tag{45b}$$

then Eq. (42) is Green's formula in the weak sense. Here, $\partial'_+\Omega$ and $\partial'_-\Omega$ are defined as the subset of $\partial'\Omega$ where $n_t + Vn_x$ is greater and less than zero, respectively. In turn, $\partial'\Omega \subset \partial\Omega$ is the lateral boundary of Ω.

Remark 4.2. It can be shown that every Green's formula in the strong sense is a Green's formula.

Proposition 4.2. When Eq. (42) is Green's formula:

(i) $$N_R = N_B \cap N_{C^*} \quad \text{while } N_{R^*} = N_{B^*} \cap N_C\,. \tag{46}$$

(ii) B and C^* are boundary operators for P; while
(iii) C and B^* are boundary operators for Q.

Proof. For the case when Eq. (42) is Green's formula in the strong sense, Eq. (46) follows from Proposition 2.1. The proof of this result for the case when (42) is only Green's formula is similar to that of Proposition 2.1 but will not be given here.

In what follows it will be assumed that P and Q are formal adjoints. Also, the notation $R = P - Q^*$ will be used. The reader must observe that the definitions and notations used in the following are dependent on having given an ordered pair of operators $\{P, Q\}$. Indeed, if the order of $\{P, Q\}$ is changed, one obtains $-R^*$ instead of R.

Definition 4.5. The boundary values relevant for P of $u \in D$ and $v \in D$ are said to be equal if and only if $Ru = Rv$. A subspace $I \subset D$ is defined by boundary values relevant for P, when $I \supset N_R$. Similarly, $I \subset D$ is defined by boundary values relevant for Q when $I \supset N_{R^*}$.

Remark 4.3. When (42) is Green's formula, the boundary values relevant for P are characterized by the pair $\{Bu, C^*u\}$, because $Ru = Rv$ if and only if $Bu = Bv$ and $C^*u = C^*V$. In addition, assume $I \subset D$ is characterized by boundary values and let $u \in D$ and $v \in D$ have the same boundary values (i.e., $Ru = Rv$); then either u and v belong to I or alternatively neither u nor v belong to I.

Definition 4.6. An abstract Green's formula in the strong sense for $P:D \to D^*$ is a canonical decomposition of D with respect to R.

Remark 4.4. Let $\{I_{11}, I_{22}\}$, $\{I_{12}, I_{21}\}$ be an abstract Green's formula; then using relation (30), it is seen that I_{11} and I_{12} are defined by boundary values relevant for P, while I_{21} and I_{22} are defined by boundary values relevant for Q.

Theorem 4.1. There is a one-to-one correspondence between Green's formulas in the strict sense and abstract Green's formulas in the strict sense. When Eq. (42) is Green's formula in the strict sense, the abstract Green's formula $\{I_{11}, I_{22}\}$, $\{I_{12}, I_{21}\}$ is given by

$$I_{11} = N_{C^*}; \qquad I_{22} = N_{B^*} \tag{47a}$$

$$I_{12} = N_B; \qquad I_{21} = N_C. \tag{47b}$$

Conversely, when the abstract Green's formula $\{I_{11}, I_{22}\}$, $\{I_{12}, I_{21}\}$ is given, then

$$\langle Bu, v\rangle = \langle Ru_{11}, v_{21}\rangle, \qquad \langle Cu, v\rangle = -\langle Rv_{12}, u_{22}\rangle.$$

Proof. This result is clear by virtue of Theorem 3.1.

Remark 4.5. A similar representation holds for Green's formulas that are not strong in the sense of Definition 4.4, but the formulation of a corresponding result is beyond the scope of this article.

V. GREEN'S FORMULAS FOR OPERATORS DEFINED IN DISCONTINUOUS FIELDS

In the systematic analysis of discrete methods (e.g., finite elements, finite differences, and boundary methods) the use of Green's formulas for operators defined in discontinuous fields is essential. Such formulas are developed here for arbitrary linear operators.

Throughout this section, the space of admissible functions will be a product space $D = D_{\mathrm{I}} \oplus D_{\mathrm{II}}$, where D_{I} and D_{II} are two linear spaces. In applications the elements of D_{I} and D_{II} will be functions defined on two neighboring regions Ω_{I} and Ω_{II} (Fig. 3), respectively. Thus, the elements of D are pairs $u = \{u_{\mathrm{I}}, u_{\mathrm{II}}\}$, where $u_{\mathrm{I}} \in D_{\mathrm{I}}$ and $u_{\mathrm{II}} \in D_{\mathrm{II}}$. The operator $R: D \to D^*$ satisfies

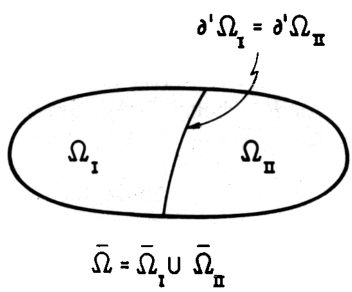


FIG. 3.

$$\langle Ru, v\rangle = \langle R_{\rm I} u_{\rm I}, v_{\rm I}\rangle + \langle R_{\rm II} u_{\rm II}, v_{\rm II}\rangle\,,$$

where $R_{\rm I}:D_{\rm I} \rightarrow D_{\rm I}^*$ and $R_{\rm II}:D_{\rm II} \rightarrow D_{\rm II}^*$ are given operators.

When (48) is satisfied, it is easy to see that

$$u = \{u_{\rm I}, u_{\rm II}\} \in N_R \Leftrightarrow u_{\rm I} \in N_{R{\rm I}} \quad \text{and} \quad u_{\rm II} \in N_{R{\rm II}} \qquad \text{(49a}$$

and

$$u = \{u_{\rm I}, u_{\rm II}\} \in N_{R^*} \Leftrightarrow u_{\rm I} \in N_{R^*{\rm I}} \quad \text{and} \quad u_{\rm II} \in N_{R^*{\rm II}}\,. \qquad \text{(49b)}$$

Here N_R, $N_{R{\rm I}}$, and $N_{R{\rm II}}$ stand for the null subspaces of R, $R_{\rm I}$, and $R_{\rm II}$, respectively. A similar notatation is used for the null subspaces of the transpose of these operators.

In the following discussion a linear subspace $S \subset D$ is considered. Elements $u \in S$ will be said to be smooth. When $u = \{u_{\rm I}, u_{\rm II}\} \in S$ (i.e., when u is smooth), $u_{\rm I} \in D_{\rm I}$ and $u_{\rm II} \in D_{\rm II}$ will be said to be smooth extensions of each other.

Definition 5.1. A linear subspace $S \subset D$ is said to be a smoothness relation, if every $u_{\rm I} \in D_{\rm I}$ possesses at least one smooth extension $u_{\rm II} \in D_{\rm II}$, and conversely every $u_{\rm II} \in D_{\rm II}$ possesses at least one smooth extension $u_{\rm I} \in D_{\rm I}$.

The mapping $\tau:D \rightarrow D$, defined for every $u = \{u_{\rm I}, u_{\rm II}\} \in D$ by

$$\tau u = \{u_{\rm I} - u_{\rm II}\} \qquad (50)$$

will be used in the sequel. Clearly, τ is self-inverse (i.e., $\tau^2 = 1$) and has the following properties:

$$\langle Ru, v\rangle = \langle R\tau u, \tau v\rangle \qquad (51)$$

and

$$N_R = \tau(N_R), \qquad N_{R^*} = \tau(N_{R^*}) \qquad ($$

Let $M \subset D$ be the image of S under $\tau:D \rightarrow D$; i.e.,

$$M = \tau(S)\,. \qquad ($$

Observe that M is necessarily a smoothness condition when S is also.

Remark 5.1. Even more, when $S \subset D$ is a smoothness relation, one has

$$D = S + M\,. \qquad ($$

Indeed, given $u = \{u_{\rm I}, u_{\rm II}\} \in D$, take $u_{\rm I}' \in D_{\rm I}$ and $u_{\rm II}' \in D_{\rm II}$ such that $\{u_{\rm I}', u_{\rm II}\} \in S$ and $\{u_{\rm I}, u_{\rm II}'\} \in S$. Define

$$\dot{u} = \frac{1}{2}\{u_{\rm I}' + u_{\rm I}, u_{\rm II}' + u_{\rm II}\} \qquad (55a)$$

and

$$[u] = \{u_{\rm I}' - u_{\rm I}, u_{\rm II}' - u_{\rm II}\}$$

Then $\dot{u} \in S$, $[u] \in M$, while

$$u = \dot{u} - \frac{1}{2}[u] \qquad (56)$$

In the general development of the theory pairs $\{S^\ell, S^r\}$ of smoothness conditions will be considered. Elements $u \in S^\ell$ will be said to be left-smooth, while those belonging to S^r are right-smooth. Similarly, there will be left- and right-smooth extensions of elements $u_I \in D_I$ or, alternatively, of elements $u_{II} \in D_{II}$. Also, we write $M^\ell = \tau(S^\ell)$ and $M^r = \tau(S^r)$. Equation (56) yields two representations:

$$u = \dot{u}^\ell - \frac{1}{2}[u]^\ell = \dot{u}^r - \frac{1}{2}[u]^r,$$

where $\dot{u}^\ell \in S^\ell$, $[u]^\ell \in M^\ell$, $\dot{u}^r \in S^r$ and $[u]^r \in M^r$.

Definition 5.2. A pair $\{S^\ell, S^r\}$ of smoothness relations will be said to be conjugate, regular, or completely regular, respectively, when the pair $\{S^\ell, S^r\}$ is conjugate, regular, or completely regular for R, in the sense of Definitions 3.2 and 3.4.

Remark 5.2. If $\{S^\ell, S^r\}$ is a regular pair of conjugate smoothness relations, then the pair $\{M^\ell, M^r\}$ is also a regular pair of conjugate smoothness relations, as can be verified using Definitions 3.2 and 5.1, together with Eqs. (51) and (52).

Theorem 5.1. Let $P{:}D \to D^*$, $B{:}D \to D^*$, $Q{:}D \to D^*$, and $C{:}D \to D^*$ be given. Assume

(a) $P - B$ and $Q - C$ are formal adjoints.
(b) $R = P - B - (Q - C)^*$.
(c) The pair $\{S^\ell, S^r\}$ is a conjugate pair of smoothness conditions, regular for R.

Then,

(i) The two pairs $\{S^\ell, S^r\}$, $\{M^\ell, M^r\}$ constitute an abstract Green's formula, in the strong sense, for $P - B$.
(ii) The equation

$$P - B - J = Q^* - C^* - K^* \tag{57}$$

is Green's formula, in the strong sense, for $P - B$, if $J{:}D \to D^*$ and $K{:}D \to D^*$ are defined by

$$2\langle Ju, v\rangle = -\langle R[u]^\ell, \dot{v}^r\rangle, \qquad 2\langle K^*u, v\rangle = \langle R\dot{u}^\ell, [v]^r\rangle \tag{58}$$

(iii)

$$S^\ell = N_J; \qquad S^r = N_K,$$
$$M^\ell = N_{K^*}; \qquad M^r = N_{J^*}$$

Proof. It is only necessary to prove (i) because once this has been shown (ii) and (iii) are a straightforward application of Theorem 4.1. In view of Remark 5.1, it is clear that

$$D = S^\ell + M^\ell = S^r + M^r.$$

Therefore, by virtue of Definitions 3.2, 3.3, and 4.6, it is only necessary to prove that the pair $\{M^\ell, M^r\}$ is regular conjugate for R. Using the fact that the pair $\{S^\ell, S^r\}$ has this property, the desired result follows from (51) and (52).

The following result will be useful when carrying out the actual computations of the operators J and K^*.

Corollary 5.1. Under the assumptions of Theorem 5.1, one has

$$\langle Ju, v\rangle = -\langle R_{\mathrm{I}}[u]^\ell, \dot{v}^r\rangle \quad \text{while } \langle K^*u, v\rangle = \langle R_{\mathrm{I}}\dot{u}^\ell, [v]^r\rangle \tag{60}$$

Proof. Using Eqs. (29) and (48), together with the fact that $\dot{v}^r \in S^r$ while $\{[u]_{\mathrm{I}}^\ell, -[u]_{\mathrm{II}}^\ell\} \in S^\ell$, it is seen that

$$\langle R[u]^\ell, \dot{v}^r\rangle = \langle R_{\mathrm{I}}[u]^\ell, \dot{v}^r\rangle + \langle R_{\mathrm{II}}[u]^\ell, \dot{v}^r\rangle, \tag{61a}$$

$$0 = \langle R_{\mathrm{I}}[u]^\ell, \dot{v}^r\rangle - \langle R_{\mathrm{II}}[u]^\ell, \dot{v}^r\rangle. \tag{61b}$$

Here as in what follows the notation is simplified by deleting subindexes whenever such practice yields unambiguous results. Thus, for example, we write $\langle R_{\mathrm{I}}[u]_{\mathrm{I}}^\ell, \dot{v}_{\mathrm{I}}^r\rangle = \langle R_{\mathrm{I}}[u]^\ell, \dot{v}^r\rangle$.

The first of Eqs. (60) can be obtained from the first one in (58) by means of Eqs. (61), since the latter imply

$$\langle R[u]^\ell, \dot{v}^r\rangle = 2\langle R_{\mathrm{I}}[u]^\ell, \dot{v}^r\rangle = 2\langle R_{\mathrm{II}}[u]^\ell, \dot{v}^r\rangle. \tag{62}$$

The second of Eqs. (60) is obtained similarly.

VI. APPLICATIONS

Two examples are here given, one relates to time-independent problems and the other one to time-dependent ones.

A. Elliptic Operators

Let $\mathcal{L}$ be a second-order elliptic differential operator which is linear. Any such operator can be written as the sum of two operators $\mathcal{L}_s$ and $\mathcal{L}_a$ defined by

$$\mathcal{L}_s u = \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + cu \tag{63a}$$

and

$$\mathcal{L}_a u = b_i\frac{\partial u}{\partial x_i} + \frac{1}{2}(\nabla \cdot \mathbf{b})u$$

Here, as in what follows, it will be assumed that the coefficients are infinitely differentiable in the closure of the region Ω of definition of the functions, unless otherwise stated explicitly; also $a_{ij} = a_{ji}$.

Take

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\, dx \quad \text{and} \quad \langle Qu, v\rangle = \int v\mathcal{L}^*u\, dx, \tag{64}$$

where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. Thus, $\mathcal{L} = \mathcal{L}_s + \mathcal{L}_a$ while $\mathcal{L}^* = \mathcal{L}_s - \mathcal{L}_a$ because $\mathcal{L}_s$ and $\mathcal{L}_a$ are formally symmetric and antisymmetric differential operators, respectively. By integration by parts it is seen that

$$\langle Pu, v\rangle - \langle Q^*u, v\rangle = \int_{\partial\Omega} \{vT_i(u) - uT_i(v) + b_i uv\} n_i \, dx. \tag{65}$$

$$T_i(u) = a_{ij}\frac{\partial u}{\partial x_j}, \tag{66}$$

while $\mathbf{n}$ is the unit normal vector. Assume $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are differential operators defined on $\partial\Omega$ such that

$$\int_\Omega \{v\mathcal{L}(u) - u\mathcal{L}^*(v)\} \, dx = \int_{\partial\Omega} \{\mathcal{B}u\mathcal{D}v - \mathcal{E}u\mathcal{C}v\} \, dx \tag{67}$$

is Green's formula in the sense of Lions and Magenes [22]. Define

$$\langle Bu, v\rangle = \int_{\partial\Omega} \mathcal{B}u\mathcal{D}v \, dx \quad \text{and} \quad \langle C^*u, v\rangle = \int_{\partial\Omega} \mathcal{E}u\mathcal{C}v \, dx. \tag{68}$$

There are many possible ways of taking the space D of admissible functions. A possibility is to take $D = H^2(\Omega)$. However, this is not essential, one can take, for example, D as the space of functions possessing continuous second-order derivatives in the closure of Ω. For these two cases, one has that Eq. (42) is satisfied and it can be shown that it is Green's formula in the strong sense (Definition 4.4).

Let us divide the region Ω into two subregions Ω_I and Ω_II (Fig. 3). The elements $u \in D$ of the space of admissible functions will be pairs $u = \{u_\mathrm{I}, u_\mathrm{II}\}$, where u_I is the restriction to Ω_I of the function belonging to $H^2(\Omega)$ and similarly u_II is the restriction to Ω_II of a function belonging to $H^2(\Omega)$. Then, functions $u \in D$ are in general discontinuous across $\partial'\Omega_\mathrm{I} = \partial'\Omega_\mathrm{II}$. Thus, Eq. (42) is no longer satisfied but instead

$$P - B - (Q - C)^* = R = R_\mathrm{I} + R_\mathrm{II}. \tag{69}$$

$$\langle R_\mathrm{I}u, v\rangle = \int_{\partial'\Omega_\mathrm{I}} \{vT_i(u) - uT_i(v) + b_i uv\} n_i \, dx, \tag{70}$$

while R_II is obtained by replacing $\partial'\Omega_\mathrm{I}$ with $\partial'\Omega_\mathrm{II}$. In order to make the meaning of integrals like the one occurring in Eq. (70) more precise, the following observation is made: "On $\partial'\Omega_\mathrm{I} = \partial'\Omega_\mathrm{II}$ two unit normals $\mathbf{n}_\mathrm{I}$ and $\mathbf{n}_\mathrm{II}$ are defined; they point outward from Ω_I and Ω_II, respectively. Also, the values on $\partial'\Omega_\mathrm{I} = \partial'\Omega_\mathrm{II}$ of u_I, u_II and their derivatives do not coincide." For the evaluation of integrals like the one occurring in Eq. (70), the convention adopted is such that the normals and values of the functions and their derivatives to be used are those corresponding to the subindex of the integral sign.

Next, we define the subspaces $S^{\ell} \subset D$ and $S^{r} \subset D$.

(a) Elements $u \in S^{\ell}$ satisfy the smoothness conditions.

(a)(1) Continuity of the function, i.e.,

$$u_{\mathrm{I}} = u_{\mathrm{II}}, \quad \text{on } \partial'\Omega_{\mathrm{I}} = \partial'\Omega_{\mathrm{II}}. \tag{71a}$$

(a)(2) Continuity of the total flux, i.e.,

$$\{T_i^{\mathrm{I}}(u) + q_i^{\mathrm{I}}(u)\}n_i = \{T_i^{\mathrm{II}}(u) + q_i^{\mathrm{II}}(u)\}n_i, \quad \text{on } \partial'\Omega_{\mathrm{I}} = \partial'\Omega_{\mathrm{II}}. \tag{71b}$$

Here, $q_i(u) = b_i u$. Also, in (71b), it is immaterial which normal is used as long as it is the same in both members of the equation.

(b) Elements $v \in S^{r}$ satisfy the smoothness conditions.

(b)(1) Continuity of the function, i.e.,

$$vv_{\mathrm{I}} = v_{\mathrm{II}}, \quad \text{on } \partial'\Omega_{\mathrm{I}} = \partial'\Omega_{\mathrm{II}}$$

(b)(2) Continuity of diffusive flux, i.e.,

$$T_i^{\mathrm{I}}(v)n_i = T_i^{\mathrm{II}}(v)n_i$$

Then $\{S^{\ell}, S^{r}\}$ is a regular conjugate pair for $R: D \to D^*$, as given by (69). Applying the results of Section V, it is seen that

$$\langle Ju, v\rangle = -\int_{\partial'\Omega_{\mathrm{I}}} \{\dot{v}[T_i(u) + q_i(u)] - [u]\dot{T}_i(v)\}n_i\,dx$$

while

$$\langle K^*u, v\rangle = -\int_{\partial'\Omega_{\mathrm{I}}} \{\dot{u}[T_i(v)] - [v]\{T_i(u) + \dot{q}_i(u)\}n_i\,dx$$

Here for any function the dot and the square bracket mean average and jump, respectively. Thus, for example,

$$\dot{q}_i(u) = \frac{1}{2}[q_i^{\mathrm{I}}(u) + q_i^{\mathrm{II}}(u)] \quad \text{and} \quad [q_i(u)] = q_i^{\mathrm{II}}(u) - q_i^{\mathrm{I}}(u) \tag{74}$$

Observe that $S^{\ell} = S^{r}$ when the coefficients b_i are continuous, i.e., when $b_i^{\mathrm{I}} = b_i^{\mathrm{II}}$ on $\partial'\Omega_{\mathrm{I}}$. However, Eqs. (73) are applicable even if $b_i^{\mathrm{I}} \neq b_i^{\mathrm{II}}$, and $S^{\ell} \neq S^{r}$ for that case.

B. Pure Advection (a Hyperbolic Operator)

As mentioned in Example 4.2, in connection with the study of transport phenomena, one considers the operator

$$\langle Pu, v\rangle = \int_{\Omega} v\left\{\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x}\right\} d\mathbf{x}, \tag{75}$$

where V is a function of x, $d\mathbf{x} = dx\,dt$, and the region Ω may be as illustrated in Figure 2. It will be assumed that V together with its first-order derivative is continuous. If $B: D \to D^*$, $Q^*: D \to D^*$, and $C^*: D \to D^*$ are defined by

Eqs. (44) and (45), then the equation

$$P - B = Q^* - C^* \tag{76}$$

is satisfied when $D = C^1(\overline{\Omega})$. Also, (76) is Green's formula. However, if the definition of the space of admissible functions is modified in such a way that the admissible functions are allowed to have simple discontinuities across a given curve such as Γ in Figure 2, then

$$P - B - (Q - C)^* = R = R_{\mathrm{I}} + R_{\mathrm{II}}. \tag{77}$$

Here

$$\langle R_{\mathrm{I}} u, v \rangle = \int_{\partial''\Omega_{\mathrm{I}}} uv(n_t + Vn_x)\,d\mathbf{x}, \tag{78}$$

while R_{II} is defined by replacing Ω_{I} with Ω_{II} in (78). The convention followed in subsection A, on the normal vector to be used in the evaluation of integrals such as (70) or (78), is recalled.

The subspaces $S^\ell \subset D$ and $S^r \subset D$ are defined by $S^\ell = S^r = S$, where elements $u \in S$ satisfy the smoothness condition

$$u_{\mathrm{I}}(n_t + Vn_x) = u_{\mathrm{II}}(n_t + Vn_x), \quad \text{on } \Gamma. \tag{79}$$

Observe that no condition is imposed upon u at points where $n_t + Vn_x = 0$. Applying Eqs. (60), one obtains

$$\langle Ju, v \rangle \qquad \int_\Gamma [u]\dot{v}(n_t + Vn_x)\,d\mathbf{x} \tag{80}$$

and

$$\langle K^*u, v \rangle = \int_\Gamma \dot{u}[v](n_t + Vn_x)\,d\mathbf{x} \qquad (81$$

where

$$[u] = u_+ - u \qquad \text{while } \dot{u} = \quad (u_+ + u$$

In Eq. (81), $\mathbf{n} = (n_x, n_t)$ is any unit normal vector to Γ, but the convention is adopted that u_+ is the value of u on the side to which $\mathbf{n}$ points to.

APPENDIX

Proof of Theorem 3.1. The following discussion supplies the proof of this result.

Lemma 3.1. Under the assumptions of Theorem 3.1, one has

(a) $u \in I_{11}$ and $\langle Ru, v \rangle = 0 \quad \forall\, v \in I_{21} \Rightarrow u \in N_R$ (A1a)

(b) $v \in I_{21}$ and $\langle Ru, v \rangle = 0 \quad \forall\, u \in I_{11} \Rightarrow v \in N_{R^*}$ (A1b)

(c) $u \in I_{12}$ and $\langle Ru, v\rangle = 0 \quad \forall\, v \in I_{22} \Rightarrow u \in N_R$ (A2a)

(d) $v \in I_{22}$ and $\langle Ru, v\rangle = 0 \quad \forall\, u \in I_{12} \Rightarrow u \in N_{R^*}$ (A2b)

Proof. We only prove (a). Let $w \in D$ be arbitrary and write $w = w_{21} + w_{22}$ with $w_{21} + w_{22}$ with $w_{21} \in I_{21}$ and $w_{22} \in I_{22}$. Assume the premise in (A1a) holds. Then $\langle Ru, w_{22}\rangle = 0$, because $\{I_{11}, I_{22}\}$ are conjugated subspaces. Thus,

$$\langle Ru, w\rangle = \langle Ru, w_{21} + w_{22}\rangle = \langle Ru, w_{21}\rangle = 0\,.$$

This shows $u \in N_R$, since $w \in D$ was arbitrary.

Proof of Proposition 3.2. Clearly $I_{11} \cap I_{12} \supset N_R$ by virtue of Definition 3.2. Assume $u \in I_{11} \cap I_{12}$ (i.e., $u \in I_{11}$ and $u \in I_{12}$), then the premise in (A1a) is satisfied. Hence $u \in N_R$, i.e., $I_{11} \in I_{12} \subset N_R$. This shows the first of Eqs. (37) and the proof of the second is similar. To prove (i), write $u = u_{11} + u_{12}$ and observe that $\langle Ru, v\rangle = \langle Ru_{12}, v\rangle\ \forall\ v \in I_{22}$. Hence, $\langle Ru, v\rangle\ \forall = 0\ v \in I_{22}$ implies $\langle Ru_{12}, v\rangle = 0\ \forall\ v \in I_{22}$, i.e., $u_{12} \in N_R$ by virtue of implication (A1a). Thus, $u = u_{11} + u_{12} \in I_{11}$, since $I_{11} \supset N_R$. This shows that (36a) holds if I_1 and I_2 are replaced by I_{11} and I_{22}, respectively. Relation (36b) for these spaces can be shown in a similar fashion. In this manner it is shown that the pair $\{I_{11}, I_{22}\}$ is completely regular. The remaining part of the lemma can be shown similarly.

Going back to Theorem 3.1, to prove existence, let us define $R_1 : D \to D^*$ and $R_2 : D \to D^*$ by means of Eqs. (34). They are well defined by virtue of Remark 3.3. Then $R = R_1 + R_2$ because, using representation (33), it can be seen that

$$\langle Ru, v\rangle = \langle Ru_{11}, v_{21}\rangle + \langle Ru_{12}, v_{22}\rangle\,. \tag{A3}$$

In order to see that Eqs. (32) are satisfied observe that $u = u_{11} + u_{12} \in N_{R2}$ if and only if

$$\langle Ru_{12}, v_{22}\rangle = 0 \quad \forall\ v_{22} \in I_{22}\,.$$

This implies $u_{12} \in I_{11}$ by Proposition 3.2. Thus, $u = u_{11} + u_{12} \in I_{11}$. Hence $I_{11} \subset N_{R2}$. From this, it is easy to see that $I_{11} = N_{R2}$. The proof of the other Eqs. (32) is similar. Once this has been shown, the rest of the proof of Theorem 3.1, follows.

Observe that part (i) of Proposition 3.2 was established while proving Theorem 3.1. Regarding Eqs. (37), this follow from Proposition 3.1, by virtue of (32).

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