Unified Approach to Numerical Methods, Part II: Finite Elements, Boundary Methods, and their Coupling

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This is the second in a series of three papers devoted to the presentation of a direct procedure of analysis of numerical methods for partial differential equations. The procedure consists of applying the method of weighted residuals and then interpreting the resulting equations by means of Green's formulas for discontinuous functions. Here, the general Green's formulas for operators defined in discontinuous fields developed in the first article, are applied to formulate the method of weighted residuals for arbitrary linear operators. Finite elements, boundary methods, and general procedures for coupling finite elements and boundary methods are discussed.

I. INTRODUCTION

Three of the most powerful numerical methods for partial differential equations are finite elements, finite differences, and boundary element methods. The foundations of each one of these as originally formulated, appeared to be unrelated. More recently, however, it has been recognized that it is desirable to develop foundations common to all these methodologies.

This article is the second in a sequence of papers devoted to presenting a direct method of analysis recently developed by the author. The approach is quite general, since it is applicable to any linear operator, symmetric or nonsymmetric, regardless of its type. In particular, the theory includes steady state and time dependent problems.

An outline of the theory was given in Part I [1]; it is based on two variational principles applicable to any linear boundary value problem. The first one is in terms of the "prescribed data."

$$\langle Pu, v\rangle - \langle Bu, v\rangle - \langle Ju, v\rangle = \langle f, v\rangle - \langle g, v\rangle - \langle j, v\rangle, \quad \forall\, v \in D \tag{1}$$

while the second one is in terms of the "sought information"

$$\langle Q^*u, v\rangle - \langle C^*u, v\rangle - \langle K^*u, v\rangle = \langle f, v\rangle - \langle g, v\rangle - \langle j, v\rangle, \quad \forall\, v \in D \tag{2}$$

where $f \in D^*$, $g \in D^*$ and $j \in D^*$ are the prescribed values of the operator Pu, the boundary operator Bu and the jump operator Ju. If Ω (Fig. 1) is the region of definition of the problem, one usually defines—but this is not essential—the operators P and Q^* by

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\, dx \quad \text{and} \quad \langle Q^*u, v\rangle = \int_\Omega u\mathcal{L}^*v\, dx \tag{3}$$

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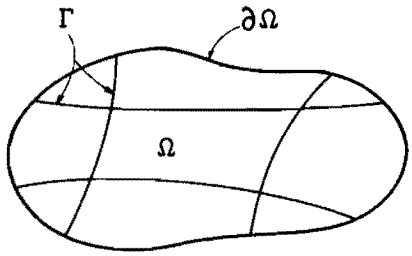

FIG. 1. Region of definition of the problem.

where $\mathcal{L}$ is a differential operator defined in Ω and $\mathcal{L}^*$ is its formal adjoint. Then, knowing Q^*u is tantamount to knowing the function u in the interior of Ω. The "complementary boundary values" C^*u were illustrated by means of examples; thus, for the Dirichlet problem of the Laplace equation in which u is prescribed on the boundary $\partial\Omega$, the complementary boundary values are the normal derivatives $\partial u/\partial n$, there. For problems of elasticity, the prescribed and complementary boundary values may be the displacements and the tractions, respectively. The average values of the exact solution across the surface Γ (Fig. 1), where Γ is the surface on which discontinuities of the functions may occur, constitute the third component of the sought information and are characterized by K^*u.

The systematic development of Green's formulas, and, in particular, the "general Green's formula for operators defined in discontinuous fields:"

$$P - B - J = Q^* - C^* - K^* \tag{4}$$

which supply the basis for the variational formulations (1) and (2), was presented in Part I. The explicit formulas for the operators J and K^*, given in Section V, were developed for the case in which the region Ω is divided into two subregions. In order to apply the theory to general numerical methods for partial differential equations, it is necessary to extend those results to the case when Ω is divided into an arbitrary number of subregions. This is done in Sections II and III. General boundary value problems are formulated in Section IV, while finite and boundary element methods are discussed in Sections V through VII. The coupling of finite elements and boundary procedures is presented in Section VIII.

An interesting application of the theory, because the treatment is quite complete, is the solution of ordinary differential equations. The third article of this series is devoted to finite differences and ordinary differential equations. Examples of such applications have already been published [2, 3].

The methodology presented here constitutes an extension and generalization of a theory I recently published in book form [4]. The work by Babuska, Oden, and Lee [5] on mixed-hybrid finite elements was inspiring for the unified formulation of numerical methods. Discussions with Professor Zienkiewicz, who repeatedly [6–10] has pointed out the possibility of having a unifying theory, prompted my interest on the matter and some of his results [7–10] are special cases of the general scheme contained here. Finally, I want to express my grati-

tude to George F. Pinder whose publications [11, 12] and specially, our joint research, motivated my work.

II. PRELIMINARY NOTIONS AND NOTATIONS

In this section some concepts and results, complementary to those given in Part I, are presented. First, a relation similar—but in a sense to be clarified soon, weaker—to that of operators that can be varied independently [1] is introduced.

Definition 2.1. The operators $P: D \to D^*$ and $Q: D \to D^*$ are disjoint, if P is a boundary operator for Q while Q is a boundary operator for P.

Proposition 2.1. Assume P and Q can be varied independently, then P^* and Q^* are disjoint.

Proof. We need to prove

$$\langle P^*u, v\rangle = 0 \quad \forall\, v \in N_Q \Rightarrow P^*u = 0 \tag{5}$$

and the implication which is obtained interchanging P and Q in (5). Assuming that P and Q can be varied independently, let the premise in (5) be satisfied. Then, given any $V \in D$ take $v \in D$ such that $Qv = 0$ while $Pv = PV$. With this choice, one has

$$\langle P^*u, V\rangle = \langle PV, u\rangle = \langle Pv, u\rangle = \langle P^*u, v\rangle = 0\,. \tag{6}$$

Hence $P^*u = 0$, since $V \in D$ was arbitrary. The other part follows by duality.

Theorem 2.1. If $B: D \to D^*$ is a boundary operator for $P: D \to D^*$, then

$$Pu + Bw = 0 \Rightarrow Pu = 0 \quad \text{and} \quad Bw = 0\,. \tag{7}$$

Proof. Assume $Pu + Bw = 0$. Then

$$0 = \langle Pu + Bw, v\rangle = \langle Pu, v\rangle \quad \forall\, v \in N_{B^*}\,. \tag{8}$$

This implies $Pu = 0$, which renders the Theorem clear.

Proposition 2.2 Let $R = P + B$, where B is a boundary operator for P. Then

$$N_R = N_P \cap N_B\,. \tag{9}$$

Proof. Clearly, $N_R \supset N_P \cap N_B$. Thus, it remains to prove

$$(P + B)u \Rightarrow Pu = 0 \quad \text{and} \quad Bu = 0\,. \tag{10}$$

This implication is the special case of (7) for which $w = u$.

Corollary 2.1. Let P and Q be disjoint. Then

(a) $Pu + Qw = 0 \Rightarrow Pu = 0 \quad \text{and} \quad Qw = 0.$ (11)

(b) When $R = P + Q$, one has

$$N_R = N_P \cap N_Q. \tag{12}$$

Proof. It is clear by virtue of Theorem 2.1 and Proposition 2.2. Observe that part (b) is a generalization of Proposition 2.1. of Part I.

Definition 2.2. A pair $\{R_1, R_2\}$ of operators is said to be fully disjoint when R_1 and R_2 are disjoint and simultaneously R_1^* and R_2^* are disjoint. Let $R = R_1 + R_2$ and the pair $\{R_1, R_2\}$ be fully disjoint, then one says that the operators R_1 and R_2 decompose weakly R.

Remark 2.1. Comparing Definition 2.2 just given, with Definition 4.4. of Part I, one gets the alternative—but equivalent—definition of Green's formula: When P and Q are formal adjoints, an equation $P - B = Q^* - C^*$ is said to be a Green's formula if the pair $\{B, C^*\}$ is fully disjoint.

Remark 2.2. Also in view of Definition 3.1 of Part I and Proposition 2.1. of this Section, it is clear that every decomposition of an operator is a weak decomposition. However, not every weak decomposition is a (strong) decomposition. In particular, not every Green's formula is a strong one, as illustrated in the following examples:

Example 2.1. Let $D = H^s(\Omega)$, $s > 2$, where Ω is the unit circle. Assume $\partial_1\Omega$ is the upper half of the circumference and $\partial_2\Omega$ its lower half. Define

$$\langle Pu, v\rangle = \int_\Omega v\Delta u\, dx; \qquad \langle Q^*u, v\rangle = \int_\Omega u\Delta v\, dx. \tag{13}$$

Take

$$\langle Bu, v\rangle = \int_{\partial_2\Omega} v\frac{\partial u}{\partial n}\, dx - \int_{\partial_1\Omega} u\frac{\partial v}{\partial n}\, dx;$$

$$\langle C^*u, v\rangle = \int_{\partial_2\Omega} u\frac{\partial v}{\partial n}\, dx - \int_{\partial_1\Omega} v\frac{\partial u}{\partial n}\, dx. \tag{14}$$

For every $u \in D$ one has $\gamma_o u \in H^{s-1/2}(\partial\Omega)$ while $\gamma_1 u \in H^{s-3/2}(\partial\Omega)$.[†]

In this example $B = C$, so that (see Eq. 47 of Part I):

$$I_{11} = I_{22} = N_{C^*} = N_{B^*}; \qquad I_{12} = I_{21} = N_B = N_C. \tag{15}$$

[†]Here γ_o and γ_1 are the trace operators [13]; i.e., $\gamma_o u = u$ and $\gamma_1 u = \partial u/\partial n$ on $\partial\Omega$.

Clearly,

$$N_{C^*} = \{u \in H^s(\Omega) \,|\, \gamma_1 u = 0 \quad \text{on} \quad \partial_1\Omega \quad \text{and} \quad \gamma_0 u = 0 \quad \text{on} \quad \partial_2\Omega\}, \tag{15a}$$

while

$$N_B = \{u \in H^s(\Omega) \,|\, \gamma_0 u = 0 \quad \text{on} \quad \partial_1\Omega \quad \text{and} \quad \gamma_1 u = 0 \quad \text{on} \quad \partial_2\Omega\}. \tag{15b}$$

It is easy to see that neither $I_{11} + I_{12}$ nor $I_{21} + I_{22}$ yield the whole space D. Thus, $P - B = Q^* - C^*$ is not a Green's formula in the strong sense. However, this equation is a Green's formula, because B and C^* are disjoint and also B^* and C are disjoint. Indeed, for example, it can be seen that

$$\langle Bu, v\rangle = 0 \quad \forall\, v \in N_C \Rightarrow Bu = 0. \tag{16}$$

Example 2.2. In Example 4.2 of Part I, the operators

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\, dx, \qquad \langle Q^*u, v\rangle = \int_\Omega u\mathcal{L}^*v\, dx,$$

where

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} \quad \text{and} \quad \mathcal{L}^*v \equiv -\frac{\partial v}{\partial t} - \frac{\partial V_v}{\partial x},$$

were considered. When the region Ω is as illustrated in Figure 2, then

$$\langle Bu, v\rangle = -\int_{\Omega_x(0)} uv\, d\mathbf{x} - \int_{\partial_-\Omega} uvV\, dt;$$

$$\langle C^*u, v\rangle = -\int_{\Omega_x(T)} uv\, d\mathbf{x} - -\int_{\partial_-\Omega} uvV\, dt. \tag{17}$$

Taking $D = C^1(\overline{\Omega})$, again the equation $P - B = Q^* - C^*$ is a Green's formula but not in the strong sense. Indeed, any function belonging to $N_B + N_{C^*}$

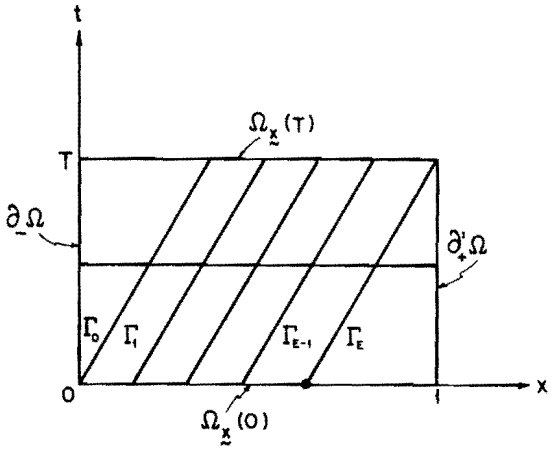

FIG. 2. The region Ω for Example 2.2.

vanishes at the upper left corner and at the lower right corner of the rectangle Ω; thus, not every function of $D = C^1(\overline{\Omega})$ belongs to $N_B + N_{C^*}$.

It is of interest to extend some of the relations among operators that have been introduced thus far, to systems of more than two operators.

Definition 2.3. Let $\{R_1, \ldots, R_E\}$ be a system of operators and let $R = \sum_{\alpha=1}^{E} R_\alpha$. Define for every $\beta = 1, \ldots, E$:

$$N_\beta = \bigcap_{\alpha \neq \beta}^{E} N_{R\alpha}; \qquad N'_\beta = \bigcap_{\alpha \neq \beta}^{E} N_{R^*\alpha} \tag{18}$$

Then one says that the system $\{R_1, \ldots, R_E\}$:

(a) Is disjoint, if for every $\beta = 1, \ldots, E$, one has

$$\langle R_\beta u, v \rangle = 0 \quad \forall\, v \in N'_\beta \Rightarrow R_\beta u = 0 \tag{19}$$

(b) Can be varied independently, if for every system $\{U_1, \ldots, U_E\} \subset D$, there exists $u \in D$ such that

$$R_\alpha u = R_\alpha U_\alpha\,, \qquad \alpha = 1, \ldots, E\,. \tag{20}$$

(c) Is fully disjoint, if $\{R_1, \ldots, R_E\}$ and also $\{R_1^*, \ldots, R_E^*\}$ are disjoint. In this case, $\{R_1, \ldots, R_E\}$ decomposes weakly R.

(d) Decomposes (strongly) R, if $\{R_1, \ldots, R_E\}$ and also $\{R_1^*, \ldots, R_E^*\}$, can be varied independently.

Proposition 2.3. Assume the system $\{R_1, \ldots, R_E\}$ can be varied independently, then $\{R_1^*, \ldots, R_E^*\}$ are disjoint.

Proof. We need to prove that for any given $\beta = 1, \ldots, E$, one has

$$\langle R_\beta^* u, v \rangle = 0 \quad \forall\, v \in N_\beta \Rightarrow R_\beta^* u = 0 \tag{21}$$

where N_β is given by (18). Assuming the premise in (21) is fulfilled, given $V \in D$ take $v \in D$ such that $R_\beta v = R_\beta V$ and $R_\alpha v = 0$ for $\alpha \neq \beta$; i.e., $R_\beta v = R_\beta V$ while $v \in N_\beta$. Then

$$\langle R_\beta^* u, V \rangle = \langle R_\beta V, u \rangle = \langle R_\beta v, u \rangle = \langle R_\beta^* u, v \rangle = 0\,. \tag{22}$$

This shows $R_\beta^* u = 0$, since $V \in D$ was arbitrary.

Corollary 2.2. If $\{R_1, \ldots, R_E\}$ is a (strong) decomposition of R then it is also a weak decomposition.

Proof. It is clear by Proposition 2.3.

Proposition 2.4. Assume the system $\{R_1, \ldots, R_E\}$ is disjoint and let $R = \sum_{\alpha=1}^{E} R_\alpha$. Then

$$\text{(a)} \quad \sum_{\alpha=1}^{E} R_\alpha u_\alpha = 0 \Rightarrow R_\alpha u_\alpha = 0 \quad \forall\, \alpha = 1, \ldots, E\,, \tag{23}$$

(b) $Ru = 0 \Rightarrow R_\alpha u = 0 \quad \forall\, \alpha = 1, \ldots, E,$ (24)

(c) $N_R = \bigcap_{\alpha=1}^{E} N_{R\alpha}.$ (25)

If the system $\{R_1, \ldots, R_E\}$ is fully disjoint, then in addition, properties (a) to (c) hold, when each operator is replaced by its transpose.

Proof. These are straight-forward generalizations of previous results.

Corollary 2.3. When the system $\{R_1^*, \ldots, R_E^*\}$ can be varied independently, properties (a) to (c) of Theorem 2.4, hold. If the system $\{R_1, \ldots, R_E\}$ decomposes (strongly) R, then properties (a) to (c) hold also when each operator is replaced by its adjoint.

Proof. By virtue of Proposition 2.3.

Definition 2.4. Assume the system $\{R_1, \ldots, R_E\}$ is fully disjoint. For any fixed $\beta = 1, \ldots, E$, let the pair $\{B_\beta, -C_\beta^*\}$ decompose R_β. Define

$$N_\beta = \bigcap_{\alpha\neq\beta}^{E} N_{R\alpha} \quad \text{and} \quad N'_\beta = \bigcap_{\alpha\neq\beta}^{E} N_{R^*\alpha}. \tag{26}$$

One says that the decomposition $\{B_\beta, -C_\beta^*\}$ is distributive in the system $\{R_1, \ldots, R_E\}$, if

$$N_\beta \cap (N_{B\beta} + N_{C^*\beta}) = N_\beta \cap N_{B\beta} + N_\beta \cap N_{C^*\beta} \tag{27}$$

and

$$N'_\beta \cap (N_{B^*\beta} + N_{C\beta}) = N'_\beta \cap N_{B^*\beta} + N'_\beta \cap N_{C\beta} \tag{28}$$

III. GREEN'S FORMULAS FOR FINITE ELEMENTS

Let Ω be a domain, not necessarily bounded, of an Euclidean space and let $\partial\Omega$ be its boundary. In general, the symbol π will stand for a partition of Ω into a collection of $E(\pi)$ subdomains Ω_e; $e = 1, \ldots, E(\pi)$. To get some notational advantages we will identify Ω_o with the original region Ω. In what follows a fixed partition π, with the property that $E = E(\pi) \geq 1$, will be considered; the argument π will be frequently deleted since it is unnecessary. First, the following notations and assumptions are adopted (the notation is similar to that used by Babuska, Oden and Lee in [5]).

A. The Partition. (Fig. 1)

(i) $$\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e; \qquad \Omega_e \cap \Omega_f = \phi, \qquad e \neq f \tag{29}$$

(ii) The boundaries. Define for $e > f$:

$$\Gamma_{ef} = \partial\Omega_e \cap \partial\Omega_f, \qquad f \geq 0. \tag{30}$$

Observe $\Gamma_{ef} \neq \phi$ only when Ω_e and Ω_f are contiguous regions.

(iii) The outer boundaries.

$$\partial_e\Omega = \Gamma_{eo}; \qquad \partial\Omega = \bigcup_{e=1}^{E} \partial_e\Omega \tag{31}$$

(iv) The interelement boundaries. When $f > 0$, Γ_{ef} is a segment of interelement boundary.

(v) The total interelement boundary is

$$\Gamma = \bigcup_{e,f\neq 0}^{E} \Gamma_{ef}. \tag{32}$$

B. The Function Spaces.

(i) For every $e = 1, \ldots, E$, there is a linear space, D_e. In most applications elements $u_e \in D_e$ will be functions—possibly vector valued, as in Elasticity—defined in Ω_e.

(ii) $D(\pi) = D_1^{\oplus} \cdots {}^{\oplus} D_E$. Thus, elements of D will be finite sequences; $u = \{u_1, \ldots, u_E\}$, with $u_e \in D_e$ for each $e = 1, \ldots, E$.

Remark 3.1. In many applications we start with a linear space D_o of functions defined in the whole region $\Omega_o = \Omega$. Then one can take

$$D_e = \{u_o|_{\Omega e} | u_o \in D_o\}; \qquad e = 1, \ldots, E \tag{33}$$

where $u_o|\Omega_e$ stands for the restriction of u_o to Ω_e. The natural immersion of D_o into $D = D_1^{\oplus} \cdots {}^{\oplus} D_E$ is supplied by the mapping which associates with every $u_o \in D_o$ the sequence of restrictions $u_o|_{\Omega e}$, $e = 1, \ldots, E$.

C. The Operators.

(i) With every $e = 1, \ldots, E$, there are associated operators P_e: $D \to D^*$, Q_e: $D \to D^*$ and R_e: $D \to D^*$, such that P_e and Q_e are formal adjoints satisfying

$$R_e = P_e - Q_e^*. \tag{34}$$

(ii) With every $e \neq f$, $e \geq 1$, and $f \geq 0$ there are defined operators R_{ef}: $D \to D^*$ such that

$$R_e = \sum_{f\neq e}^{E} R_{ef}. \tag{35}$$

Generally, $R_{ef} \neq 0$ only when Ω_e and Ω_f are contiguous regions.

(iii) For every $e > f$ and $f \geq 1$, define

$$\hat{R}_{fo} = R_{fo}; \qquad \hat{R}_{ef} = R_{ef} + R_{fe}. \tag{36}$$

(iv) Let

$$P = \sum_{e=1}^{E} P_e; \qquad Q = \sum_{e=1}^{E} Q_e \tag{37}$$

$$R = \sum_{e=1}^{E} R_e; \qquad R_\partial = \sum_{e=1}^{E} R_{eo}; \qquad R_\Gamma = \sum_{e>f\geq 1} \hat{R}_{ef}. \tag{38}$$

Then

$$P - Q^* = R = R_\partial + R_\Gamma. \tag{39}$$

Remark 3.2. In most applications

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\,dx; \qquad \langle Q^*u, v\rangle = \int_\Omega u\mathcal{L}^*v\,dx \tag{40}$$

while

$$\langle P_e u_e, v_e\rangle = \int_{\Omega e} v_e\mathcal{L}u_e\,dx; \qquad \langle Q_e^* u_e, v_e\rangle = \int_{\Omega e} u_e\mathcal{L}^*v_e\,dx. \tag{41}$$

(v) The system $\{\hat{R}_{10}, \ldots, \hat{R}_{EO}, \hat{R}_{21}, \ldots, \hat{R}_{E1}, \ldots, \hat{R}_{E,E-1}\}$ is fully disjoint.

(vi) For every $e > f \geq 1$, there is a pair $\{S_{ef}^l, S_{ef}^r\}$ of conjugate smoothness relations which are regular (see Part I) for $\hat{R}_{ef}$.

(vii) For every $e = 1, \ldots, E$, there is a pair $\{B_e, -C_e^*\}$ of operators which decompose strongly R_{eo}.

(viii) Define

$$B = \sum_{e=1}^{E} B_e; \qquad C = \sum_{e=1}^{E} C_e. \tag{42}$$

(ix) For every $e > f \geq 1$, let $\{J_{ef}, -K_{ef}^*\}$ be the pair of operators associated with the conjugate smoothness relations $\{S_{ef}^l, S_{ef}^r\}$ by means of Theorem 5.1, Eq. (59) of Part I. The pair $\{J_{ef}, -K_{ef}^*\}$, decomposes $\hat{R}_{ef}$. Define

$$J = \sum_{e>f\geq 1} J_{ef}; \qquad K = \sum_{e>f\geq 1} K_{ef}. \tag{43}$$

(x) Assume every one of the decompositions $\{B_e, -C_e^*\}$ and $\{J_{ef}, -K_{ef}^*\}$, is distributive (Definition 2.4) in the system of operators $\{\hat{R}_{10}, \ldots, \hat{R}_{EO}, \hat{R}_{21}, \ldots, \hat{R}_{E1}, \ldots, \hat{R}_{E,E-1}\}$.

Observe, thus far we have introduced the following systems of operators $\{\hat{R}_{10}, \ldots, \hat{R}_{EO}, \hat{R}_{21}, \ldots, \hat{R}_{E1}, \ldots, \hat{R}_{E,E-1}\}$, $\{B_1, \ldots, B_E, J_{21}, \ldots, J_{E1}, \ldots, J_{E,E-1}\}$, and $\{C_1, \ldots, C_E, K_{21}, \ldots, K_{E1}, \ldots, K_{E,E-1}\}$. Each one of them consists of $E(E+1)/2$ operators (some of them may be the zero operator) and in order to simplify a little, the alternative notation $\{\hat{R}_1, \ldots, \hat{R}_F\}$, $\{B_1, \ldots, B_F\}$, and

$\{C_1, \ldots, C_F\}$, respectively, will be adopted. Here, $F = E(E + 1)/2$. Also for every $\beta = 1, \ldots, F$, let be

$$\hat{N}_\beta = \bigcap_{\alpha \neq \beta}^{F} \hat{N}_{R\alpha}; \qquad \hat{N}'_\beta = \bigcap_{\alpha \neq \beta}^{F} \hat{N}_{R^*\alpha} \tag{44}$$

where $\hat{N}_{R\alpha}$ and $\hat{N}_{R^*\alpha}$ are the null subspaces of $\hat{R}_\alpha$ and $\hat{R}^*_\alpha$, respectively.

Remark 3.3. If the space of admissible functions D is taken as in Remark 3.1, the operators P and Q as in Remark 3.2 while

$$\langle Bu, v\rangle = \int_{\partial\Omega} \mathcal{B}u\mathcal{D}v\, dx \quad \text{and} \quad \langle C^*u, v\rangle = \int_{\partial\Omega} \mathcal{E}u\mathcal{C}v\, dx \tag{45}$$

are such that $P - B = Q^* - C^*$ is a Green's formula in the original space D_o, then in finite element applications it is usually convenient to define

$$\langle B_e u, v\rangle = \int_{\partial_e\Omega} \mathcal{B}u\mathcal{D}v\, dx; \qquad \langle C^*_e u, v\rangle = \int_{\partial_e\Omega} \mathcal{E}u\mathcal{C}v\, dx\,. \tag{46}$$

The smoothness relations $\{S^l_{ef}, S^r_{ef}\}$ can be taken as

$$S^l_{ef} = \tau(D_o) + \hat{N}_{Ref}; \qquad S^r_{ef} = \tau(D_o) + \hat{N}_{R^*ef} \tag{47}$$

where τ: $D_o \to D$ is the natural immersion (Remark 3.1) of D_o into D. When dealing with partial differential equations, one frequently has $\hat{N}_{Ref} = \hat{N}_{R^*ef}$ so that $S^l_{ef} = S^r_{ef}$ when (47) applies.

Example 3.1. As an illustration, consider the case for which Ω is a circle (Fig. 3) divided into five subregions Ω_e ($e = 1, \ldots, 5$). Proceed as in Remark 3.3 with $D_o = H^s(\Omega)$, $s \geq 2$, $\mathcal{L} = \mathcal{L}^* = \Delta$ and

$$\langle B_e u, v\rangle = -\int_{\partial_e\Omega} u \frac{\partial v}{\partial n}\, dx \tag{48}$$

while C_e: $D_e \to D^*_e$ satisfies $C_e = B_e$. The operators $\hat{R}_{ef}$ are

$$\langle \hat{R}_{ef} u, v\rangle = \int_{\Gamma_{ef}} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} dx + \int_{\Gamma_{fe}} \left\{ v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right\} dx \tag{49}$$

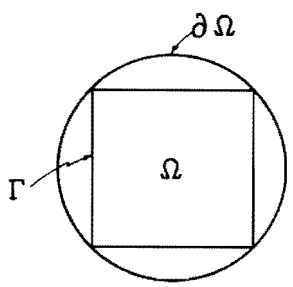

FIG. 3. The region Ω for Example 3.1.

where the sense of the normal unit vector $\mathbf{n}$ is indicated by the order of the subindexes in Γ. The smoothness relations given by (47) satisfy $S^l_{ef} = S^r_{ef}$ and are made up of the functions $D = D_1^{\oplus} \cdots {}^{\oplus} D_5$ which are continuous together with their normal derivative across Γ_{ef}. It is possible to show that the distributive assumption x) is fulfilled (see Eqs. 27a,b); however, that proof will not be given here.

Theorem 3.1. Under the assumptions of this Section the relation

$$P - (B + J) = Q^* - (C + K)^* \tag{50}$$

holds and it is a Green's formula. Even more, each one of the systems $\{B_1, \ldots, B_E, J_{21}, \ldots, J_{E1}, \ldots, J_{E,E-1}\}$ and $\{C_1, \ldots, C_E, K_{21}, \ldots, K_{E1}, \ldots, K_{E,E-1}\}$ is fully disjoint (Definition 2.3).

Proof. Using the notation introduced previously, let us first prove that $\{B_1, \ldots, B_F\}$ is fully disjoint; i.e., we prove that each one of the systems $\{B_1, \ldots, B_F\}$ and $\{B_1^*, \ldots, B_F^*\}$ is disjoint. The first one of these properties is equivalent to the family of implications

$$\langle B_\beta u, v\rangle = 0 \quad \forall\, v \in \bigcap_{\alpha \neq \beta}^{E} N_{B^*\alpha} \Rightarrow B_\beta u = 0, \qquad \beta = 1, \ldots, F. \tag{51}$$

Since $N_{B^*\alpha} \supset \hat{N}_{R^*\alpha}$, it is clear that

$$\langle B_\beta u, v\rangle = 0 \quad \forall\, v \in \bigcap_{\alpha \neq \beta}^{E} N_{B^*\alpha} \Rightarrow \langle B_\beta u, v\rangle = 0 \quad \forall\, v \in \hat{N}'_\beta. \tag{52}$$

However, there exists $u_{11} \in N_{C^*\beta}$ such that $B_\beta u = \hat{R}_\beta u_{11}$, because $\{B_\beta, -C_\beta^*\}$ decompose (strongly) $\hat{R}_\beta$. The desired result is now clear since

$$\langle \hat{R}_\beta u_{11}, v\rangle = 0 \quad \forall\, v \in \hat{N}'_\beta \Rightarrow \hat{R}_\beta u_{11} = 0 \tag{53}$$

because the system of operators $\{\hat{R}_1, \ldots, \hat{R}_F\}$ is disjoint. We leave the proof of the fact that each one of the systems $\{B_1, \ldots, B_F\}$ and $\{C_1^*, \ldots, C_F^*\}$ are fully disjoint at this point, since the remaining part of the proof is similar.

Once this has been shown, it is seen that

$$N_{(B+J)} = N_B \cap N_J = \bigcap_{\alpha=1}^{F} N_{B\alpha} \tag{54a}$$

$$N_{(B^*+J^*)} = N_{B^*} \cap N_{J^*} = \bigcap_{\alpha=1}^{F} N_{B^*\alpha} \tag{54b}$$

$$N_{(C+K)} = N_C \cap N_K = \bigcap_{\alpha=1}^{F} N_{C\alpha} \tag{54c}$$

$$N_{(C^*+K^*)} = N_{C^*} \cap N_{K^*} = \bigcap_{\alpha=1}^{F} N_{C^*\alpha} \tag{54d}$$

by virtue of Proposition 2.4, Eq. (25).

In order to prove that (50) is a Green's formula, it is necessary to show that the system $\{B + J, C^* + K^*\}$ is fully disjoint. As a first step we need to show that $C^* + K^*$ is a boundary operator for $B + J$. This is tantamount to showing that

$$\sum_{\alpha=1}^{F} \langle B_\alpha u, v\rangle = 0 \quad \forall\, v \in \bigcap_{\alpha=1}^{F} N_{C\alpha} \Rightarrow B_\beta u = 0 \quad \forall\, \beta = 1, \ldots, F. \tag{55}$$

To prove (55), we establish first an auxiliary result.

Lemma 3.1. Under the assumptions of Theorem 3.1 for every fixed $\beta = 1, \ldots, F$, one has

(a) There exists $u_{11} \in N_{C^*\beta}$ such that

$$B_\beta u = \hat{R}_\beta u_{11} \tag{56}$$

(b)
$$\sum_{\alpha=1}^{F} \langle B_\alpha u, v\rangle = \langle \hat{R}_\beta u_{11}, v\rangle \quad \forall\, v \in \hat{N}'_\beta \tag{57}$$

where u_{11} is taken as in (a).

(c)
$$\langle B_\beta u, v\rangle = 0 \quad \forall\, v \in N'_\beta \Rightarrow B_\beta u = 0 \tag{58}$$

Proof. Part (a) follows from the fact that the system $\{B_\beta, -C^*_\beta\}$ decomposes (strongly) $\hat{R}_\beta$. Part (b) can be seen using the fact that $N_{B^*\alpha} \supset N'_\beta$ whenever $\alpha \neq \beta$. In view of part (a), part (c) follows from

$$u_{11} \in N_{C^*\beta} \quad \text{and} \quad \langle \hat{R}_\beta u_{11}, v\rangle = 0 \quad \forall\, v \in \hat{N}'_\beta \Rightarrow \hat{R}_\beta u_{11} = 0$$

which is clear because the system $\{\hat{R}_1, \ldots, \hat{R}_E\}$ is disjoint. This completes the proof of the Lemma.

Observe, for every $\alpha = 1, \ldots, F$, one has $N_{C\alpha} \supset \hat{N}_{R^*\alpha}$. Thus $\cap_{\alpha\neq\beta}^{F} N_{C\alpha} \supset \cap_{\alpha\neq\beta}^{F} \hat{N}_{R^*\alpha} = \hat{N}'_\beta$. Hence $\cap_{\alpha=1}^{F} N_{C\alpha} \supset N_{C\beta} \cap \hat{N}'_\beta$. This implies that

$$\sum_{\alpha=1}^{F} \langle B_\alpha u, v\rangle = 0 \quad \forall\, v \in \bigcap_{\alpha=1}^{F} N_{C\alpha} \Rightarrow \sum_{\alpha=1}^{F} \langle B_\alpha u, v\rangle = 0 \quad \forall\, N_{C\beta} \cap \hat{N}'_\beta. \tag{59}$$

However,

$$\sum_{\alpha=1}^{F} \langle B_\alpha u, v\rangle = \langle \hat{R}_\beta u_{11}, v\rangle \quad \forall\, v \in N_{C\beta} \cap \hat{N}'_\beta \subset \hat{N}'_\beta \tag{60}$$

for some $u_{11} \in N_{C^*\beta}$, by virtue of Lemma 3.1. Therefore, implication (55) will follow from part (c) of the Lemma, if we establish that

$$\langle B_\beta u, v\rangle = \langle \hat{R}_\beta u_{11}, v\rangle = 0 \quad \forall\, v \in N_{C\beta} \cap \hat{N}'_\beta \Rightarrow \langle B_\beta u, v\rangle 0 \quad \forall\, v \in \hat{N}'_\beta. \tag{61}$$

This latter proposition follows from the distributivity of the decomposition $\{B_\beta, -C_\beta^*\}$ in the system $\{\hat{R}, \ldots, \hat{R}_F\}$. Indeed

$$N_{B^*\beta} \cap \hat{N}'_\beta + N_{C\beta} \cap \hat{N}'_\beta = (N_{B^*\beta} + N_{C\beta}) \cap \hat{N}'_\beta = \hat{N}'_\beta \tag{62}$$

because $N_{B^*\beta} + N_{C\beta} = D$ since B_β^* and C_β can be varied independently. Equation (62) implies that every $v \in \hat{N}'_\beta$ can be written as $v = v_{21} + v_{22}$, where $v_{21} \in N_{B^*\beta}$ while $v_{22} \in N_{C\beta}$. Replacing in (61) one gets

$$\langle B_\beta u, v\rangle = \langle \hat{R}_\beta u_{11}, v\rangle = \langle \hat{B}_\beta u_{11}, v_{22}\rangle = \langle \hat{R}_\beta u_{11}, v_{22}\rangle$$

which vanishes when the premise of (61) is satisfied. This completes the proof of implication (55). The proof that $B + J$ is a boundary operator for $(C + K)^*$ is similar. To show also that $(B + J)^*$ and $C + K$ are disjoint, one can use dual arguments.

IV. BOUNDARY VALUE PROBLEMS

Definition 4.1. Let $B: D \to D^*$ be a boundary operator for $P: D \to D^*$. Given $U \in D$ and $V \in D$, define

$$f = PU \quad \text{and} \quad g = BV. \tag{63}$$

The abstract boundary value problem to be considered consists in finding $u \in D$ such that

$$Pu = f \tag{64a}$$

and simultaneously

$$Bu = g. \tag{64b}$$

Remark 4.1. In view of (63), attention is restricted to problems for which $f \in D^*$ and $g \in D^*$, are in the range of $P: D \to D^*$ and $B: D \to D^*$, respectively. Clearly, any problem which possesses at least one solution fulfills this condition.

Theorem 4.1. An element $u \in D$ is a solution of the abstract boundary problem, if and only if

$$(P - B)u = f - g. \tag{65}$$

Even more, if $\{B_1, \ldots, B_F\}$ is a weak decomposition of B, and $V_1, \ldots, V_F \in D$ are such that $B_\alpha V_\alpha = g_\alpha$, $\alpha = 1, \ldots, F$. Then

$$B_\alpha u = g_\alpha, \qquad \alpha = 1, \ldots, F. \tag{66}$$

whenever $u \in D$ satisfies (65).

Proof. Equations (64a, b) clearly imply (65). Conversely, using Eq. (63), it is seen that equation (65) implies

$$P(u - U) - B(u - V) = 0. \tag{67}$$

By Theorem 2.1, from equation (67) it follows that

$$P(u - U) = 0 \quad \text{and} \quad B(u - V) = 0. \tag{68}$$

Hence, Eqs. (64a, b). The second part of this Theorem follows from Proposition 2.4.

Theorem 4.2. Let the equation

$$P - B = Q^* - C^* \tag{69}$$

be a Green's formula. Then, $u \in D$ is a solution of the bondary problem, if and only if

$$(Q^* - C^*)u = f - g. \tag{70}$$

Proof. Because (65) is equivalent to (70).

Remark 4.2. Let $P - B = Q^* - C^*$ be a Green's formula and $R = P - Q^*$. Then, in view of Remark 4.3 of Part I, the boundary values Ru associated with P are characterized by the pair $\{Bu, C^*u\}$. When a boundary value problem is formulated, one prescribes Bu. C^*u can be evaluated only after the solution $u \in D$ of the problem has been obtained.

Definition 4.2. When Eq. (69) is a Green's formula, Bu and C^*u will be called the prescribed and complementary values of u, respectively.

A result which is stronger than that of Theorem 4.2, is given next.

Proposition 4.1. Let Eq. (69) be a Green's formula. Then $u \in D$ is solution of the boundary problem, if and only if, there exists $v \in D$ such that

$$Q^*u - C^*v = f - g. \tag{71}$$

Even more, when (71) holds, one necessarily has

$$C^*v = C^*u. \tag{72}$$

Proof. When $u \in D$ is a solution of the boundary problem, it is clear that (71) holds, with $v = u$, by virtue of Theorem 4.2. Conversely, let $u \in D$ and $v \in D$ be such that (71) is fulfilled, then

$$(P - B)u - C^*(v - u) = f - g. \tag{73}$$

Here, (69) has been used. In view of Definition 4.1, this equation can be written as

$$P(u - U) + B(V - u) - C^*(v - u) = 0. \tag{74}$$

Let $R = P - Q^*$ and recall that $N_{R^*} = N_{B^*} \cap N_C$ by virtue of the second of Eqs. (46) of Part I. Thus

$$\langle B(V - u) - C^*(v - u), w\rangle = 0 \quad \forall\, w \in N_{R^*} = N_{B^*} \cap N_C\,. \tag{75}$$

Therefore, Eq. (74) implies

$$\langle P(u - U), w\rangle = 0 \quad \forall\, w \in N_{R^*}\,. \tag{76}$$

Hence $P(u - U) = 0$, because R is a boundary operator for P. Once this has been shown, Eq. (94) reduces to

$$B(V - u) - C^*(v - u) = 0 \tag{77}$$

which implies that $Bu = BV = g$ and $C^*v = C^*u$, by virtue of Corollary 2.1, since B and C^* are disjoint.

Remark 4.3. Proposition 4.1 exhibits the essential difference between Q^* and the boundary operator C^*. Let $u \in D$ and $u' \in D$ be any two elements of D, then

$$Q^*u' = Q^*u \Rightarrow C^*u' = C^*u\,.$$

However, the converse is not true in general.

Remark 4.4. Equation (65) is equivalent to the variational principle

$$\langle (P - B)u, v\rangle - \langle f - g, v\rangle = 0 \quad \forall\, v \in D \tag{78}$$

while Eq. (70) supplies the alternative variational principle

$$\langle (Q^* - C^*)u, v\rangle - \langle f - g, v\rangle = 0 \quad \forall\, v \in D\,. \tag{79}$$

The first of these principles involves the functionals Pu and Bu which are prescribed; indeed, the prescribed data are f and g, respectively. On the other hand, the second one of these principles involves the functionals Q^*u and C^*u, whose values are not prescribed and which can be evaluated only after the problem has been solved. Indeed, when (40) hold, knowing Q^*u is tantamount to knowing u in the interior of the region Ω.

Definition 4.3. When Eq. (69) is a Green's formula, the complementary boundary values C^*u together with the functional Q^*u, will be called the sought information.

Definition 4.4. The variational principle (78) will be called the variational principle in terms of data while (79) will be called the variational principle in terms of sought information.

In conclusion, in this Section we have associated with the abstract boundary value problem two variational formulations: the direct one, involving the data of the problem; and, the derived or indirect one, involving the sought information.

V. FORMULATION AND PRELIMINARY ANALYSIS OF DISCRETE METHODS

Taking the assumptions about the operators, function spaces, and partition as in Section III., it is required to find $u \in D$ such that

$$(P - B - J)u = f - g - j \tag{80}$$

where $f \in D^*$, $g \in D^*$ and $j \in D^*$ are three given functionals belonging to the ranges of P, B and J, respectively. When hypotheses of Theorem 3.1 hold, Eq. (80) is equivalent to

$$Pu = f, \qquad Bu = g \quad \text{and} \quad Ju = j \tag{81}$$

by virtue of Theorem 4.1. If $\{B_1, \ldots, B_I\}$ and $\{J_1, \ldots, J_M\}$ are weak decompositions of B and J, respectively (I and M being any integers), then Eq. (81) is equivalent to

$$Pu = f, \qquad B_\alpha u = g_\alpha (\alpha = 1, \ldots, I), \qquad J_\beta u = j_\beta (\beta = 1, \ldots, M). \tag{82}$$

In view of Green's formula (50), Eq. (80) can also be written as

$$(Q^* - C^* - K^*)u = f - g - j. \tag{83}$$

Equation (80) yields the direct variational formulation of the problem:

$$\langle (P - B - J)u, v \rangle = \langle f - g - j, v \rangle \quad \forall\, v \in D. \tag{84}$$

On the other hand, Eq. (83) yields the indirect or derived variational formulation of the problem:

$$\langle (Q^* - C^* - K^*)u, v \rangle = \langle f - g - j, v \rangle \quad \forall\, v \in D. \tag{85}$$

Usually, the operator P: $D \to D^*$ is related with a linear differential operator $\mathcal{L}$ by means of an equation such as

$$\langle Pu, v \rangle = \int_\Omega v \mathcal{L} u \, dx = \sum_{e=1}^{E} \int_{\Omega_e} v \mathcal{L} u \, dx. \tag{86}$$

Recall that the differential operator $\mathcal{L}$ is understood in an elementary sense; indeed, $\mathcal{L}$ is not defined on surfaces of discontinuity (the interelement boundaries) and the integral over Ω is understood as the sum of integrals over the subregions Ω_e where the derivability of the functions $u \in D_e$ is assumed to be high enough to have $\mathcal{L}$ well-defined. A similar observation applies to the definition of Q: $D \to D^*$ which is usually such that

$$\langle Q^* u, v \rangle = \langle Qv, u \rangle = \int_\Omega u \mathcal{L}^* v \, dx = \sum_{e=1}^{E} \int_{\Omega_e} u \mathcal{L}^* v \, dx \tag{87}$$

where $\mathcal{L}^*$ is the formal adjoint, in the usual sense applicable to differential operators, of $\mathcal{L}$. Generally, knowing Q^*u is tantamount to knowing the function u in the interior of each one of the subregions Ω_e.

Recall that the direct variational formulation (84) involves the functionals Pu, associated the prescribed value $\mathcal{L}u$ of the differential operator, Bu associ-

ated with the prescribed "boundary conditions" and Ju associated with the "prescribed jumps." On the other hand the indirect variation principle (85) involves the functionals Q^*u, associated with the sought values of the function u in the interior of the "finite elements" Ω_e, C^*u associated with the "complementary boundary values," and K^*u associated with the "generalized averages." In this manner, the sought information has been separated into three distinct parts: the "values in the interior" of the subregions Ω_e, the "complementary boundary values" on $\partial\Omega$ and the "generalized averages" on the interelement boundaries Γ. Usually, one looks for smooth solutions, so that $j = 0$; i.e., the jump condition is $Ju = 0$ which implies

$$u \in \bigcap_{e>f\geq 1}^{E} S^l_{ef} = S^l. \tag{88}$$

Applying the method of weighted residuals [14], one chooses a system $\{\varphi^1, \ldots, \varphi^N\}$ of "weighting or test functions" in order to define approximate solutions.

Definition 5.1. Let the system $\{\varphi^1, \ldots, \varphi^N\}$ of "weighting functions" be given. Then, any function $u' \in D$ which satisfies

$$\langle (P - B - J)u', \varphi^\alpha \rangle = \langle f - g - j, \varphi^\alpha \rangle, \qquad \alpha = 1, \ldots, N \tag{89}$$

will be said to be an approximate solution.

Remark 5.1. Clearly, any (exact) solution $u \in D$, is an approximate solution.

In addition, it is customary to impose the "representation constraint"

$$u' = \sum_{\alpha=1}^{N} a_\alpha \Phi^\alpha \tag{90}$$

where $\{\Phi^1, \ldots, \Phi^N\}$ is a system of "base functions." However, this latter condition is alien to the problem, while the system of Eqs. (89) is necessarily satisfied by the exact solution, as has already been pointed out in Remark 5.1. Indeed, condition (90) is a "mathematical artifice" introduced mainly to specify uniquely the sought approximate solution. Therefore, it is of interest to analyze the restrictions implied by equations (89) when no other assumption is made.

The system of Eqs. (89) satisfied by any approximate solution is not informative because it was obtained by application of the direct variational formulation (84) which only involves the prescribed functionals. A more informative form is obtained by applying the indirect variational formulation (85), which involves only the sought information. This yields

$$\langle Q^*u', \varphi^\alpha \rangle - \langle C^*u', \varphi^\alpha \rangle - \langle K^*u', \varphi^\alpha \rangle = \langle f, \varphi^\alpha \rangle - \langle g, \varphi^\alpha \rangle, \qquad \alpha = 1, \ldots, N. \tag{91}$$

In view of previous discussion, it is clear that Eqs. (91) imply restrictions on the possible values of u' on the interior of the subregions Ω_e, of the comple-

mentary boundary values on $\partial\Omega$ and of the averages across the interelement boundaries Γ. Of course, the particular choice of the family $\{\varphi^1, \ldots, \varphi^N\}$ of test functions determines the specific conditions imposed by Eqs. (91).

Since Eqs. (91) are also satisfied by the exact solution u, by subtraction one gets

$$\langle Q^*u', \varphi^\alpha\rangle - \langle C^*u', \varphi^\alpha\rangle - \langle K^*u', \varphi^\alpha\rangle = \langle Q^*u, \varphi^\alpha\rangle - \langle C^*u, \varphi^\alpha\rangle - \langle K^*u, \varphi^\alpha\rangle, \quad \alpha = 1, \ldots, N. \tag{92}$$

Equations (92) show that given any approximate solution $u' \in D$, one can compute correctly (i.e., exactly) the functionals

$$\langle Q^*u, \varphi^\alpha\rangle - \langle C^*u, \varphi^\alpha\rangle - \langle K^*u, \varphi^\alpha\rangle, \qquad \alpha = 1, \ldots, N \tag{93}$$

independently of the representation (90) chosen. Thus, the N functionals (93) may be interpreted as all the "information" contained on an approximate solution, while representation (90) may be interpreted as a manner of interpolating this information. Of course, the specific manner in which this interpolation is carried out depends on the specific choice of the system $\{\Phi^1, \ldots, \Phi^N\}$ of base functions while the information contained in an approximate solution depends on the family of weighting functions chosen.

VI. FINITE ELEMENTS

In general, one can use the previous results to develop more efficient numerical schemes. In Section V, the information about the sought solution was separated into three distinct parts: the values in the interior of the finite elements Ω_e, the complementary boundary values and the generalized averages at the interelement boundaries Γ (usually, these become the values of the solution u and its derivatives when the sought solution is smooth). By a suitable choice of the weighting functions $\{\varphi^1, \ldots, \varphi^N\}$, it is possible to eliminate one or more of these parts of the sought information, thus concentrating the available information in the remaining ones.

As a first example, let us apply our theory to a problem recently treated by Zielinski and Zienkiewicz [10]. Consider the equation of torsion of a prismatic bar;

$$\mathcal{L}u \equiv \nabla \cdot \left(\frac{1}{G}\nabla u\right) = -2\theta \quad \text{in } \Omega \tag{94}$$

with the boundary conditions

$$u = g_{\partial 1} \quad \text{on } \partial_1\Omega \tag{95a}$$

and

$$\frac{1}{G}\frac{\partial u}{\partial n} = g_{\partial 2} \quad \text{on } \partial_2\Omega. \tag{95b}$$

Here, $\partial\Omega = \partial_1\Omega + \partial_2\Omega$, u is stress function, G shear modulus, and θ rate of twist.

To apply our method to this problem define P: $D \to D^*$ and Q^*: $D \to D^*$ by

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\,dx; \qquad \langle Q^*u, v\rangle = \int_\Omega u\mathcal{L}^*v\,dx \tag{96}$$

where D is constructed as explained in Section III, using the procedure of Remark 3.1 with $D_o = H^s(\Omega)$. Consider first the case when there is no partition (i.e., $E = 1$, $D = D_o = H^s(\Omega)$), then

$$\langle (P - Q^*)u, v\rangle = \int_{\partial\Omega} \frac{1}{G}\left\{v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right\} d\mathbf{x}. \tag{97}$$

Then, in order to accommodate the boundary conditions (95), it is convenient to define

$$\langle Bu, v\rangle = \int_{\partial_2\Omega} \frac{v}{G}\frac{\partial u}{\partial n}dx - \int_{\partial_1\Omega} \frac{u}{G}\frac{\partial v}{\partial n}dx \tag{98a}$$

and

$$\langle C^*u, v\rangle = -\int_{\partial_1\Omega} \frac{v}{G}\frac{\partial u}{\partial n}dx + \int_{\partial_2\Omega} \frac{u}{G}\frac{\partial v}{\partial n}dx. \tag{98b}$$

The equation $P - B = Q^* - C^*$ is essentially the Green's formula established in Example 2.1 for the case when Ω is a unit circle and $G \equiv 1$. The validity of this result for more general conditions only requires sufficient regularity of the boundary and the function G, but such restrictions will not be discussed here.

Consider now the case when $E(\pi) > 1$ and apply the results of Section VI.A. of Part I (Eqs. 73), to obtain

$$\langle Ju, v\rangle = \int_\Gamma \left\{\frac{[u]}{G}\frac{\partial \dot{v}}{\partial n} - \frac{\dot{v}}{G}\left[\frac{\partial u}{\partial n}\right]\right\} dx \tag{99}$$

and

$$\langle K^*u, v\rangle = \int_\Gamma \left\{\frac{[v]}{G}\frac{\partial \dot{u}}{\partial n} - \frac{\dot{u}}{G}\left[\frac{\partial v}{\partial n}\right]\right\} dx \tag{100}$$

In view of Eqs. (94) to (96) and (98), it is clear that one must define

$$\langle f, v\rangle = -2\int_\Omega v\theta\,dx, \qquad \langle g, v\rangle = \int_{\partial_2\Omega} vg_{\partial 2}\,dx - \int_{\partial_1\Omega} \frac{g_{\partial 1}}{G}\frac{\partial v}{\partial n}dx. \tag{101}$$

In view of Eqs. (96), (98), and (99), it is clear that the prescribed data are $\mathcal{L}u$ in Ω (through Pu), the boundary values on $\partial_1\Omega$ and the normal derivatives on $\partial_2\Omega$ (through Bu) and the jumps across Γ of the stress function u and of its normal derivative (through Ju). On the other hand, the sought information is the stress function u in the interior of Ω (through Q^*u), the complementary boundary values, $\partial u/\partial n$ on $\partial_1\Omega$ and the function u on $\partial_2\Omega$ (through C^*u) and the averages across Γ of the stress function and of its normal derivative (through K^*u).

The direct variational formulation (84) is

$$\int_\Omega v\nabla\cdot\left(\frac{1}{G}\nabla u\right)dx + \int_{\partial_1\Omega}\frac{u}{G}\frac{\partial v}{\partial n}dx - \int_{\partial_2\Omega}\frac{v}{G}\frac{\partial u}{\partial n}dx + \int_\Gamma\left\{\frac{\dot v}{G}\left[\frac{\partial u}{\partial n}\right] - \frac{[u]}{G}\frac{\partial \dot v}{\partial n}\right\}dx = -2\int_\Omega v\theta\,dx + \int_{\partial_1\Omega}\frac{g_{\partial 1}}{G}\frac{\partial v}{\partial n}dx - \int_{\partial_2\Omega} g_{\partial 2}v\,dx, \qquad \forall\, v\in D. \tag{102}$$

In addition, the indirect variational formulation (85) is

$$\int_\Omega u\nabla\cdot\left(\frac{1}{G}\nabla v\right)dx + \int_{\partial_1\Omega}\frac{v}{G}\frac{\partial u}{\partial n}dx - \int_{\partial_2\Omega}\frac{u}{G}\frac{\partial v}{\partial n}dx + \int_\Gamma\left\{\frac{\dot u}{G}\left[\frac{\partial v}{\partial n}\right] - \frac{[v]}{G}\frac{\partial \dot u}{\partial n}\right\}dx = -2\int_\Omega v\theta\,dx + \int_{\partial_1\Omega}\frac{g_{\partial 1}}{G}\frac{\partial v}{\partial n}dx - \int_{\partial_2\Omega} g_{\partial 2}v\,dx. \tag{103}$$

A variational principle intermediate between (102) and (103) can be derived using the relation†

$$\langle (P - B - J)u, v\rangle = -\int_\Omega\frac{1}{G}\nabla v\cdot\nabla u\,dx + \int_{\partial_1\Omega}\left\{\frac{u}{G}\frac{\partial v}{\partial n} + \frac{v}{G}\frac{\partial u}{\partial n}\right\}dx - \int_\Gamma\frac{[u]}{G}\frac{\partial \dot v}{\partial n}dx \tag{104}$$

This is usually applied to functions $u'\in D$ which are continuous and satisfy the boundary conditions on $\partial_1\Omega$ and with weighting functions $\{\varphi^1,\ldots,\varphi^N\}$ also continuous and vanishing on $\partial_1\Omega$, in which case it reduces to

$$\int_\Omega\frac{1}{G}\nabla u'\cdot\nabla\varphi^\alpha dx = 2\int_\Omega\varphi^\alpha\theta\,dx + \int_{\partial_2\Omega} g_{\partial 2}\varphi^\alpha dx; \qquad \alpha = 1,\ldots,N \tag{105}$$

The variational principle (105) is most frequently derived from the standard maximum or minimum principle for elliptic equations (see for example [13, 15]).

As mentioned in Section V, the direct variational formulation is not informative—and the same is true of Eq. (105)—about the relation between any approximate solution u' and the exact one u. However, the more informative variational principle (103) has remained unnoticed (see for example [13, 15, 16]). With discontinuous weighting functions, this yields

$$\int_\Omega u'\nabla\cdot\left(\frac{1}{G}\nabla\varphi^\alpha\right)dx + \int_{\partial_1\Omega}\frac{\varphi^\alpha}{G}\frac{\partial u'}{\partial n}dx - \int_{\partial_2\Omega}\frac{u'}{G}\frac{\partial\varphi^\alpha}{\partial n}dx + \int_\Gamma\left\{\frac{u'}{G}\left[\frac{\partial\varphi^\alpha}{\partial n}\right] - \frac{[\varphi^\alpha]}{G}\frac{\partial \dot u'}{\partial n}\right\}dx = -2\int_\Omega\varphi^\alpha\theta\,dx + \int_{\partial_1\Omega}\frac{g_{\partial 1}}{G}\frac{\partial\varphi^\alpha}{\partial n}dx - \int_{\partial_2\Omega} g_{\partial 2}\varphi^\alpha dx, \qquad \alpha = 1,\ldots,N \tag{106}$$

† Equation (104) can be derived applying a decomposition of the kind considered in Sections IV and V of Part I, but the reader can verify it by integration by parts.

where the approximate solution u' is assumed to be continuous. The relation between the exact solution u and the approximate solution u', implied by (106) is given by (92):

$$\int_\Omega (u - u')\nabla \cdot \left(\frac{1}{G}\nabla\varphi^\alpha\right) dx + \int_{\partial_1\Omega} \frac{\varphi^\alpha}{G}\frac{\partial(u - u')}{\partial n} dx - \int_{\partial_2\Omega} \frac{(u - u')}{G}\frac{\partial\varphi^\alpha}{\partial n} dx$$
$$+ \int_\Gamma \left\{\frac{(u - u')}{G}\left[\frac{\partial\varphi^\alpha}{\partial n}\right] - \frac{[\varphi^\alpha]}{G}\frac{\partial(u - u')}{\partial n}\right\} dx; \qquad \alpha = 1, \ldots, N. \tag{107}$$

Under the conditions for which (105) was derived, this reduces to

$$\int_\Omega (u - u')\nabla \cdot \left(\frac{1}{G}\nabla\varphi^\alpha\right) dx - \int_{\partial_2\Omega} \frac{(u - u')}{G}\frac{\partial\varphi^\alpha}{\partial n} dx$$
$$+ \int_\Gamma \frac{(u - u')}{G}\left[\frac{\partial\varphi^\alpha}{\partial n}\right] dx = 0. \tag{108}$$

This exhibits explicitly the information contained in an approximate solution satisfying (105). However, this is not usually analyzed. Comparing (107) and (108), it is seen that by taking the weighting functions continuous, all the information about the exact solution has been centered on the function u itself and the derivatives have been eliminated. However, if information about the normal derivatives on $\partial_1\Omega$ or about the normal derivatives on the interelement boundaries is desired, the restriction $\varphi^\alpha = 0$ on $\partial_1\Omega$ or $[\varphi^\alpha] = 0$ on Γ, for the weighting functions must be removed, respectively.

To illustrate the application of our methodology to time dependent problems consider the initial value problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = f_\Omega; \qquad \text{in } \Omega \tag{109}$$

$$u - \nu\frac{\partial u}{\partial x} = g_\partial \quad \text{on } \partial'\Omega = \partial'_+\Omega + \partial'_-\Omega \tag{110a}$$

$$u = g_o, \qquad \text{at } t = 0 \tag{110b}$$

which governs advective convection (see for example [2]). Here, Ω is the rectangle illustrated in Figure 2, of space time.

Define

$$\langle Pu, v\rangle = \int_\Omega v\mathcal{L}u\, d\mathbf{x}; \qquad \langle Q^*u, v\rangle = \int_\Omega u\mathcal{L}^*v\, d\mathbf{x} \tag{111}$$

where

$$\mathcal{L}u = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2}; \qquad \mathcal{L}^*v = -\left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + \nu\frac{\partial^2 v}{\partial x^2}\right). \tag{112}$$

Take

$$\langle Bu, v\rangle = \int_{\partial'\Omega} v\left(u - \nu\frac{\partial u}{\partial x}\right) dt - \int_o^1 u(0)v(0)\, dx \tag{113a}$$

$$\langle C^*u, v\rangle = -v\int_{\partial'\Omega} u\frac{\partial v}{\partial x}\,dt - \int_o^1 u(T)v(T)\,dx \tag{113b}$$

$$\langle J_\alpha u, v\rangle = \int_{\Gamma_\alpha}\left\{\dot{v}\left(\left(v\left[\frac{\partial u}{\partial x}\right] - [u]\right)dt - [u]dx\right) - v[u]\frac{\partial v}{\partial x}\,dt\right\} \tag{114a}$$

$$\langle K_\alpha^* u, v\rangle = \int_{\Gamma_\alpha}\left\{\dot{u}\left(\left([v] + v\left[\frac{\partial v}{\partial x}\right]\right)dt + [v]dx\right) - v[v]\frac{\partial \dot{u}}{\partial x}\,dt\right\} \tag{114b}$$

where the notation for line integrals of elementary calculus has been used. Then, define

$$J = \sum_{\alpha=1}^{E-1} J_\alpha; \qquad K^* = \sum_{\alpha=1}^{E-1} K_\alpha^*. \tag{115}$$

In view of (111), (113b), (114b), and (115), the sought information is made of the function u in the interior of the elements (through Q^*u), the prediction of the values of u at time T and the values of the function at the lateral boundary $\partial'\Omega$ (through C^*u) and the averages of u and its spatial derivative $\partial u/\partial x$ across the interelement boundaries Γ_α (through K^*u).

It has special interest to consider the case when $\Gamma_1, \ldots, \Gamma_{E-1}$ are characteristic for the advection equation (i.e., $n_x + n_t = 0$ on Γ_β). Then

$$\langle K_\alpha^* u, \varphi^\beta\rangle = v\int_{\Gamma_\alpha}\left\{\dot{u}\left[\frac{\partial \varphi^\beta}{\partial x}\right] - [\varphi^\beta]\frac{\partial \dot{u}}{\partial x}\right\}dx. \tag{116}$$

Assume further that $Q\varphi^\alpha = 0$ (i.e., $\mathcal{L}^*\varphi^\beta = 0$), while $\partial\varphi^\alpha/\partial x = 0$ at $x = 0$ and $x = 1$, then the indirect variational formulation (91) becomes

$$\int_o^1 u(T)\varphi^\alpha(T)\,dx + v\sum_{\beta=1}^{E-1}\int_{\Gamma_\beta}\left\{[\varphi^\alpha]\frac{\partial \dot{u}}{\partial x} - \dot{u}\left[\frac{\partial \varphi^\alpha}{\partial x}\right]\right\}dx$$
$$= \int_\Omega \varphi^\alpha f_\Omega\,dx - -\int_{\partial'\Omega} g_\partial\varphi^\alpha\,dt - \int_o^1 g_o\varphi^\alpha(0)\,dx. \tag{117}$$

VII. BOUNDARY ELEMENT METHODS

One way in which one can use the variational formulation in terms of sought information (86) is by eliminating part of it from the equations and concentrating all the information in the remaining parts. For example, one can eliminate the function in the interior of the elements by setting $\langle Q^*u', \varphi^\alpha\rangle = 0$. This is the essence of boundary methods.

Observe $\langle Q^*u', \varphi^\alpha\rangle = \langle Q\varphi^\alpha, u'\rangle$, so that if the weighting functions are chosen so that

$$Q\varphi^\alpha = 0 \tag{118}$$

the indirect variational principle (85), reduces to

$$\langle C^*u', \varphi^\alpha\rangle + \langle K^*u', \varphi^\alpha\rangle = \langle g - f, \varphi^\alpha\rangle, \qquad \alpha = 1, \ldots, N. \tag{119}$$

Here, it has been assumed that the sought solution u is smooth; i.e., $j = 0$. In most applications $\langle Q\varphi^\alpha, v\rangle = \int_\Omega v\mathcal{L}^*\varphi^\alpha\, dx$, in which case Eq. (118) is tantamount to requiring that the weighting functions satisfy the adjoint differential equations. When (119) holds, the sought information which is involved consists of the complementary boundary values C^*u and the generalized averages, only. Applying (92), it is seen that

$$\langle C^*(u' - u), \varphi^\alpha\rangle + \langle K^*(u' - u), \varphi^\alpha\rangle = 0 \tag{120}$$

This equation exhibits explicitly the information about the exact solution contained in any approximate solution of the boundary procedure. When the system of weighting functions $\{\varphi^1, \ldots, \varphi^N\}$ is T-complete [4], the author has shown that the system (119) or equivalently (120), implies $C^*u' = C^*u$ and $K^*u' = K^*u$; i.e., the complementary boundary values and the generalized averages of any approximate solution are the exact ones.

Application of the variational principle (119) allows formulating two classes of boundary methods; the first one, to be called boundary methods in an extended sense, only requires that Eq. (118) be satisfied by the weighting functions $\{\varphi^1, \ldots, \varphi^N\}$. A more restricted class of boundary methods is obtained when, in addition to Eq. (118), one requires that the terms $\langle K^*u', \varphi^\alpha\rangle$, $\alpha = 1, \ldots, N$, vanish. This is granted taking the test functions so that

$$K\varphi^\alpha = 0, \qquad \alpha = 1, \ldots, N. \tag{121}$$

In view of Theorem 3.1, Eq. (121) are tantamount to requiring that the test functions be right-smooth (or simply smooth when $S^r = S^l$). When Eqs. (121) are satisfied, Eqs. (119) reduce to

$$\langle C^*u', \varphi^\alpha\rangle = \langle g - f, \varphi^\alpha\rangle, \qquad \alpha = 1, \ldots, N \tag{122}$$

This is Trefftz method [4], for nonsymmetric operators.

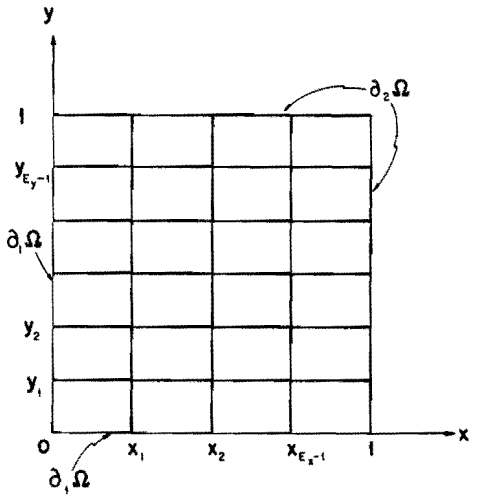

FIG. 4. The region Ω.

As an illustration of boundary methods in an extended sense, let us go back to the example of Section VI, and in order to be specific, assume the region Ω is a unit square (Fig. 4), $\partial_1\Omega$ the lower left corner of the square ($x = 0, 0 \leq y \leq 1$; $y = 0, 0 \leq x \leq 1$) while $\partial_2\Omega$ is the upper-right corner of the square ($x = 1, 0 \leq y \leq 1$; $y = 1, 0 \leq x \leq 1$). The partition will be made of $E = E_x \cdot E_y$ elements and we associate one weighting function with each one of the interior nodes; thus, there will be exactly $N = (E_x - 1)(E_y - 1)$ such functions. It will be assumed further, that Eqs. (118) are satisfied; these are

$$\nabla \cdot \left(\frac{1}{G}\nabla\varphi^\alpha\right) = 0, \qquad \alpha = 1, \ldots, N. \tag{123}$$

For simplicity it will be assumed that $C\varphi^\alpha = 0$; i.e.,

$$\varphi_\alpha = 0 \quad \text{on } \partial_1\Omega; \quad \text{and} \quad \frac{\partial\varphi_\alpha}{\partial n} = 0 \quad \text{on } \partial_2\Omega. \tag{124}$$

Also, let G be constant ($G \equiv 1$) and observe that in this case (123) reduces to the Laplace equation.

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be any ordering of the nodes, then φ^α are chosen as continuous, satisfying the boundary conditions (124), bilinear on each finite element (i.e., linear combination of 1, x, y and xy, there) and satisfying

$$\varphi^\alpha(\mathbf{x}_\beta) = \delta_{\alpha\beta}. \tag{125}$$

It is easy to see that this defines $\{\varphi^1, \ldots, \varphi^N\}$ uniquely and that the only discontinuities of the derivatives of φ^α can occur on the interelement boundaries Γ.

With this choice, the system of Eqs. (106), reduces to

$$\int_\Gamma u'\left[\frac{\partial\varphi^\alpha}{\partial n}\right] dx = -2\int_\Omega \varphi^\alpha\theta\, dx + \int_{\partial_1\Omega} g_{\partial 1}\frac{\partial\varphi^\alpha}{\partial n}\, dx - \int_{\partial_2\Omega} g_{\partial 2}\varphi^\alpha\, dx. \tag{126}$$

Observe that all the information about the exact solution has been concentrated on the interelement boundaries Γ. Let Γ_α be that part of Γ on which $[\partial\varphi^\alpha/\partial n]$ does not vanish, then Γ_α is as illustrated in Figure 5. If all the finite elements

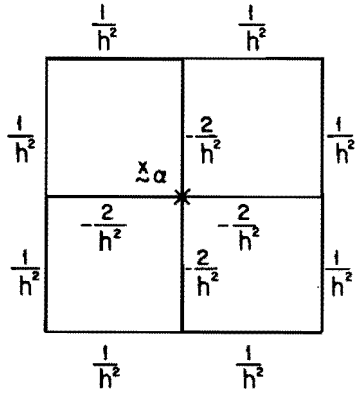

FIG. 5. The interelement boundary Γ_α and the coefficients c_α when $\mathbf{x}_\alpha = (x_\alpha, y_\alpha)$ is not next to the boundary $\partial\Omega$.

are squares with sides of length h, then the values $[\partial\varphi^\alpha/\partial n]$ are

$$\left[\frac{\partial\varphi^\alpha}{\partial n}\right] = C_\alpha(1 - |x - x_\alpha|) \qquad \text{on horizontal segments} \tag{127a}$$

$$\left[\frac{\partial\varphi^\alpha}{\partial n}\right] = C_\alpha(1 - |y - y_\alpha|) \qquad \text{on vertical segments} \tag{127b}$$

The coefficients C_α are as indicated in Figure 5.

Finally, to illustrate Trefftz method we impose the conditions $Q\varphi^\alpha = 0$ and $K\varphi^\alpha = 0$. For the example here considered, this means that φ^α is a solution of the Laplace equation (i.e., harmonic function) and continuous together with its first derivatives across the interelement boundaries. Equation (122) is

$$\int_{\partial_1\Omega} \varphi^\alpha \frac{\partial u'}{\partial n}\,dx - \int_{\partial_2\Omega} u' \frac{\partial\varphi^\alpha}{\partial n}\,dx = \int_{\partial_2\Omega} \varphi^\alpha g_{\partial 2}\,dx - \int_{\partial_1\Omega} g_{\partial 1} \frac{\partial\varphi^\alpha}{\partial n}\,dx + 2\int_\Omega \theta\varphi^\alpha\,dx \tag{128}$$

by virtue of (98b) and (101). Equation (128) corresponds to the second choice of weighting functions presented by Zielinski and Zienkiewicz [10], except that to apply (128) there is no need of breaking the sought solution into two parts as they did. The information contained in an approximate solution satisfying the variational principle (128) is exhibited by

$$\int_{\partial_1\Omega} \varphi^\alpha\left(\frac{\partial u'}{\partial n} - \frac{\partial u}{\partial n}\right) dx - \int_{\partial_2\Omega} (u' - u)\frac{\partial\varphi^\alpha}{\partial n}\,dx = 0. \tag{129}$$

This refers, of course, to the complementary boundary values, as is frequently the case when Trefftz method is applied [4]. If the system is T-complete [4],

$$\frac{\partial u'}{\partial n} = \frac{\partial u}{\partial n} \quad \text{on } \partial_1\Omega \quad \text{and} \quad u' = u \quad \text{on } \partial_2\Omega. \tag{130}$$

However, T-complete systems are usually infinite. T-complete systems discussed by the author [3, 4], and recently applied by Zielinski and Zienkiewicz [10] are given in Tables I and II.

TABLE I. T-complete systems in two dimensions.

Bounded Ω	Ω = exterior of a bounded region
Laplace Equations	
$\{1, r^n \cos n\theta, r^n \sin n\theta\}$	$\{\ln r, r^{-n} \cos n\theta, r^{-n} \sin n\theta\}$
Reduced Wave Equation $\Delta u + u = 0$	
$\{J_o(r), J_n(r) \cos n\theta, J_n(r) \sin n\theta\}$	$\{H_o^{(1)}(r), H_n^{(1)}(r) \cos n\theta, H_n^{(1)}(r) \sin n\theta\}$
$n = 1, 2, \ldots$.	

TABLE II. T-complete systems in three dimensions.

Bounded Ω	Ω = exterior of a bounded region
Laplace Equations	
$\{r^n P_n^q(\cos\theta)e^{iq\phi}\}$	$\{r^{-n-1}P_n^q(\cos\theta)e^{iq\phi}\}$
Reduced Wave Equation	
$\{j_n(r)P_n^q(\cos\theta)e^{iq\phi}\}$	$\{h_n^{(1)}(r)P_n^q(\cos\theta)e^{iq\phi}\}$
$n = 0, 1, 2, \ldots; -n \leq q \leq n$	

There is an important case for which T-complete systems are finite. They will be discussed in Part 3 of this work.

VIII. THE COUPLING OF FINITE ELEMENTS AND BOUNDARY METHODS

The formulation and analysis of problems in which finite elements and boundary procedures are coupled is straightforward when the framework of the theory here presented is used.

Indeed, the discussion of Sections III through VI applies to this case but in the partition π one separates a certain number of subregions Ω_e in which the boundary method is going to be applied. For simplicity we consider the case in which only one such region $-\Omega_E$ to be specific—is singled out (Fig. 6). Then using the notation of Section III, one defines

$$\Omega_I = \text{Interior of } \left(\bigcup_{e=1}^{E-1} \overline{\Omega}_e\right); \qquad \Omega_{II} = \Omega_E. \tag{131}$$

In applications Ω_E is usually much larger then the other elements and it is said to be a macroelement (Fig. 6). Then one defines

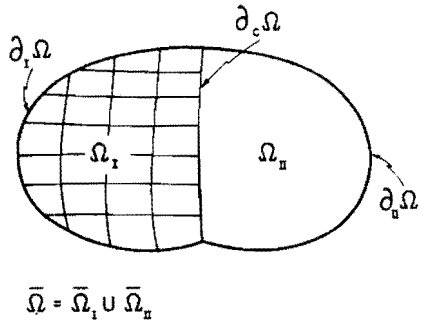

FIG. 6. The coupling of finite elements with boundary methods.

$$P_{\mathrm{I}} = \sum_{e=1}^{E-1} P_e; \qquad P_{\mathrm{II}} = P_E; \qquad Q_{\mathrm{I}} = \sum_{e=1}^{E-1} Q_e; \qquad Q_{\mathrm{II}} = Q_E \tag{132a}$$

$$B_{\mathrm{I}} = \sum_{e=1}^{E-1} B_e; \qquad B_{\mathrm{II}} = B_E; \qquad C_{\mathrm{I}} = \sum_{e=1}^{E-1} C_e; \qquad C_{\mathrm{II}} = C_E \tag{132b}$$

$$J_{\mathrm{I}} = \sum_{e>f\geq 1}^{e=1-1} J_{ef}; \qquad J_C = \sum_{f=1}^{E-1} J_{Ef} \tag{132c}$$

and similarly for K_{I} and K_C, where the notation of Section III has been used once more. The weighting functions $\{\varphi^1, \ldots, \varphi^{N'}, \ldots, \varphi^N\}$ are taken so that Q

$$Q_{\mathrm{II}}\varphi^\alpha = 0, \qquad N' < \alpha \leq N_E \tag{133}$$

while φ^α are taken with support in Ω_{I} for $1 \leq \alpha \leq N'$. Then, the indirect variational principle (91) yields

$$\langle Q_{\mathrm{I}}^* u', \varphi^\alpha\rangle - \langle C_{\mathrm{I}}^* u', \varphi^\alpha\rangle - \langle K_{\mathrm{I}}^*, \varphi^\alpha\rangle - \langle K_C^*, \varphi^\alpha\rangle = \langle f, \varphi^\alpha\rangle - \langle g, \varphi^\alpha\rangle, \qquad \alpha = 1, \ldots, N' \tag{134a}$$

and

$$\langle C_{\mathrm{II}}^* u', \varphi^\alpha\rangle + \langle K_C^* u', \varphi^\alpha\rangle = \langle g, \varphi^\alpha\rangle - \langle f, \varphi^\alpha\rangle, \qquad N' < \alpha \leq N \tag{134b}$$

When the boundary conditions $C_{\mathrm{II}}\varphi^\alpha = 0$ are satisfied, Eq. (134b) reduces to

$$\langle K_C^* u', \varphi^\alpha\rangle = \langle g, \varphi^\alpha\rangle - \langle f, \varphi^\alpha\rangle, \qquad N' < \alpha \leq N \tag{135}$$

Observe that when Eqs. (134a) and (135) hold, region Ω_{II} is eliminated from the analysis.

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