Solution of General Ordinary Differential Equations Using the Algebraic Theory Approach

Michael Celia
Department of Civil Engineering,
Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139

Ismael Herrera
Instituto de Geofísica, UNAM, (National University of Mexico),
Apdo. Postal 21-524, México, D.F., México

The algebraic theory for numerical methods, as developed by Herrera [3–7], provides a broad theoretical framework for the development and analysis of numerical approximations. To this point, the technique has only been applied to ordinary differential equations with constant coefficients. The present work extends the theory by developing a methodology for equations with variable coefficients. Approximation of the coefficients by piecewise polynomials forms the foundation of the approach. Analysis of the method provides firm error estimates. Furthermore, the analysis points to particular procedures that produce optimal accuracy. Example calculations illustrate the computational procedure and verify the theoretical convergence rates.

I. INTRODUCTION

Standard approaches to the finite element method which are based on the theory of distributions, do not permit the use of discontinuous trial and test functions, simultaneously [1, 2]. Recently, this limitation has been overcome by a unified theory of numerical methods developed by Herrera [3–7]. In particular, the results which supply the theoretical foundations for the formulation of the finite element method with discontinuous trial and test functions are given in Refs. 4 and 7. They are completely general Green's formulas which are applicable to arbitrary (symmetric or nonsymmetric) linear operators for which both the trial and the test function can be fully discontinuous.

Among the results which the theory yields, there are two variational principles applicable to any boundary value problem: the first one in terms of the "prescribed data" and the second one in terms of the "sought information." The general version of the finite element method mentioned above is obtained when the method of weighted residuals is applied, using these variational formulations.

Any approximate solution to a boundary value problem possesses some amount of information about the exact one. As a matter of fact, its usefulness depends essentially on it. In previous articles [5, 6], as an application of the

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general theory, a procedure for analyzing such information has been presented. It has been shown that, in some sense which is made precise there, the information about the exact solution contained in an approximate one is independent of the trial functions used and depends solely on the weighting functions that are applied. In this context, the trial functions supply means for interpolating (or extrapolating), more or less effectively, the actual information.

Usually that information is incorporated into the approximate solution by assuming a specific set of basis functions, in terms of which it is represented. However, it is also possible, at least in some cases, to extract that information without assuming any representation. For ordinary differential equations this leads to finite difference methods. In Ref. 6, algorithms which yield the exact values at the nodes were constructed.

The theory shows that the information supplied by an approximate solution can be decomposed into three parts: the weighted averages of the solution in the interior of the elements, the complementary boundary values (i.e., that part of the "relevant" boundary values [3, 4] which is not prescribed as data of problem), and the values and derivatives of the solution at the partition nodes. The information can be concentrated in any one of these three parts by suitably choosing the weighting functions. Finite difference approximations are obtained when attention is restricted to nodal values. In particular, if the weighting functions satisfy the adjoint differential equation exactly at the interior of the subintervals of the partition, the corresponding algorithm yields the exact values at the nodes [6].

However, the applicability of such a procedure is too restricted. The practical value of the method is very much enhanced when the condition that the adjoint differential equation be satisfied by the test functions is relaxed. Two procedures for achieving this goal are being investigated which lead to finite difference algorithms yielding the values of the solution and its derivative at any degree of accuracy. The approach in this first article is to approximate the operator $\mathcal{L}$ by one for which the exact solutions of its adjoint are easily computed.

It must be mentioned that the numerical treatment of advection dominated flows has become the subject of considerable research. The main approaches have been using finite differences [8, 9], approximate symmetrization [10], and Petrov-Galerkin methods [11]. It is clear that the results reported here, those of Ref. 6 and the general methodology have a bearing on that problem, but those aspects will be discussed elsewhere.

II. PRELIMINARY NOTIONS AND NOTATIONS

The method is based on *two variational principles* applicable to any linear boundary value problem [4–7]. The first one is in terms of the "prescribed data"

$$\langle Pu, v\rangle - \langle Bu, v\rangle - \langle Ju, v\rangle = \langle f, v\rangle - \langle g, v\rangle - \langle j, v\rangle, \qquad \forall\, v \in D \tag{1}$$

while the second one is in terms of the "sought information"

$$\langle Q^*u, v\rangle - \langle C^*u, v\rangle - \langle K^*u, v\rangle = \langle f, v\rangle - \langle g, v\rangle - \langle j, v\rangle, \qquad \forall\, v \in D \tag{2}$$

Here $f \in D^*$, $g \in D^*$, and $j \in D^*$ are the prescribed values of the operator Pu, the boundary operator Bu, and the jump operator Ju, respectively. In addition, K^*u, C^*u, and Q^*u are the generalized averages, the complementary boundary values, and the sought solution at interior points, respectively. The equivalence between these variational principles is granted when

$$P - B - J = Q^* - C^* - K^* \tag{3}$$

is a Green's formula in the sense of the theory. This has been applied to ordinary differential equations of arbitrary order in Ref. 6 and some results presented there will be used here.

Consider the differential operator

$$\mathcal{L}u \equiv \frac{d^2u}{dx^2} + 2a\frac{du}{dx} + \left(b + \frac{da}{dx}\right)u \tag{4a}$$

and its formal adjoint

$$\mathcal{L}^*v \equiv \frac{d^2v}{dx^2} - 2a\frac{dv}{dx} + \left(b - \frac{da}{dx}\right)v, \tag{4b}$$

where the functions u and v are assumed to be defined in the interval $[0, 1]$ Define

$$\langle Pu, v\rangle = \int_0^1 v\mathcal{L}u\,dx \qquad \langle Q^*u, v\rangle = \int_0^1 u\mathcal{L}^*v\,dx \tag{5}$$

Then

$$\langle (P - Q^*)u, v\rangle = \left(v\frac{du}{dx} - u\frac{dv}{dx} + 2auv\right)\Bigg|_0^1 \tag{6}$$

if u and v are sufficiently differentiable.

In the most general boundary value problem which is linear, the differential equation

$$\mathcal{L}u = f_\Omega \qquad \text{in } \Omega = (0, 1) \tag{7}$$

is subjected to boundary conditions

$$e_1^0 u + e_2^0\frac{du}{dx} = g_{\partial 0} \qquad \text{at } x = 0 \tag{8a}$$

$$e_1^1 u + e_2^1\frac{du}{dx} = g_{\partial 1} \qquad \text{at } x = \tag{8b}$$

where the pairs $\{e_1^\beta, e_2^\beta\}$, $\beta = 0, 1$, can be taken normalized (i.e., $(e_1^\beta)^2 + (e_2^\beta)^2 = 1$).

The choice of the operators B and C^* depends on the boundary conditions to be satisfied. For the boundary conditions (8), it is convenient to decompose the operator B and also the operator C^*, associated with the complementary boundary values, into two parts; each one corresponding to one of the boundary

Thus, we write

$$B = \sum_{\gamma=0}^{\gamma=1} B_\gamma; \qquad C^* = \sum_{\gamma=0}^{\gamma=1} C_\gamma^*, \tag{9}$$

where

$$\langle B_\gamma u, v\rangle = (-1)^\gamma \left(e_2^\gamma \frac{du}{dx} + e_1^\gamma u\right)_\gamma \left\{e_1^\gamma \frac{dv}{dx} - (2ae_1^\gamma + e_2^\gamma)v\right\} \qquad \gamma = 0, \tag{9a}$$

and

$$\langle C_\gamma^* u, v\rangle = (-1)^\gamma \left(e_1^\gamma \frac{du}{dx} - e_2^\gamma u\right)_\gamma \left(e_2^\gamma \frac{dv}{dx} + (e_1^\gamma - 2ae_2^\gamma)v\right)_\gamma; \qquad \gamma = 0, 1 \tag{9b}$$

Introduce a partition of Ω into E subintervals $\Omega_\alpha = (x_{\alpha-1}, x_\alpha)$, $\alpha = 1, \ldots, E$. Here, $x_0 = 0$, while $x_E = 1$. The interior boundary Γ is made of a finite number of points; these are the interior nodes. Then [4]

$$J = J^0 + J^1 = \sum_{\alpha=1}^{E-1} J_\alpha \quad \text{and} \quad K^* = (K^0)^* + (K^1)^* = \sum_{\alpha=1}^{E-1} K_\alpha^*. \tag{10}$$

$$J_\alpha = J_\alpha^0 + J_\alpha^1; \qquad K_\alpha^* = (K_\alpha^0)^* + (K_\alpha^1)^*; \qquad \alpha = 1, \quad , E - \tag{11a}$$

$$J^j = \sum_{\alpha=1}^{E-1} J_\alpha^j; \qquad (K^j)^* = \sum_{\alpha=1}^{E-1} (K_\alpha^j)^*; \qquad j = 0, \tag{11b}$$

In addition

$$\langle J_\alpha^0 u, v\rangle = -\dot{q}(v)_\alpha [u]_\alpha; \qquad \langle J_\alpha^1 u, v\rangle = -\dot{v}_\alpha \left[\frac{du}{dx}\right]_\alpha; \qquad \alpha = \qquad , E - 1 \tag{12a}$$

$$\langle (K_\alpha^0)^* u, v\rangle = \dot{u}_\alpha [q(v)]_\alpha; \qquad \langle (K_\alpha^1)^* u, v\rangle = [v]_\alpha \frac{d\dot{u}_\alpha}{dx} \qquad \alpha = \qquad , E - \tag{12b}$$

with

$$q(v) = 2av - \frac{dv}{dx} \tag{13}$$

In (12) the square brackets stand for the jump of the function involved, while the dots refer to its average across the corresponding node. Observe that $J_\alpha^0 u$ characterizes the jump of u at node α, while $J_\alpha^1 u$ characterizes the jump of its derivative. Similarly, $(K_\alpha^0)^* u$ characterizes the average of u at node α, while

$(K_\alpha^1)^* u$ characterizes the average of its derivative. Notice also that these latter averages coincide with the values of the function and the derivative, respectively, for continuous solutions.

The prescribed value of the functional g corresponding to the boundary conditions (8) will be written as $g = g_0 + g_1$, where

$$\langle g_\gamma, v\rangle = (-1)^\gamma \left\{ e_1^\gamma \frac{dv}{dx} - (2ae_1^\gamma + e_2^\gamma)v \right\}_{x=\gamma} g_{\partial\gamma}; \qquad \gamma = 0, 1. \tag{14}$$

III. FINITE DIFFERENCE METHODS

In this section, the finite difference algorithms using the nontraditional approach mentioned in the Introduction, are derived. This nontraditional approach is based on the unified theory presented in Refs. 4–7. In the unified approach to numerical methods for partial differential equations, the algorithms satisfied by an approximate solution (denoted by u') are derived using the variational formulation (2) in terms of the sought information, and applying the method of weighted residuals.

Let $\{\varphi^1, \ldots, \varphi^N\}$ be the system of "weighting or test functions," then the system of equations satisfied by any approximate solution is

$$\langle (Q^* - C^* - K^*)u', \varphi^\alpha\rangle = \langle f - g - j, \varphi^\alpha\rangle, \qquad \alpha = 1, \ldots, N. \tag{15}$$

A way in which this variational formulation in terms of the sought information can be used, is by eliminating part of the information from the equations and concentrating it in the remaining parts. For example, one can eliminate the function in the interior of the elements by setting $\langle Q^* u', \varphi^\alpha\rangle = 0$, concentrating in this manner the information in the boundary. This is the essence of generalized boundary methods [12]. However, notice that the boundary is not only the exterior boundary, but it also includes the interelement boundaries (i.e., the interior nodes for this case).

In Ref. 6, four algorithms were developed. Each algorithm is characterized by the components of the sought information on which the algorithm focuses. Only information at interior nodes was sought. The sought information is, for

Algorithm 1 —the value of the function and its derivative;
Algorithm 2 —the value of the function only;
Algorithm 3 —the value of the derivative only;
Algorithm 4 —the function at some nodes and the derivative at some others.

For every one of the algorithms, the equation $Q\varphi^\alpha = 0$ was satisfied. In addition, it was required that $C\varphi^\alpha = 0$ and

$$K^1\varphi^\alpha = 0; \qquad \text{for algorithm 2}, \tag{16a}$$

$$K^0\varphi^\alpha = 0; \qquad \text{for algorithm 3}, \tag{16b}$$

and

$$K_\beta^{\delta\beta}\varphi^\alpha = 0, \qquad \beta = 1, \ldots, E-1; \qquad \text{for algorithm 4} \tag{16c}$$

Here, $\delta_\beta = 0$ if the derivative is sought at x_β and $\delta_\beta = 1$ if the function is sought at x_β.

In view of Eqs. (9)–(13), the requirements $C\varphi^\alpha = 0$ are boundary conditions, while Eqs. (16) imply alternative continuity conditions to be satisfied by the weighting functions. For algorithm 1, the functions φ^α are fully discontinuous at interior nodes and at every subinterval $\Omega_\alpha = (x_{\alpha-1}, x_\alpha)$ there are two linearly independent solutions $\{\varphi_1^\alpha, \varphi_2^\alpha\}$ of $Q\varphi^\alpha = 0$. They are taken to be identically zero outside Ω_α. This yields $2(E-1)$ weighting functions; i.e., $N = 2(E-1)$ (Fig. 1). For the other three algorithms, there is one continuity condition to be satisfied at each one of the interior nodes. Due to this fact, we have $E-1$ weighting functions (see Fig. 1). The results presented in Ref. 6 supply finite difference schemes, which yield the exact values of the sought information in all these cases.

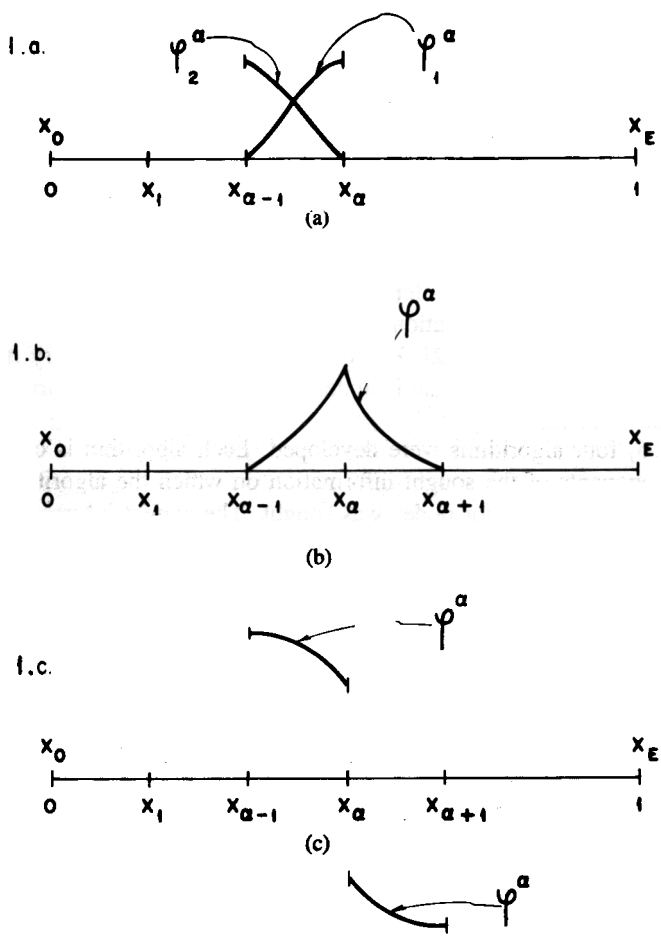

Fig. Test functions. (a) Algorithm 1, (b) Algorithm 2, and (c) Algorithm 3.

However, the condition $Q\varphi^{\alpha} = 0$ is tantamount to requiring

$$\mathcal{L}^{*}\varphi^{\alpha} = 0 \tag{17}$$

at the interior of every one of the subintervals, separately. This condition is difficult to satisfy exactly, except for special classes of operators. The purpose of this article is to present a systematic procedure for developing highly accurate finite difference algorithms, in which one fulfils (17), in an approximate manner only. This method is a natural extension of the results developed in Ref. 6, which considerably enhance their practical value.

IV. THE APPROXIMATE METHOD

The applicability of the method presented in Ref. 6 is restricted because it is generally not possible to exactly satisfy the homogeneous adjoint differential equation (17) for problems of practical interest. Thus, in this section an approximate method of solution is presented in which the original differential equation (7) is replaced by another one. This is obtained by replacing the original differential operator $\mathcal{L}$ by the operator $\overline{\mathcal{L}}$, for which the test functions satisfying the adjoint differential equations (17) are more easily constructed. Thus, the approximate solution u' of the boundary value problem (7), (8), is taken as the exact solution of

$$\overline{\mathcal{L}}u' = f_{\Omega} \tag{18}$$

subjected to the original boundary conditions (8). The exact values of the sought information, for any one of the algorithms 1 to 4, is then obtained using the method presented in Ref. 6.

Since the modified operator $\overline{\mathcal{L}}$ does not coincide with the original one $\mathcal{L}$, an error results in the numerical solution. This section details the derivation of the approximate operator. It is followed in the next section by an analysis of the associated error.

Consider the original linear differential equation (7)

$$\mathcal{L}u \equiv \frac{d^2u}{dx} + 2a\frac{du}{dx} + \left(b + \frac{da}{dx}\right)u = f_{\Omega}(x)\,. \tag{19}$$

A more compact notation for the operator $\mathcal{L}$ is

$$\mathcal{L}u \equiv \frac{d^2u}{dx^2} + c\frac{du}{dx} + du\,. \tag{20a}$$

The functions $c(x)$ and $d(x)$ are defined by direct comparison of Eqs. (19) and (20a). The adjoint operator is then given by

$$\mathcal{L}^{*}v \equiv \frac{d^2v}{dx^2} - c\frac{dv}{dx} + \left(d - \frac{dc}{dx}\right)v\,. \tag{20b}$$

The approximate operator can be derived by replacing the coefficients $c(x)$, $d(x)$ by *piece wise constant* approximations, which are denoted by $\bar{c}(x)$, $\bar{d}(x)$. Assuming the domain is already discretized into subintervals, it is convenient to

define $\bar{c}(x)$, $\bar{d}(x)$ as constants within each subinterval. As is shown in the next section, choice of $\bar{c}(x)$, $\bar{d}(x)$ as being equal to $c(x_\alpha + \Delta x_\alpha/2)$, $d(x_\alpha + \Delta x_\alpha/2)$ within each subinterval $[x_\alpha, x_{\alpha+1}]$ produces a second-order accurate numerical approximation.

The procedure can be generalized to include approximations to $c(x)$, $d(x)$ using piecewise Lagrange polynomials of arbitrary degree. That is, within each subinterval $[x_\alpha, x_{\alpha+1}]$, choose $N + 1$ interpolation points, $\{x_{\alpha j}\}_{j=1}^{N+1}$, at which $\bar{c}(x_{\alpha j}) = c(x_{\alpha j})$, $\bar{d}(x_{\alpha j}) = d(x_{\alpha j})$, and define the associated Lagrange interpolation polynomial of degree N. Then the coefficients of the equation $\mathcal{L}^*\varphi^\alpha = 0$ are also piecewise polynomials of the same degree N and can be solved within each subinterval by standard series solution techniques and the methodology explained in Sections II and III can be applied. The case of $N = 0$ corresponds to the previous example of piecewise constant approximations.

The weak form solution allows the coefficients to be $C^{-1}[0, 1]$. Thus, simple discontinuities in the approximate coefficients $\bar{c}(x)$, $\bar{d}(x)$ across element interfaces (node points) are acceptable. The remaining task is to decide on the optimal locations for the interpolation knots, $\{x_{\alpha j}\}$, within each element. Insight into this choice can be gained by an analysis of the error term inherent in the numerical approximation.

V. AN ERROR ESTIMATE

The exact solution satisfies

$$\mathcal{L}u = f_\Omega, \tag{21}$$

while the approximate one

$$\bar{\mathcal{L}}u' = f_\Omega.$$

Hence, the error $w = u' - u$ fulfills

$$\mathcal{L}w = (\delta\mathcal{L})u',$$

where $\delta\mathcal{L} = \mathcal{L} - \bar{\mathcal{L}}$.

Using an integral representation theorem, we have

$$w(y) = \int_0^1 (\delta\mathcal{L}u)G(x, y)\,dx$$

Here $G(x, y)$ is the Green's function corresponding to boundary conditions (8).

Clearly

$$(\delta\mathcal{L})u' = \{c(x) - \bar{c}(x)\}\frac{du'}{dx} + \{d(x) - \bar{d}(x)\}u'.$$

Thus

$$w(y) = \sum_{\alpha=1}^{E} e_\alpha \tag{26}$$

with

$$e_\alpha = \int_{x_{\alpha-1}}^{x_\alpha} \left\{ [c(x) - \bar{c}(x)] \frac{du'}{dx} + [d(x) - \bar{d}(x)] u' \right\} G(x,y)\, dx . \qquad (27)$$

Expansion of the differences $c - \bar{c}$, $d - \bar{d}$ in divided differences (see Ref. 13, pp. 57–63 and Ref. 14) about the interpolation knots $\{x_{\alpha j}\}$, and subsequent grouping of terms, produces

$$e_\alpha = \int_{x_{\alpha-}}^{x_\alpha} \left[\prod_{j=1}^{N+1} (x - x_{\alpha j}) \right] F(x)\, dx \qquad (28)$$

Expansion of $F(x)$ in a Taylor series about the point x_α produces

$$F(x) = F_\alpha + F'_\alpha (x - x_\alpha) + \frac{F''_\alpha}{2}(x - x_\alpha)^2 + \ldots \frac{F_\alpha^{(n)}}{n}(x - x_\alpha)^n + \qquad (29)$$

Since all differences $(x - x_{\alpha j})$ appearing in Eq. (28) are less than $\Delta x = x_\alpha - x_{\alpha-1}$, the leading term in Eq. (28), in light of Eq. (29) must be $0[(\Delta x)^{N+2}]$. However, if the$\{x_{\alpha j}\}$ are chosen as the Gaussian integration points within $[x_{\alpha-1}, x_\alpha]$, then significant improvement results. Since the Gauss points are the roots of the (shifted) Legendre polynomial of degree N ([13], p. 131), the term $\Pi_{j=1}^{N+1}(x - x_{\alpha j})$ differs from the $N + 1$ degree Legendre polynomial by only a constant factor. The $N + 1$ degree Legendre polynomial has the property that it is orthogonal, over its interval of definition, to all polynomials of degree N or less. Thus, it must be that

$$\int_{x_{\alpha-1}}^{x_\alpha} \left[\prod_{j=1}^{N+1} (x - x_{\alpha j}) \right] F_\alpha\, dx = \ldots$$

$$= \int_{x_{\alpha-1}}^{x_\alpha} \left[\prod_{j=1}^{N+1} (x - x_{\alpha j}) \right] \frac{F_\alpha^{(N)}}{N!}(x - x_\alpha)^N dx = 0 . \qquad (30)$$

Thus, the leading error term is

$$\int_{x_{\alpha-1}}^{x_\alpha} \left[\prod_{j=1}^{N+1} (x - x_{\alpha j}) \right] \frac{F^{(N+1)}}{(N+1)!}(x - x_\alpha)^{N+1} dx \sim 0[(\Delta x)^{2N+3}] .$$

Summation over each subinterval produces $0[(\Delta x)^{N+1}]$ in the general case, and $0[(\Delta x)^{2N+2}]$ for the choice of Gaussian roots.

VI. NUMERICAL RESULTS

To demonstrate the applicability of the computational algorithm, and to verify the theoretical results of the previous section, several numerical examples are solved. The test problems are linear differential equations with spatially varying coefficients. Constant coefficient problems lead to exact nodal values for all choices of discretization, and are thus not discussed further.

As a first example, consider the following differential equation

$$\frac{d^2u}{dx^2} + [e^x(1 + \sin \pi x)]u = 0, \qquad 0 \le x \le 1,$$

$$u(0) = 1,$$

$$u(1) = 0. \tag{31}$$

Observe that in this case the operator $\mathcal{L}$ is symmetric. The numerical procedure begins with a standard spatial discretization using E subintervals and $E + 1$ nodes. The weak form statement is next written as

$$\int_0^1 \left[\frac{d^2u}{dx^2} + s(x)u\right]\phi(x)\,dx = 0, \tag{32}$$

where $s(x) \equiv e^x(1 + \sin \pi x)$. For $u(x)\varepsilon C^1[0, 1]$ and $\phi(x)\varepsilon C^{-1}[0, 1]$, Eq. (32) can be written as

$$\sum_{\alpha=1}^{E} \int_{x_{\alpha-1}}^{x_\alpha} \left[\frac{d^2u}{dx^2} + s(x)u\right]\phi(x)\,dx = 0 \tag{33}$$

Integration by parts produces

$$\sum_{\alpha=1}^{E} \left\{\left[\frac{du}{dx}\phi - u\frac{d\phi}{dx}\right]_{x_{\alpha-1}}^{x_\alpha} + \int_{x_{\alpha-1}}^{x_\alpha} \left[\frac{d^2\phi}{dx^2} + s(x)\phi\right]u\,dx\right\} = 0 \tag{34}$$

or more compactly

$$\left[\frac{du}{dx}\phi - u\frac{d\phi}{\delta x}\right]_0^1 + \sum_{\alpha=1}^{E-1} \left[\frac{d\phi}{dx}\right]_\alpha u_\alpha - [\phi]_\alpha \frac{du_\alpha}{dx} + \langle \mathcal{L}^*\phi, u\rangle = 0, \tag{35}$$

where $[\cdot]_\alpha \equiv (\cdot)_{x_\alpha^+} - (\cdot)_{x_\alpha^-}$ is a jump operator. Definition of $\phi(x)$ such that $\mathcal{L}^*\phi = 0$ leads directly to a linear algebraic equation in terms of the nodal values $\{u_\alpha, du_\alpha/dx\}$.

Since the equation $\mathcal{L}^*\phi = 0$ cannot be solved exactly, let us approximate $s(x)$ using piecewise constants. Then

$$\bar{\mathcal{L}}^*\phi = \frac{d^2\phi}{dx^2} + \bar{s}(x)\phi \tag{36}$$

with $\bar{s}(x)$ constant within each element. The homogeneous equation

$$\bar{\mathcal{L}}^*\phi = 0 \tag{37}$$

has two independent solutions over each element, which can be written as

$$\phi_1^\alpha = \cos[\sqrt{\bar{s}}\,(x - x_{\alpha-1})],$$

$$\phi_2^\alpha = \sin[\sqrt{\bar{s}}\,(x - x_{\alpha-1})].$$

A set of test functions is thus defined by

$$\phi_1^\alpha(x) = \begin{cases} \cos[\sqrt{\bar{s}}\,(x - x_{\alpha-1})], & x_{\alpha-1} < x < x_\alpha \\ 0, & x < x_{\alpha-1}; \quad x > x_\alpha, \end{cases} \tag{38a}$$

$$\phi_2^\alpha(x) = \begin{cases} \sin[\sqrt{s}(x - x_{\alpha-1})], & x_{\alpha-1} < x < x_\alpha \\ 0 \quad , & x < x_{\alpha-1}; \quad x > x_\alpha , \end{cases} \tag{38b}$$

with $1 \leq \alpha \leq E$. Insertion of each of these ϕ_k^α $(k = 1, 2)$ into Eq. (35) with the approximation that $\mathcal{L}^*\phi_k^\alpha = 0$, produces $2E$ algebraic equations. Coupled with two boundary conditions, there results $2E + 2$ linear algebraic equations for the $2E + 2$ nodal unknowns $\{u_\alpha, du_\alpha/dx\}_{\alpha=0}^{E}$.

When piecewise polynomials of degree greater than zero are used to approximate $s(x)$, a series solution is formulated for Eq. (37). Two independent test functions are again generated for each element, and the numerical solution proceeds exactly as described above.

Numerical results computed using this procedure are shown in the convergence plots of Figure 2. Three sets of curves are presented, corresponding to

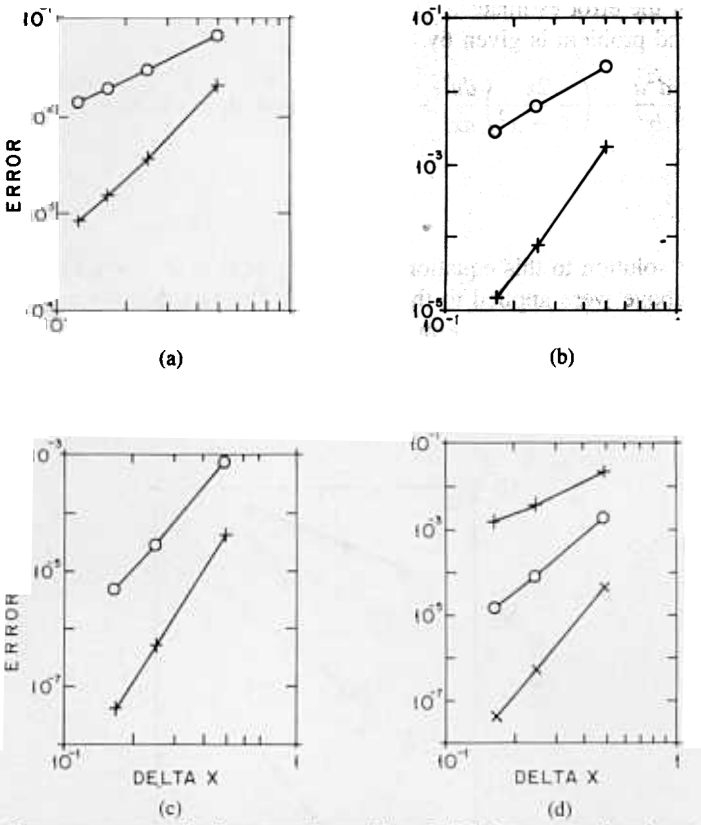

Fig. 2. Convergence results for example problem 1. Solution errors for piecewise constant approximations using one endpoint (0 − 0, slope = 1) and the Gauss point (+ − +, slope = 2) values are shown in (a); piecewise linear approximations using the two endpoints (0 − 0, slope = 2) and the two Gauss points (+ − +, slope = 4) are shown in (b); and piecewise quadratic approximations using the midpoint and two endpoints (0 − 0, slope = 4) and the three Gauss points (+ − +, slope = 6) are shown in (c). Figure (d) assembles the Gauss point results for constant (+ − +), linear (0 − 0), and quadratic (x − x) approximations.

the cases of piecewise constant, piecewise linear, and piecewise quadratic approximations. Each plot shows solution error as a function of grid spacing for cases using different choices of interpolation knots. The exact solution was taken as that produced by an orthogonal collocation solution with a very fine mesh (1000 nodes). All convergence rates correspond to the theoretical predictions of the previous section.

The quadratic curve of Figure 2(c) that uses the interval midpoint and two endpoints as interpolation knots produces $0[(\Delta x)^4]$ convergence, which is one order higher than the rate predicted in the last section. This occurs because whenever the three interpolation knots one chosen to be symmetric about the midpoint of the interval, the leading error term in Eq. (30) vanishes, producing one order higher convergence. This is a consequence of the fact that the (cubic) polynomial of the leading term is anti-symmetric (positive and negative) about the midpoint, and thus integrates to zero. This result is further evidence of the validity of the error estimates.

A second problem is given by

$$\frac{d^2u}{dx^2} - \left(\frac{2x}{1+x^2}\right)\frac{du}{dx} + \left(\frac{2}{1+x^2}\right)u = 0, \qquad 0 \le x \le 1,$$

$$u(0) = 1,$$

$$u(1) = 0.$$

The exact solution to this equation is given by $u(x) = 1 - x^2$. The procedures outlined above were applied to this equation. The fact that the operator is not self-adjoint does not affect the procedure in any way. Convergence results are presented in Figure 3. The rates again match the theoretically predicted values.

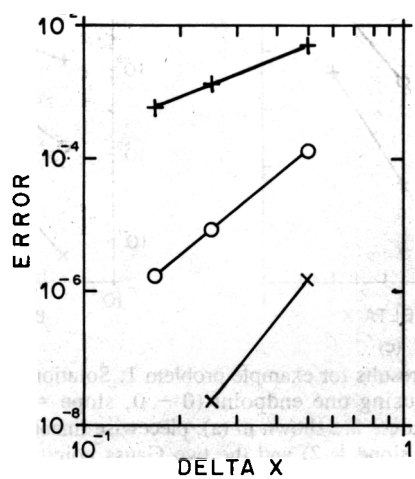

Fig. 3. Convergence results for example problem 2, using Gauss points to define piecewise constant (+ − +, slope = 2), piecewise linear (0 − 0, slope = 4), and piecewise quadratic ($x - x$, slope = 6) interpolation for the variable coefficients.

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