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GENERALIZED BOUNDARY METHODS

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ABSTRACT

According to the abstract theory of boundary value problems developed by the author, the method of weighted residuals admits two alternative (but equivalent) variational formulations. The first one is the standard formulation in terms of data of the problem, while the second one is in terms of the "sought information". This latter formulation is quite useful because it allows analyzing the information about the exact solution contained in any approximate one. By choosing conveniently the weighting functions one can focus the informtion supplied by the corresponding approximate solution. Generalized boundary methods corresponds to the case for which the information is concentrated on the boundaries (external and internal). To illustrate the procedure, an exhaustive analysis is carried out of finite difference algorithms which yield exact values at the nodes, for ordinary differential equations.

INTRODUCTION

Boundary methods use previously known solutions to construct the sought solutions of boundary value problems. There are two ways in which previously known solutions can be used to carry out such construction. The first one consists in adding them up to represent the desired solution. The second one is less direct and consists in using reciprocity relations to derive information about the sought solution [Herrera, 1984].

Recently, the author has derived reciprocity relations of complete generality which are applicable to any problem which is linear [Herrera, 1985a,b; Herrera, et al., 1985]; they include Green's formulas for

$$\langle Pu,v\rangle = \int_\Omega v\,\mathcal{L}\,u\,dx \quad \text{and} \quad \langle Q^*u,v\rangle = \int_\Omega u\,\mathcal{L}^*v\,dx \qquad (2.3)$$

where $\mathcal{L}$ is a differential operator defined in an elementary sense, in Ω and $\mathcal{L}^*$ is its formal adjoint. Then, knowing Q^*u is tantamount to know the function u in the interior of Ω. The "complementary boundary values" C^*u are illustrated by means of examples; thus, for Dirichlet problem of Laplace equation in which u is prescribed on the boundary $\partial\Omega$, the complementary boundary values are the normal derivatives $\partial u/\partial n$, there. For problems of Elasticity, the prescribed and complementary boundary values, may be the displacements and the tractions, respectively. The average values of the exact solution across the surface Γ (Fig. 1), where Γ is the surface on which discontinuities of the functions may occur, constitute the third component of the sought information and are characterized by K^*u.

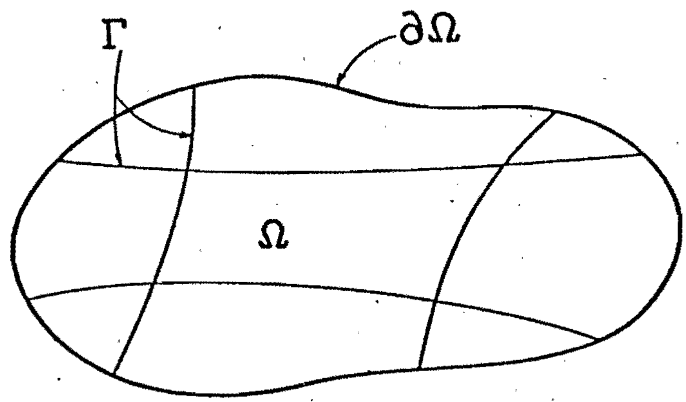

Fig. 1. The region Ω and its subdomains.

The equivalence between the variational principles (2.1) and (2.2), is granted when

$$P - B - J = Q^* - C^* - K^* \qquad (2.4)$$

is a Green's formula, in the sense of the theory. The systematic construction of such formulas in a manner which is applicable to fully discontinuous functions, has been given elsewhere [Herrera, 1985a].

According to the method of weighted residuals [Herrera, 1985b; Finlayson, 1972], an approximate solution $u' \in D$, satisfies

$$\langle (P - B - J)u', \varphi^\alpha\rangle = \langle f - g - j, \varphi^\alpha\rangle \ , \ \alpha = 1,\ldots,N \qquad (2.5)$$

where $\{\varphi^1,\ldots,\varphi^N\} \subset D$ is a system of weighting functions. Clearly, any exact solution also satisfies (2.5). A more informative form of (2.5) is obtained by applying the variational formulation in terms of the sought information (2.2). This yields

$$\langle Q^*u',\varphi^\alpha\rangle - \langle C^*u',\varphi^\alpha\rangle - \langle K^*u',\varphi^\alpha\rangle = \langle f,\varphi^\alpha\rangle - \langle g,\varphi^\alpha\rangle \;; \qquad \alpha = 1,\ldots,N \qquad (2.6)$$

One also has

$$\langle Q^*u',\varphi^\alpha\rangle - \langle C^*u',\varphi^\alpha\rangle - \langle K^*u',\varphi^\alpha\rangle = \langle Q^*u,\varphi^\alpha\rangle - \langle C^*u,\varphi^\alpha\rangle - \langle K^*u,\varphi^\alpha\rangle \;; \quad \alpha = 1,\ldots,N, \qquad (2.7)$$

because the exact solution $u \in D$, also satisfies (2.6). Thus, the N functionals $\langle Q^*u',\varphi^\alpha\rangle - \langle C^*u',\varphi^\alpha\rangle - \langle K^*u',\varphi^\alpha\rangle$ can be interpreted as "all the information" contained in an approximate solution.

BOUNDARY ELEMENT METHODS

One way in which one can use the variational formulation in terms of the sought information is by eliminating part of it from the equations and concentrating all the information in the remaining parts. For example, one can eliminate the function in the interior of the elements by choosing the weighting functions so that $\langle Q^*u',\varphi^\alpha\rangle = 0$. This is the essence of boundary methods.

Observe $\langle Q^*u',\varphi^\alpha\rangle = \langle Q\varphi^\alpha,u'\rangle$; thus, when

$$Q\varphi^\alpha = 0 \qquad (3.1)$$

equations (2.6) become

$$\langle C^*u',\varphi^\alpha\rangle + \langle K^*u',\varphi^\alpha\rangle = \langle g-f,\varphi^\alpha\rangle \;, \quad \alpha = 1,\ldots,N \qquad (3.2)$$

Here, it has been assumed that the sought solution u is smooth; i.e. $j = 0$. In most applications $\langle Q\varphi^\alpha,v\rangle = \int_\Omega v\,\mathcal{L}^*\varphi^\alpha dx$, in which case (3.1) is $\mathcal{L}^*\varphi^\alpha = 0$. When (3.2) holds, the sought information which is involved consists of the complementary boundary values and the generalized averages, only. Applying (2.7), it is seen that

$$\langle C^*(u'-u),\varphi^\alpha\rangle + \langle K^*(u'-u),\varphi^\alpha\rangle = 0 \qquad (3.3)$$

This equation exhibits explicitly the information about the exact solution contained in any approximate solution of the boundary procedure.

By a T-complete system, we mean a system $\{\varphi^\alpha\}$ such that

$$<(C^* + K^*)u',\varphi^\alpha> = <g-f,\varphi^\alpha> \ \forall\ \alpha \Rightarrow C^*u' = C^*u \ \& \ K^*u' = K^*u \tag{3.4}$$

When the existence of solution of the adjoint boundary value problem is satisfied, the existence of such system is granted [Herrera, 1985c].

Application of the variational principle (3.2) allows formulating several classes of boundary methods; the first one, to be called boundary methods in an extended sense, only requires that equation (3.1) be satisfied by the weighting functions $\{\varphi^1,\ldots,\varphi^N\}$. A more restricted class of boundary methods is obtained when, in addition to equation (3.1), one requires that the terms $<K^*u',\varphi^\alpha>$, $\alpha = 1,\ldots,N$, vanish. This is granted taking the test functions so that

$$K\varphi^\alpha = 0 \quad , \quad \alpha = 1,\ldots,N \tag{3.5}$$

Equations (3.5) are tantamount to require that the test functions be right-smooth [Herrera, 1985b] (or simply smooth when $S^r = S^\ell$). When equations (3.5) are satisfied, equations (3.2) reduce to

$$<C^*u',\varphi^\alpha> = <g-f,\varphi^\alpha> \ , \quad \alpha = 1,\ldots,N \tag{3.6}$$

This is Trefftz method [Herrera, 1984], for non-symmetric operators.

In the following sections we illustrate these procedures by applying them to ordinary differential equations, in which case the analysis can be carried out exhaustively.

GREEN'S FORMULAS FOR ORDINARY DIFFERENTIAL OPERATORS

Let Ω be the unit interval and introduce a partition of Ω into E subintervals $\Omega_\alpha = (x_{\alpha-1}, x_\alpha)$, $\alpha = 1,\ldots,E$. Here, $x_o = 0$ while $x_E = 1$. Define

$$\mathcal{L}u \equiv \sum_{k=0}^{M} a_k \frac{d^k u}{dx^k} \ ; \ \mathcal{L}^* v \equiv \sum_{k=0}^{M} (-1)^k \frac{d^k a_k v}{dx^k} \tag{4.1}$$

with the operators P and Q* given by (2.3). Then

$$J = \sum_{\alpha=1}^{E-1} J_\alpha = \sum_{j=0}^{M-1} J^j \ ; \ K^* = \sum_{\alpha=1}^{E-1} K^*_\alpha = \sum_{j=0}^{M-1} (K^j)^* \tag{4.2}$$

Here, we have written

$$J_\alpha = \sum_{j=0}^{M-1} J_\alpha^j \quad ; \quad K_\alpha^* = \sum_{j=0}^{M-1} (K_\alpha^j)^* \tag{4.3a}$$

and

$$J^j = \sum_{\alpha=1}^{E-1} J_\alpha^j \quad ; \quad (K^j)^* = \sum_{\alpha=1}^{E-1} (K_\alpha^j)^* \tag{4.3b}$$

with

$$\begin{aligned} \langle J_\alpha^j u, v\rangle &= -\dot{q}_{M-j-1}(v)_\alpha \left[\frac{d^j u}{dx^j}\right]_\alpha \quad ; \\ \langle (K^j)^* u, v\rangle &= \left[q_{M-j-1}(v)\right] \frac{d^j \dot{u}_\alpha}{dx^j} \end{aligned} \tag{4.4}$$

In turn

$$q_j(v) = \sum_{k=0}^{j} (-1)^k \frac{d^k a_{M+K-j} v}{dx^k} \; ; \; j = 0, \ldots, M-1 \tag{4.5}$$

and the subindex α implies that the corresponding quantity must be evaluated at the node x_α.

If the operators B and C* are conveniently chosen, then the equation $P - B - J = Q^* - C^* - K^*$ is satisfied and it is the desired Green's formula for functions with arbitrary jump discontinuities at the interior nodes [Herrera, 1985b; Herrera, et al., 1985]. Observe that the representations in terms of J_α and K_α^* for J and K*, respectively, supplied in Eqs. (4.2), decompose these operators in terms of the contributions at each one of the interior nodes. On the other hand, J^j and $(K^j)^*$ decompose the jumps and the averages in their components corresponding to the derivatives of order j.

DIFFERENTIAL EQUATIONS OF ARBITRARY ORDER

Consider the boundary value problem (initial value problems are included as a special case) which consists in finding $u \in D$ (continuous together with its first derivative), such that

$$\mathcal{L}u = f_\Omega \quad \text{in } \Omega = (0,1) \tag{5.1}$$

and satisfying suitable boundary conditions.

By a convenient choice of the operators $B: D \to D^*$ and

$C^*: D \to D^*$, and of the functional $g \in D^*$, the variational formulations (2.1) and (2.2) are applicable, with $\langle f,v\rangle = \int_\Omega v f_\Omega dx$. The method of weighted residuals, yields equations (3.2) when it is assumed that $Q\varphi^\alpha = 0$, i.e. $\mathcal{L}^*\varphi^\alpha = 0$. Clearly, in equations (3.2) the information about the sought solution is concentrated in the generalized averages (K^*u) and the complementary values (C^*u).

It must be observed that the generalized averages $\langle K^*u,\varphi^\alpha\rangle$ of the exact solution coincide with the values at the nodes when the sought solution is smooth. Recalling equations (4.2) to (4.5), it is seen that the information about the derivative of order j at node x_α is given by $(K^j_\alpha)^*u$. Thus, by suitably choosing the weighting functions $\{\varphi^1,\ldots,\varphi^N\}$, one can focus the sought information even more. For example, if the only desired information is about the function itself (i.e. about the zero order derivative), we must choose the weighting functions $\{\varphi^1,\ldots,\varphi^N\}$ so that

$$\langle (K^j)^*u,\varphi^\alpha\rangle = \langle K^j\varphi^\alpha,u\rangle = 0 \quad , \; j = 1,\ldots,M-1 \qquad (5.2)$$

This is granted if $K^j\varphi^\alpha = 0$, for $j = 1,\ldots,M-1$. In view of equations (4.4), this latter condition requires that

$$[\, q_\gamma(\varphi^\alpha)\,]_\beta = 0, \;\; \alpha = 1,\ldots,N \;\; ; \;\; \beta = 1,\ldots,E-1; \qquad \gamma = 0,\ldots,M-2 \qquad (5.3)$$

When the coefficients $\{a_1,\ldots,a_M\}$ are continuous together with their derivatives, equations (5.3) are tantamount to require that the weighting functions and their derivatives up to order M-2 be continuous; i.e. discontinuities of the derivatives of order M-1 are the only admissible ones. This is precisely the condition for the method to be conforming.

When the system $\{\varphi^1,\ldots,\varphi^N\}$ of weighting functions is T-complete (relation 3.4), one has $C^*u' = C^*u$ and $K^*u' = K^*u$, so that the complementary boundary values and generalized averages are predicted correctly. For ordinary differential equations the application of this procedure yields finite difference algorithms which supply the desired information exactly, because $\langle C^*u',\varphi^\alpha\rangle$ and $\langle K^*u',\varphi^\alpha\rangle$ involve nodal values only and the corresponding T-complete systems are finite. The finite difference algorithms derived in this manner yield the exact values of the solution and its derivatives at the nodes. If some of the continuity requirements $K^j\varphi^\alpha = 0$ are satisfied for some j's, then some of the values of the derivatives are eliminated from the sought information about the solution. In this manner many alternative algorithms, are constructed. An exhaustive analysis of them has been carried out in [Herrera, 1985b].

SECOND ORDER DIFFERENTIAL EQUATIONS

In this section attention is restricted to the differential operator

$$\mathcal{L}u \equiv \frac{d^2u}{dx^2} + 2a\,\frac{du}{dx} + \left(b + \frac{da}{dx}\right)u \qquad (6.1)$$

whose formal adjoint is

$$\mathcal{L}^*v \quad \frac{d^2v}{dx^2} - 2a\,\frac{dv}{dx} + \left(b - \frac{da}{dx}\right)u \qquad (6.2)$$

The coefficients a and b may have jump discontinuities nodes. The operator P and Q* are defined by (2.3). In the most general boundary value problem which is linear, one prescribes

$$e_1^o u + e_2^o \frac{du}{dx} = g_{\partial 0} \quad \text{at } x = 0 \qquad (6.3a)$$

$$e_1^1 u + e_2^1 \frac{du}{dx} = g_{\partial 1} \quad \text{at } x = 1 \qquad (6.3b)$$

where the pairs $\{e_1^\beta, e_2^\beta\}$, $\beta = 0,1$, can be taken normalized (i.e. $(e_1^{\beta 2}) + (e_2^{\beta 2}) = 1$). For the boundary value problem, it is convenient to decompose each one of the operators B and C*, into two parts, thus, we write

$$B = \sum_{\gamma=0}^{\gamma=1} B_\gamma \quad ; \quad C^* = \sum_{\gamma=0}^{\gamma=1} C_\gamma^* \qquad (6.4)$$

where

$$\langle B_\gamma u, v\rangle = (-1)^\gamma \left(e_2^\gamma \frac{du}{dx} + e_1^\gamma u\right)_\gamma \left\{e_1^\gamma \frac{dv}{dx} - (2ae_1^\gamma + e_2^\gamma)v\right\}_\gamma \;;$$

$$\gamma = 0,1 \qquad (6.5a)$$

and

$$\langle C_\gamma^* u, v\rangle = (-1)^\gamma \left(e_1^\gamma \frac{du}{dx} - e_2^\gamma u\right)_\gamma \left\{e_2^\gamma \frac{dv}{dx} + (e_1^\gamma - 2ae_2^\gamma)v\right\}_\gamma \;;$$

$$\gamma = 0,1 \qquad (6.5b)$$

For this case equations (4.4) are

$$\langle J^o_\alpha u, v\rangle = -\dot{q}(v)_\alpha[u]_\alpha \; ; \; \langle J^1_\alpha u, v\rangle = -\dot{v}_\alpha[\frac{du}{dx}]_\alpha \; ;$$

$$\alpha = 1,\ldots,E-1 \qquad (6.6a)$$

$$\langle (K^o)^* u, v\rangle = \dot{u}_\alpha[q(\dot{v})]_\alpha \; ; \; \langle (K^1_\alpha)^* u, v\rangle = [v]_\alpha[\frac{d\dot{u}_\alpha}{dx}] \; ;$$

$$\alpha = 1,\ldots,E-1 \qquad (6.6b)$$

where

$$q(v) = 2av - \frac{dv}{dx} \qquad (6.7)$$

In addition, the decompositions (4.2) and (4.3) are available. The test functions satisfy $Q\varphi^\alpha = 0$; i.e.

$$\frac{d^2\varphi^\alpha}{dx^2} - 2a\,\frac{d\varphi^\alpha}{dx} + (b - \frac{da}{dx})\varphi^\alpha = 0, \quad \alpha = 1,\ldots,N \qquad (6.8)$$

at every interior point of the subintervals Ω_e $(e = 1,\ldots,E)$.

Four algorithms will be considered. All of them yield the exact values of the sought information at the nodes, because the system of weighting functions to be used will be chosen to be T-complete [Herrera, 1985b]. The components of the sought information on which each algorithm focusses, differs for each one of them.

Algorithm 1.- The value of the function and its derivative.

Eq. (6.8) is the only constraint imposed on the weighting functions; thus, they are fully discontinuous. At every subinterval $\Omega_\alpha = (x_{\alpha-1}, x_\alpha)$ there are two linearly independent solutions (φ^α_1 and φ^α_2) which are taken to be identically zero outside Ω_α (Fig. 2a). The system $\{\varphi^1_\beta,\ldots,\varphi^E_\beta\}$ $(\beta = 1,2)$, so defined, posesses 2E linearly independent functions and can be shown to be T-complete. Equations (3.2) reduce to an $E \times E$ system:

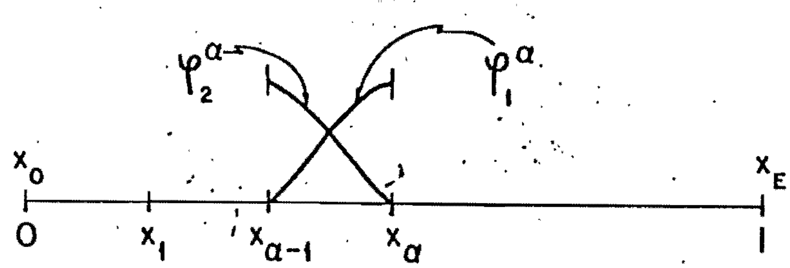

Fig. 2a. Weighting functions for algorithm 1.

$$<K_{\alpha-1}\varphi_\gamma^\alpha\,,\,u'> + <K_\alpha\varphi_\gamma^\alpha\,,\,u'> = -<f,\varphi_\gamma^\alpha> \;;\; \alpha = 2,\ldots,E-1 \;; \quad \gamma = 1,2 \tag{6.9}$$

$$<C_o\varphi_\gamma^1\,,\,u'> + <K_1\varphi_\gamma^1\,,\,u'> = <g_o\,,\,\varphi_\gamma^1> - <f,\varphi_\gamma^1> \;; \quad \gamma = 1,2 \tag{6.10a}$$

and

$$<C_1\varphi_\gamma^E\,,\,u'> + <K_{E-1}\varphi_\gamma^E\,,\,u'> = <g_1,\varphi_\gamma^E> - <f,\varphi_\gamma^E>; \quad \gamma = 1,2 \tag{6.10b}$$

Carrying out the computations in these equations yields

$$k_{\alpha-}^{o\gamma}u_{\alpha-1} + k_\alpha^{o\gamma}\,u_\alpha + k_{\alpha-}^{1\gamma}\,\frac{du_{\alpha-1}}{dx} + k_\alpha^{1\gamma}\,\frac{du_\alpha}{dx} = -<f,\varphi_\gamma^\alpha> \;; \quad \alpha = 2,\ldots,E-1 \;;\; \gamma = 1,2 \tag{6.11}$$

$$c_o^\gamma u_o^c + k_1^{o\gamma}u_1 + k_1^{1\gamma}\,\frac{du_1}{dx} = <g_o,\varphi_\gamma^1> - <f,\varphi_\gamma^1> \;;\; \gamma = 1,2 \tag{6.12a}$$

$$c_1^\gamma u_E^c + k_{E-}^{o\gamma}\,u_{E-1} + k_{E-}^{1\gamma}\,\frac{du_{E-1}}{dx} = <g_1,\varphi_\gamma^E> - <f,\varphi_\gamma^E> \;; \quad \gamma = 1,2 \tag{6.12b}$$

where

$$k_{\alpha-}^{o\gamma} = [\,2a\varphi_\gamma^\alpha - \frac{d\varphi_\gamma^\alpha}{dx}]_{\alpha-1} \quad ; \quad k_\alpha^{o\gamma} = [\,2a\varphi_\gamma^\alpha - \frac{d\varphi_\gamma^\alpha}{dx}]_\alpha \tag{6.13a}$$

$$k_{\alpha-}^{1\gamma} = [\,\varphi_\gamma^\alpha]_{\alpha-1} \quad ; \quad k_\alpha^{1\gamma} = [\,\varphi_\gamma^\alpha]_\alpha \tag{6.13b}$$

and

$$c_o^\gamma = e_2^o\,(\frac{d\varphi_\gamma^1}{dx})_o + (e_1^o - 2ae_2^o)_o\,\varphi_\gamma^1(0) \;;\; c_1^\gamma = -\{e_2^1(\frac{d\varphi_\gamma^E}{dx})_1 + (e_1^1 - 2ae_2^1)_1\,\varphi_\gamma^E(1)\} \tag{6.14}$$

$$u_o^c = (e_1^o\,\frac{du}{dx} - e_2^o u)_o \quad \text{and} \quad u_E^c = (e_1^1\,\frac{du}{dx} - e_1^1 u)_{x=x_E} \tag{6.15}$$

Here, the dots were eliminated because the sought solution is continuous. As illustration, the results for the constant coefficients case are summarized in Tables I.

Algorithm 2. The value of the function only.

In order to obtain a system of equations involving the function only, the test functions are required to

TABLE I. SOUGHT INFORMATION: THE FUNCTION AND ITS DERIVATIVE

TABLE I.1

$$\varphi_1^\alpha(x) = e^{a(x-x_{\alpha-1})} \sin\Delta(x-x_{\alpha-1}) \quad ; \quad \alpha = 1,\ldots,E$$

$$\varphi_2^\alpha(x) = e^{a(x-x_{\alpha-1})} \sin\Delta(x-x_{\alpha-1}+\delta) \quad ; \quad \alpha = 1,\ldots,E$$

$$k_{\alpha-}^{01} = -\Delta \qquad k_\alpha^{01} = \sqrt{b}\, e^{ah_\alpha}\sin\Delta(\delta-h_\alpha)$$

$$k_{\alpha-}^{02} = 0 \qquad k_\alpha^{02} = -e^{ah_\alpha}\sin\Delta h_\alpha$$

$$k_{\alpha-}^{11} = 0 \qquad k_\alpha^{11} = -\sqrt{b}\, e^{ah_\alpha}\sin\Delta h_\alpha$$

$$k_{\alpha-}^{12} = \frac{\Delta}{\sqrt{b}} \qquad k_\alpha^{12} = -e^{ah_\alpha}\sin\Delta(\delta+h_\alpha)$$

$$c_0^1 = \sqrt{b}\; e_2^0 \sin\Delta\delta \qquad c_0^2 = e_1^0 \sin\Delta\delta$$

$$c_1^1 = e^{ah_E}\{e_2^1 \sqrt{b} \sin\Delta(h_E-\delta) - e_1^1 \sin\Delta h_E\}$$

$$c_1^2 = e^{ah_E}\{e_2^1 \sqrt{b} \sin\Delta h_E - e_1^1 \sin\Delta(h_E+\delta)\}$$

Here $\mathrm{tg}\ \Delta\delta = \Delta/a$.

TABLE I.2

$$\varphi_1^\alpha(x) = e^{a(x-x_{\alpha-1})} \sinh\Delta(x-x_{\alpha-1}) \quad ; \quad \alpha = 1,\ldots,E$$

$$\varphi_2^\alpha(x) = e^{a(x-x_{\alpha-1})} \{a\sinh\Delta(x-x_{\alpha-1}) + \Delta\cosh\Delta(x-x_{\alpha-1})\} \quad ; \quad \alpha = 1,\ldots E$$

$$k_{\alpha-}^{01} = -\Delta \qquad k_\alpha^{01} = e^{ah_\alpha}\{\Delta\cosh\Delta h_\alpha - a\sinh\Delta h_\alpha\}$$

$$k_{\alpha-}^{02} = 0 \qquad k_\alpha^{02} = -e^{ah_\alpha}\sinh\Delta h_\alpha$$

$$k_{\alpha-}^{11} = 0 \qquad k_\alpha^{11} = -be^{ah_\alpha}\sinh\Delta h_\alpha$$

$$k_{\alpha-}^{12} = \Delta \qquad k_\alpha^{12} = -e^{ah_\alpha}\{\Delta\cosh\Delta h_\alpha + a\sinh\Delta h_\alpha\}$$

$$c_0^1 = e_2^0\, \Delta \quad ; \qquad c_0^2 = e_1^0\, \Delta$$

$$c_1^1 = e^{ah_E}\{e_2^1(a\sinh\Delta h_E - \Delta\cosh\Delta h_E) - e_1^1 \sinh\Delta h_E\}$$

$$c_1^2 = e^{ah_E}\{e_2^1\, b\sinh\Delta h_E - e_1^1\, (a\sinh\Delta h_E + \Delta\cosh\Delta h_E)\}$$

TABLE I.3

$$\varphi_1^{\alpha}(x) = (x-x_{\alpha-1})\, e^{a(x-x_{\alpha-1})} \quad ; \quad \alpha = 1,\ldots,E$$

$$\varphi_2^{\alpha}(x) = \{a(x-x_{\alpha-1}) + 1\}\, e^{a(x-x_{\alpha-1})} \quad ; \quad \alpha = 1,\ldots,E$$

$k_{\alpha-}^{01} = -1$	$k_{\alpha}^{01} = (1-ah_{\alpha})\, e^{ah_{\alpha}}$
$k_{\alpha-}^{02} = 0$	$k_{\alpha}^{02} = -h_{\alpha}\, e^{ah_{\alpha}}$
$k_{\alpha-}^{11} = 0$	$k_{\alpha}^{11} = -a^2\, h_{\alpha}\, e^{ah_{\alpha}}$
$k_{\alpha-}^{12} = 1$	$k_{\alpha}^{12} = -(ah_{\alpha} + 1)\, e^{ah_{\alpha}}$
$c_0^1 = e_2^0$	$c_0^2 = e_1^0$
$c_1^1 = e_2^1(ah_E - 1)e^{ah_E} - e_1^1 h_E e^{ah_E}$;	$c_1^2 = e_2^1 a^2 h_E e^{ah_E} - e_1^1(ah_E+1)e^{ah_E}$

satisfy $C\varphi^{\alpha} = 0$ and $K^1\varphi^{\alpha} = 0$. Such functions are illustrated in Fig. 2b. It must be mentioned that there are exceptional cases for which the construction illustrated in Fig. 2b may fail.

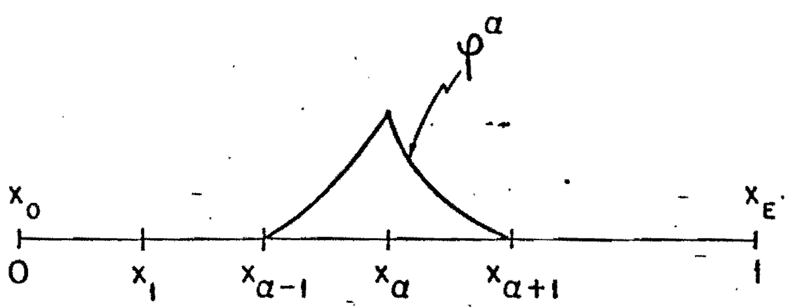

Fig. 2b. Weighting functions for algorithm 2.

Carrying out the computations one gets [Herrera, et al., 1985]:

$$\rho_{\alpha-}\, u_{\alpha-1} + \rho_{\alpha+}\, u_{\alpha+1} - 2u_{\alpha} = \mu_{\alpha} f^{\alpha} \;;\; \alpha = 2,\ldots,E-2 \qquad (6.16a)$$

$$\rho_{1+}\, u_2 - 2u_1 = \mu_1 (f^1 - g^o) \qquad (6.16b)$$

$$\rho_{(E-1)-}\, u_{E-2} - 2u_{E-1} = \mu_{E-1}(f^{E-1} - g^E) \qquad (6.16c)$$

where

$$\rho_{\alpha-} = -\frac{2[\frac{d\varphi^\alpha}{dx} - 2a\varphi\ ]_{\alpha-1}}{[\frac{d\varphi^\alpha}{dx} - 2a\varphi^\alpha]_\alpha} \ ; \ \rho_{\alpha+} = -\frac{2[\frac{d\varphi^\alpha}{dx} - 2a\varphi^\alpha]_{\alpha+1}}{[\frac{d\varphi^\alpha}{dx} - 2a\varphi^\alpha]_\alpha} \qquad (6.17a)$$

$$\mu_\alpha = -\frac{2}{[\frac{d\varphi^\alpha}{dx} - 2a\varphi^\alpha]} \ ; \ g^o = \{e_1^o \frac{d\varphi^1}{dx} - (2ae_1^o + e_2^o)\varphi^1\}_{x=0}\ g_{\partial 0} \qquad (6.17b)$$

$$f^\alpha = \int_{x_{\alpha-1}}^{x^{\alpha+1}} f\varphi^\alpha\, dx \ ; \ g^E = -\{e_1^1 \frac{d\varphi^{E-1}}{dx} - (2ae_1^1 + e_2^1)\varphi^{E-1}\}_{x=1}\ g_{\partial 1} \qquad (6.17c)$$

and it is assumed that $[d\varphi^\alpha/dx - 2a\varphi^\alpha]_\alpha \neq 0$. The manner of writing Eqs. (6.16) is suitable for comparison with central differences. However, there is "up winding" because $\rho_{\alpha+}$ and $\rho_{\alpha-}$ are different to 1 when $a \neq 0$. The results for the constant coefficients case are summarized in Tables II.

Algorithm 3. The value of the derivative only.

The weighting functions satisfy $K^o\varphi^\alpha = 0$ and $C\varphi^\alpha = 0$. The algorithm is

$$\rho_{\alpha-} \frac{du_{\alpha-1}}{dx} + \rho_{\alpha+} \frac{du_{\alpha+1}}{dx} - 2\frac{du_\alpha}{dx} = \mu_\alpha\ f^\alpha \qquad (6.18a)$$

$$\rho_{1+} \frac{du_2}{dx} - 2\frac{du_1}{dx} = \mu_1(f^1 - g^o) \qquad (6.18b)$$

$$\rho_{(E-1)-} \frac{du_{E-2}}{dx} - 2\frac{du_{E-1}}{dx} = \mu_{E-1}(f^{E-1} - g^E) \qquad (6.18c)$$

Here, it is assumed that $[\varphi^\alpha]_\alpha \neq 0$. Then

$$\rho_{\alpha-} = -2\frac{[\varphi^\alpha]_{\alpha-1}}{[\varphi^\alpha]_\alpha} \ ; \ \rho_{\alpha+} = -2\frac{[\varphi^\alpha]_{\alpha+1}}{[\varphi^\alpha]_\alpha} \qquad (6.19a)$$

$$\mu_\alpha = -\frac{2}{[\varphi^\alpha]_\alpha} \qquad (6.19b)$$

It must be mentioned that operator $\mathcal{L}u \equiv \frac{d^2u}{dx^2}$ is exceptional for this algorithm.

TABLE II. SOUGHT INFORMATION: THE FUNCTION

TABLE II.1

For $\alpha = 2,\ldots,E-2$

$$\varphi^{\alpha}(x)=\begin{cases} e^{a(x-x_{\alpha})}\sin\Delta h_{\alpha+}\sin\Delta(x-x_{\alpha-1}) \; ; & x_{\alpha-1}<x<x_{\alpha} \\ -e^{a(x-x_{\alpha})}\sin\Delta h_{\alpha}\sin\Delta(x-x_{\alpha+1}) \; ; & x_{\alpha}<x<x_{\alpha+1} \end{cases}$$

$$\varphi^{1}(x)=\begin{cases} e^{a(x-x_{1})}\sin\Delta h_{2}\sin\Delta(x+\varepsilon_{0}) \; ; & 0<x<x_{1} \\ -e^{a(x-x_{1})}\sin\Delta(\varepsilon_{0}+h_{1})\sin\Delta(x-x_{2}) \; ; & x_{1}<x<x_{2} \end{cases}$$

$$\varphi^{E-1}(x)=\begin{cases} e^{a(x-x_{E-1})}\sin\Delta(\varepsilon_{1}-h_{E})\sin\Delta(x-x_{E-2}) \; ; & x_{E-2}<x<x_{E-1} \\ -e^{a(x-x_{E-1})}\sin\Delta h_{E-1}\sin\Delta(x-x_{E}+\varepsilon_{1}) \; ; & x_{E-1}<x<1 \end{cases}$$

For $\alpha = 2,\ldots,E-2$

$$\rho_{\alpha-} = \frac{2e^{-ah_{\alpha}}\sin\Delta h_{\alpha+1}}{\sin\Delta(h_{\alpha+1}+h_{\alpha})} \qquad \rho_{\alpha+} = \frac{2e^{ah_{\alpha+}}\sin\Delta h_{\alpha}}{\sin\Delta(h_{\alpha}+h_{\alpha+1})}$$

$$\mu_{\alpha} = \frac{2}{\Delta\sin\Delta(h_{\alpha}+h_{\alpha+1})}$$

$$\rho_{1+} = \frac{2e^{ah_{2}}\sin\Delta(h_{1}+\varepsilon_{o})}{\sin\,(h_{1}+h_{2}+\varepsilon_{o})} \qquad \mu_{1} = \frac{2}{\Delta\sin\Delta(h_{1}+h_{2}+\varepsilon_{o}}$$

$$\rho_{(E-1)-} = \frac{2e^{-ah_{E-1}}\sin\Delta(\varepsilon_{1}-h_{E})}{\sin\Delta(\varepsilon_{1}-h_{E-1}-h_{E})} \qquad \mu_{E-1} = \frac{2}{\Delta\sin\Delta(\varepsilon_{1}-h_{E-1}-h_{E})}$$

Here $tg\Delta\varepsilon_{\gamma} = \dfrac{\Delta e_{2}^{\gamma}}{ae_{2}^{\gamma}-e_{1}^{\gamma}}$.

TABLE II.2

For $\alpha = 2,\ldots,E-2$

$$\varphi^{\alpha}(x) = \begin{cases} e^{a(x-x_\alpha)} \sinh\Delta h_{\alpha+1} \sinh\Delta(x-x_{\alpha-1}) \; ; \; x_{\alpha-1}<x<x_\alpha \\ -e^{a(x-x_\alpha)} \sinh\Delta h_\alpha \sinh\Delta(x-x_{\alpha+1}) \; ; \; x_\alpha<x<x_{\alpha+1} \end{cases}$$

$$\varphi^{1}(x) = \begin{cases} e^{a(x-x_1)} \sinh\Delta h_2((ae_2^o - e_1^o)\sinh\Delta x + \Delta e_2^o\cosh\Delta x); \; 0<x<x_1 \\ -e^{a(x-x_1)}((ae_2^o - e_1^o)\sinh\Delta h_1 + \Delta e_2^o \cosh\Delta h_1)\sinh\Delta(x-x_2); \\ \qquad x_1 < x < x_2 \end{cases}$$

$$\varphi^{E-1}(x) = \begin{cases} e^{a(x-x_{E-1})}((ae_2^1 - e_1^1)\sinh\Delta h_E - \Delta e_2^1 \cosh\Delta h_E)\sinh\Delta(x-x_{E-2}); \\ \qquad x_{E-2} < x < x_{E-1} \\ -e^{a(x-x_{E-1})} \sinh\Delta h_{E-1}((ae_2^1-e_1^1)\sinh\Delta(x-x_E)+\Delta e_2^1\cosh\Delta(x-x_E)); \\ \qquad x_{E-1} < x < 1 \end{cases}$$

For $\alpha = 2,\ldots,E-2$

$$\rho_{\alpha-} = \frac{2e^{-ah_\alpha}\sinh\Delta h_{\alpha+1}}{\sinh\Delta(h_{\alpha+1} + h_\alpha)} \quad ; \quad \rho_{\alpha+} = \frac{2e^{ah_{\alpha+1}}\sinh\Delta h_\alpha}{\sinh\Delta(h_{\alpha+1}+h_\alpha)}$$

$$\mu_\alpha = \frac{2}{\Delta\sinh\Delta(h_{\alpha+1} + h_\alpha)}$$

$$\rho_{1+} = \frac{2e^{ah_2}((ae_2^o - e_1^o)\sinh\Delta h_1 \mp \Delta e_2^o\cosh\Delta h_1)}{\Delta((ae_2^o-e_1^o)\sinh\Delta(h_1+h_2)+e_2^o\Delta\cosh\Delta(h_1+h_2))}$$

$$\mu_1 = \frac{2}{\Delta((ae_2^o-e_1^o)\sinh\Delta(h_1+h_2)+e_2^o\Delta\cosh\Delta(h_1+h_2))}$$

$$\rho_{(E-1)-} = \frac{2e^{-ah_{E-1}}((ae_2^1-e_1^1)\sinh\Delta h_E-\Delta e_2^1\cosh\Delta h_E)}{(ae_2^1-e_1^1)\sinh\Delta(h_{E-1}+h_E)-\Delta e_2^1\cosh\Delta(h_{E-1}+h_E)}$$

$$\mu_{E-1} = \frac{2}{((ae_2^1-e_1^1(\sinh\Delta(h_{E-1}+h_E)-\Delta e_2^1\cosh\Delta(h_{E-1}+h_E))}$$

is 0 if the derivative is sought at x_β and $\delta_\beta = 1$ if the function is sought at x_β. Choose $\{\varphi^1,\ldots,\varphi^{E-1}\}$ satisfying

$$K_\beta^{\delta\beta}\varphi^\alpha = 0 \quad \text{and} \quad C\varphi^\alpha = 0 \quad ; \quad \alpha,\beta = 1,\ldots,E-1 \qquad (6.20)$$

Using a construction similar to the previous two algorithms, one gets

$$k_{\alpha-}u_{\alpha-1}+k_\alpha u_\alpha+k_{\alpha+}u_{\alpha+1} = f^\alpha \quad ; \quad \alpha = 2,\ldots,E-2 \qquad (6.21a)$$

$$k_1u_1+k_{1+}u_2=f^1-g^o \quad ; \quad k_{(E-1)-}u_{E-2}+k_{E-1}u_{E-1} = f^{E-1}-g^E \qquad (6.21b)$$

where

$$k_\beta = -[q(\varphi^\alpha)]_\beta \quad ; \quad \text{when } \gamma_\beta = 0 \qquad (6.22a)$$

$$k_\beta = -[\varphi^\alpha]_\beta \quad ; \quad \text{when } \gamma_\beta = 1 \qquad (6.22b)$$

Here, $\beta = \alpha-$, α, $\alpha+$ and $\gamma_{\alpha\pm}$ is identified with $\gamma_{\alpha\pm1}$.

As has been mentioned previously Tables I and II, refer to the constant coefficient case. The notation is $\Delta = |\sqrt{a^2-b}\,|$. Three cases are distinguished: $a^2 - b < 0$ (case 1), $a^2 - b > 0$ (case 2) and $a^2 - b = 0$ (case 3). We have also written $h_\alpha = h_\alpha - h_{\alpha-1}$.

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