The Algebraic Theory Approach For Ordinary Differential Equations: Highly Accurate Finite Differences

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This article reports further developments of Herrera's algebraic theory approach to the numerical treatment of differential equations. A new solution procedure for ordinary differential equations is presented. Finite difference algorithms of $0(h^r)$, for arbitrary "r" are developed. The method consists in constructing local approximate solutions and using them to extract information about the sought solution. Only nodal information is derived. The local approximate solutions are constructed by collocation, using polynomials of degree G. When "n" collocation points are used at each subinterval, $G = n + 1$ and the order of accuracy is $0(h^{2n-1})$. The procedure here presented is very easy to implement. A program in which n can be chosen arbitrarily, was constructed and applied to selected examples.

I. INTRODUCTION

In previous articles [1–5], it has been shown that Herrera's algebraic theory of boundary value problems (including initial value problems) for arbitrary differential equations, has great potential as a tool of analysis of numerical methods for partial differential equations.

Standard approaches to the finite element method which are based on the theory of distributions do not permit the use of discontinuous trial and test functions simultaneously [6, 7]. This limitation is overcome by the unified theory of numerical methods developed by the author. In particular, the results which supply the theoretical foundations for the formulation of the finite element method with discontinuous trial and test functions are given in [2, 3, 5]. They are completely general Green's formulas which are applicable to arbitrary (symmetric or non-symmetric) linear operators for which both the trial and the test functions can be fully discontinuous.

Among the results which the theory yields, there are two variational principles applicable to any boundary value problem: the first one in terms of the "prescribed data" and the second one in terms of the "sought information." The general version of the finite element method mentioned above is obtained when the method of weighted residuals is applied, using these variational formulations (see [3, 4]).

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Any approximate solution to a boundary value problem possesses some amount of information about the exact one. As a matter of fact, its usefulness depends essentially on it. In previous articles [3, 4], as an application of the general theory, a procedure for analyzing such information has been presented. It has been shown that, in some sense which is made precise there, the information about the exact solution contained in an approximate one is independent of the trial functions used and depends solely on the test functions that are applied. In this context, the trial functions supply means for interpolating (or extrapolating), more or less effectively, the actual information.

Usually that information is incorporated into the approximate solution by assuming a specific set of basis functions, in terms of which it is represented. However, it is also possible, at least in some cases, to extract that information without assuming any representation. For ordinary differential equations, this leads to finite difference methods [4, 8].

The theory shows that the information supplied by an approximate solution can be decomposed into three parts: the weighted averages of the solution in the interior of the elements, the complementary boundary values (i.e., that part of the "relevant" boundary values which is not prescribed as data of problem), and the values and derivatives of the solution at the partition nodes. The information can be concentrated in any one of these three parts by suitably choosing the weighting functions. Finite difference approximations are obtained when the information is concentrated in the nodes. This was done in [4] obtaining algorithms which yield the exact values of the function and its derivative at the nodes. Such procedure, however, requires constructing test functions which exactly satisfy the adjoint differential equation in the interior of the subintervals of the partition. Thus, this restricts the applicability of the procedure to simple equations, such as those with constant coefficients.

The versatility of the method is very much enhanced when the condition that the adjoint differential equation be satisfied by the test functions is relaxed. Two procedures for achieving this goal are being investigated which lead to finite difference algorithms yielding the values of the solution and its derivative at any degree of accuracy. In a first article [8], the operator $\mathcal{L}$ was approximated by one for which the exact solutions of the adjoint equation are easily computed. The second procedure, which is treated here, consists in constructing test functions which approximately satisfy the adjoint equation. In this article, this is done using collocation. The numerical efficiency of the method is tested applying it to specific examples, obtaining very satisfactory results.

Polynomial approximations are used to construct the test functions and they are required to satisfy the adjoint differential equation at n collocation points at each subinterval of the partition. The degree of accuracy of the resulting algorithm is $0(h^{2n-1})$, where "h" is a norm of the partition. Since n is arbitrary and the procedure was easily programmed, very accurate and efficient algorithms are constructed in this manner.

Numerical simulation of the advective-diffusive transport equation is a problem of extreme inportance and also one of great difficulty. When the diffusive process dominates the advective process, the equation is relatively easy to solve

by virtually any standard numerical scheme. However, when the problem is advection-dominated, the standard numerical approximations become problematic. Either nonphysical oscillations appear in the vicinity of sharp fronts, or excessive numerical diffusion is introduced and the ability to capture a sharp front is precluded.

When a transport problem shifts from being diffusion-dominated to being advection-dominated, the fundamental nature of the equation changes. A numerical approximation that fails to appropriately incorporate the changing character of the equation has little chance of success. This is precisely the case for most standard numerical approximations. On the contrary, in Herrera's approach this is incorporated through the test functions that are used.

Because of its difficulty and importance, the numerical treatment of advection dominated flows has been the subject of considerable research. The main approaches have been using finite differences [9, 10], approximate symmetrization (Morton's approach [11]) and Petrov-Galerkin methods (Hughes' approach [12]). The numerical results of the new method have been compared with other methods in [13], including steady and non-steady state problems, and its considerable superiority has been exhibited there.

II. PRELIMINARY REMARKS

The method is based on two variational principles applicable to any linear boundary value problem [2–4]. The first one is in terms of the "prescribed data"

$$\langle Pu, \psi\rangle - \langle Bu, \psi\rangle - \langle Ju, \psi\rangle = \langle f, \psi\rangle - \langle g, \psi\rangle - \langle j, \psi\rangle, \quad \forall\, \psi \in D, \tag{1}$$

while the second one is in terms of the "sought information"

$$\langle Q^*u, \psi\rangle - \langle C^*u, \psi\rangle - \langle K^*u, \psi\rangle = \langle f, \psi\rangle - \langle g, \psi\rangle - \langle j, \psi\rangle, \quad \forall\, \psi \in D \tag{2}$$

Here D is the space of admissible functions (a linear space), D^* is its algebraic dual, and $f \in D^*$, $g \in D^*$, and $j \in D^*$ are the prescribed values of the operator Pu, the boundary operator Bu, and the jump operator Ju. In addition, K^*u, C^*u, and Q^*u are the generalized averages, the complementary boundary values, and the sought solution at interior points. The equivalence between these variational principles is granted when

$$P - B - J = Q^* - C^* - K^* \tag{3}$$

is a Green's formula in the sense of the theory [2, 4]. Green's formulas of completely general validity, which apply to any (possibly) non-symmetric operators defined in fully discontinuous trial and test functions have been given in [2] for the case when the trial and test function spaces are the same. Such formulas have been extended in [5] to the case when the trial and test functions spaces are different.

Using the author's approach, which is applicable to equations of arbitrary order, finite difference formulas have been given for second-order differential equations with variable coefficients, in terms of the corresponding test functions [4]. Thus, for the purposes of the present article, it would be sufficient to take the results presented in Section 6 of [4] and derive in this manner the finite differences algorithms that will be discussed. However, in order to make the paper more accessible, such formulas will be derived using an elementary approach, but in the sequel the relation between this procedure and the more systematic abstract method which is based on Herrera's algebraic theory [1–5] will be pointed out. Only second-order equations will be considered, although the same procedure is applicable to equations of arbitrary order (see [4]).

III. SECOND-ORDER EQUATIONS

For convenience, the differential equation will be written as

$$\mathcal{L}u \equiv \frac{d}{dx}\left(D\frac{du}{dx}\right) + 2a\frac{du}{dx} + \left(b + \frac{da}{dx}\right)u = f_\Omega(x), \qquad 0 \le x \le l, \tag{4}$$

$$u(0) = g_\partial 0; \qquad u(l) = g_{\partial l}, \tag{5}$$

where the coefficients are not required to be constant. The notation $g_{\partial 0}$ and $g_{\partial l}$ is used to represent the prescribed values of u at 0 and l, respectively. First-type boundary conditions are given for convenience only; the treatment of general boundary conditions was given in [4].

The author's procedure evolves as follows. First, the domain $[0, l]$ is partitioned into E subintervals with $E + 1$ node points $\{x_e\}_{e=0}^{E}$ used to delineate the subintervals. Next, a weak form statement is written for Eq. (4) as

$$\sum_{e=1}^{E} \int_{x_{e-1}}^{x_e} (\mathcal{L}u)\psi(x)\,dx = \sum_{e=1}^{E} \int_{x_{e-1}}^{x_e} f_\Omega(x)\psi(x)\,dx. \tag{6}$$

The reason for using a weak statement in the form of Eq. (6) is that in this manner it is applicable even when u is fully discontinuous at the nodes, as long as it is $\mathcal{C}^2$ within each subinterval. Even more, if such function u satisfies Eq. (6) for every ψ which is $\mathcal{C}^2$ within each subinterval (but is fully discontinuous at the nodes), then Eq. (4) is fulfilled at the interior of each subinterval, separately. Thus, in what follows, the admissible functions will be $\mathcal{C}^2$ within every subinterval separately, but they may be fully discontinuous at the nodes.

A variational statement for the boundary conditions can also be given. This can be done in many different ways; one possibility is

$$u(l)q(\psi)_l - u(0)q(\psi)_0 = g_{\partial l}q(\psi)_l - g_{\partial_0}q(\psi)_0, \tag{7}$$

where the notation

$$q(\psi) = D\frac{d\psi}{dx} - 2a\psi \tag{8}$$

has been adopted. It is easy to see that a function u satisfies the boundary conditions (5), if and only if, Eq. (7) is fulfilled for every admissible function ψ.

Usually, for a second-order differential equation such as (4), the sought solution together with its derivative are required to be continuous. At the nodes, not every admissible function satisfies this condition and it is necessary to impose this requirement in some manner. There are many alternative ways of imposing this condition variationally. The following variational statement is one of them

$$\sum_{e=1}^{E-1}\left\{\dot{q}(\psi)_{x_e}[\![u]\!]_{xe} - (\dot{D}\dot{\psi})_{x_e}\left[\!\left[\frac{du}{dx}\right]\!\right]_{x_e}\right\} = 0. \tag{9}$$

Here, the double bracket notation and the dot denote the jump and the average, respectively, of the corresponding function at the designated point; i.e.

$$[\![\cdot]\!]_{x_e} \equiv (\cdot)_{x_e^+} - (\cdot)_{x_e^-}; \qquad (\dot{\cdot})_{x_e} = \frac{1}{2}\left\{(\cdot)_{x_e} + (\cdot)_{x_e^-}\right\}. \tag{10}$$

Adding Eqs. (6), (7), and (9), it is obtained

$$\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}(\mathcal{L}u)\psi(x)\,dx + u(l)q(\psi)_l - u(0)q(\psi)_0 + \sum_{e=1}^{E-1}\left\{\dot{q}(\psi)_{x_e}[\![u]\!]_{x_e} - (D\dot{\psi})_{x_e}\left[\!\left[\frac{du}{dx}\right]\!\right]_{x_e}\right\} = \sum_{e=1}^{E}\int_{x_{e-1}}^{x_e} f\psi\,dx + g_{\partial_1}q(\psi)_l - g_{\partial_0}q(\psi)_0 \tag{11}$$

It is easy to see that Eq. (11) is satisfied for every admissible function ψ, if and only if, the same is true of Eqs. (6), (7), and (9). Thus, in conclusion, given an admissible function u, the variational statement (11) holds for every admissible function ψ, if and only if, u satisfies (4) at every interior point of the subintervals, fulfills the boundary conditions (5), and it is continuous, together with its derivative, at every one of the interior nodes.

This is the author's variational principle in terms of the data of the problem [3, 4] for the boundary value problem defined by (4) and (7), and corresponds to Eq. (1), as is further explained later.

If the differential operator $\mathcal{L}$ is defined as per Eq. (4), then application of integration by parts produces

$$\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}\{\psi\mathcal{L}u - u\mathcal{L}^*\psi\}\,dx = \sum_{e=1}^{E}\left[D\psi\frac{du}{dx} - uq(\psi)\right]_{x_{e-1}}^{x_e}, \tag{12}$$

where $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$, given by

$$\mathcal{L}^*\psi = \frac{d}{dx}\left(D\frac{d\psi}{dx}\right) - 2a\frac{d\psi}{dx} + \left(b - \frac{da}{dx}\right)\psi$$

and notation (8) has been used. By an algebraic manipulation, it can be seen that

$$\sum_{e=1}^{E}\left[D\psi\frac{du}{dx} - uq(\psi)\right]_{x_{e-1}}^{x_e} = \left[D\psi\frac{du}{dx} - uq(\psi)\right]_0^l + \sum_{e=1}^{E}\left\{[\![q(\psi)]\!]_{x_e}\dot{u}_e - [\![D\psi]\!]\frac{d\dot{u}_e}{dx} + \dot{q}(\psi)_{x_e}[\![u]\!] - (D\dot{\psi})_{x_e}\left[\!\left[\frac{du}{dx}\right]\!\right]_{x_e}\right\}$$

Substitution of (14) into (12) and further rearrangement yields

$$\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}\psi\mathcal{L}u\,dx+[uq(\psi)]_0^l-\sum_{e=1}^{E}\left\{\dot{q}(\psi)_{x_e}[\![u]\!]_{x_e}-(D\psi)_{x_e}\left[\!\left[\frac{du}{dx}\right]\!\right]_{x_e}\right\}$$
$$=\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}u\mathcal{L}^*\psi\,dx+\left[D\psi\frac{du}{dx}\right]_0^l+\sum_{e=1}^{E}\left\{[\![q(\psi)]\!]_{x_e}\dot{u}_e-[\![D\psi]\!]_{x_e}\frac{d\dot{u}_e}{dx}\right\}\qquad(15)$$

Equation (15) is a particular case of the author's Green formula for operators in discontinuous trial and test functions [2, 4, 5], and corresponds to Eq. (3).

Using (15), it is clear that the variational principle (11) is equivalent to

$$\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}u\mathcal{L}^*\psi\,dx+\left[D\psi\frac{du}{dx}\right]_0^l+\sum_{e=1}^{E}\left\{[\![q(\psi)]\!]_{x_e}\dot{u}_{x_e}-[\![D\psi]\!]_{x_e}\frac{d\dot{u}_e}{dx}\right\}$$
$$=\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}f\psi\,dx+g_{\partial l}q(\psi)_l-g_{\partial_0}q(\psi)_0.\qquad(16)$$

This is the author's [4, 5] variational principle in terms of the sought information.

By inspection of Eqs. (11) and (16), the motivation for this terminology is apparent; the left-hand side of Eq. (11) involves only quantities (or functions) which have been prescribed as data of the problem; namely, $\mathcal{L}u$, the boundary values of u which have been prescribed and the jumps of u and its derivative at the interior nodes; these latter jumps are required to be zero when the sought solution is smooth. On the other hand, on the left-hand side of Eq. (16) there are only properties of the solution which are not known beforehand. Observe that the sought information appearing on the left-hand side of Eq. (16) can be divided into three parts: the complementary boundary values (i.e., the derivatives at 0 and l); the averages of u and its derivative at interior nodes (which for a smooth solution coincide with their values), and the values of the function at the interior of the subintervals of the partition.

Finally, one can write these two variational principles in the form (1) and (2) if the operators and functionals are suitably defined. Indeed, define

$$\langle Pu,\psi\rangle=\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}\psi\mathcal{L}u\,dx;\qquad\langle Q^*u,\psi\rangle=\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}u\mathcal{L}^*\psi\,dx,\qquad(17)$$

$$\langle Bu,\psi\rangle=u(0)q(\psi)_0-u(l)q(\psi)_l;\qquad\langle C^*u,\psi\rangle=D\psi(0)\frac{du}{dx}(0)-D\psi(l)\frac{du}{dx}(l),$$

$$\langle Ju,\psi\rangle=\sum_{e=1}^{E-1}\left\{(D\dot{\psi})_{x_e}\left[\!\left[\frac{du}{dx}\right]\!\right]_{x_e}-\dot{q}(\psi)_{x_e}[\![u]\!]_{x_e}\right\};$$

$$\langle K^*u,\psi\rangle=\sum_{e=1}^{E}\left\{[\![D\psi]\!]_{x_e}\frac{d\dot{u}_e}{dx}-[\![q(\psi)]\!]_{x_e}\dot{u}_{x_e}\right\},$$

$$\langle f,\psi\rangle=\sum_{e=1}^{E}\int_{x_{e-1}}^{x_e}f_\Omega\psi\,dx;\qquad\langle g,\psi\rangle=g_{\partial_0}q(\psi)_0-g_{\partial l}q(\psi)_l;\qquad\langle j,\psi\rangle=0$$

Then, Eq. (2) becomes (11), while (2) is (16). Even more, Eq. (3) means that (15) is valid for every admissible u and ψ.

As pointed out in [4], one can write

$$J = J^0 + J^1 \quad \text{and} \quad K^* = (K^0)^* + (K^1)^* \tag{21}$$

$$J^0 = \sum_{e=1}^{E} J_e^0; \qquad J^1 = \sum_{e=1}^{E-1}; \qquad (K^0)^* = \sum_{e=1}^{E-1} (K_e^0)^*; \qquad (K^1)^* = \sum_{e=1}^{E-1} (K_e^1)^* \tag{22}$$

$$\langle J_e^0 u, \psi \rangle = -\dot{q}(\psi)_{x_e} [\![u]\!]_{x_e}; \qquad \langle J_e^1 u, \psi \rangle = (\dot{D}\psi)_{x_e} \left[\!\!\left[\frac{du}{dx}\right]\!\!\right]_{x_e} \tag{23a}$$

$$\langle K_e^0 \psi, u \rangle = -[\![q(\psi)]\!]_{x_e} \dot{u}_{x_e}; \qquad \langle K_e^1 \psi, u \rangle = [\![D\psi]\!]_{x_e} \frac{d\dot{u}_e}{dx} \tag{23b}$$

The interest of this representation is that $(K_e^0)^* u$ and $(K_e^1)^* u$ give information about the function u and its derivative at node e, respectively.

Recall that the variational principles in terms of the prescribed data (11) and in terms of the sought information (3.13), were derived using an elementary approach. Thus, the introduction of the variational conditions (7) and especially (9), in an *ad hoc* manner, may seem to be rather artificial. The beauty of the author's algebraic theory [1–5] is due, in part, to the fact that it supplies explicit formulas for the operators P, B, J, Q^*, C^*, K^* and the functionals f, g, and j, which can be applied automatically, once a differential equation and corresponding boundary conditions are given.

IV. HIGHLY ACCURATE FINITE DIFFERENCES

The next step will be to eliminate the function at the interior of the subintervals of the partition from the variational principle (16) in terms of the sought information. To the end, a family of test functions $\{\varphi_\alpha\}$, $\alpha = 1, \ldots, N$, each of which satisfies the homogeneous adjoint equation

$$\mathcal{L}^* \varphi = 0 \tag{24}$$

within each one of the subintervals, is applied in (16). In this manner, all the information is concentrated in the nodes and (16) reduces to

$$D\varphi_\alpha(l) \frac{du}{dx}(l) - D\varphi_\alpha(0) \frac{du}{dx}(0) + \sum_{e=1}^{E} \left\{ [\![q(\varphi_\alpha)]\!]_{x_e} \dot{u}_{x_e} - [\![D\varphi_\alpha]\!]_{x_e} \frac{d\dot{u}e}{dx} \right\} = \sum_{e=1}^{E} \int_{x_{e-1}}^{x_e} f\varphi_\alpha \, dx + g_{\partial l} q(\varphi_\alpha)_l - g_{\partial 0} q(\varphi_\alpha)_0 \tag{25}$$

As explained by Herrera and co-workers [4], a variety of choices exists concerning the smoothness of the test functions $\varphi_\alpha(x)$. These additional conditions

further concentrate the information. Thus, for example, if the test functions φ_α are continuous at a given node "e," then $[\![D\varphi_\alpha]\!]_{x_e}$ vanishes (when the coefficient D is continuous there), and from (23b) and (25), it is clear that the information about the derivative at that node is eliminated.

In [4], four algorithms were considered. Each algorithm is characterized by the components of the sought information on which the algorithm focuses. Only information at the nodes is sought. The sought information is, for

Algorithm 1—The value of the function and its derivative;
Algorithm 2—The value of the function only;
Algorithm 3—The value of the derivative only;
Algorithm 4—The function at some nodes and the derivative at some others.

The smoothness conditions satisfied by the test functions were

$$K^1\varphi^\alpha = 0; \qquad \text{for Algorithm 2;} \tag{26a}$$

$$K^0\varphi^\alpha = 0; \qquad \text{for Algorithm 3;} \tag{26b}$$

$$K_\beta^{\delta_\beta}\varphi^\alpha = 0, \beta = 1, \ldots, E-1; \qquad \text{for Algorithm 4.} \tag{26c}$$

Here, $\delta_\beta = 0$ if the derivative is sought at x_β and $\delta_\beta = 1$ if the function is sought at x_β.

For Algorithm 1, the functions φ^α are fully discontinuous at interior nodes. At every subinterval $\Omega_\alpha = (x_{\alpha-1}, x_\alpha)$ there are two linearly independent solutions $\{\varphi_1^\alpha, \varphi_2^\alpha\}$ of $\mathcal{L}^*\varphi^\alpha = 0$, which are taken to be identically zero outside Ω_α. This yields $2E$ test functions; i.e., $N = 2E$ for this case (Fig. 1a). For the other three algorithms, there is one continuity condition to be satisfied at each one of the interior nodes, and due to this fact, each test function φ^α has support in the interval $(x_{\alpha-1}, x_{\alpha+1})$ and vanishes elsewhere as depicted. Figures 1a and 1b correspond to Algorithms 1 and 2, respectively. If, in addition, the boundary conditions $\varphi^\alpha(0) = \varphi^\alpha(l) = 0$ are satisfied, then there are $E-1$ linearly independent test functions (i.e., $\alpha = 1, \ldots, E-1$). The system so constructed is T-complete [1, 3, 5] and the finite difference algorithms derived in this manner yield exact values of the sought information at interior nodes only (due to the boundary conditions satisfied by the test functions).

However, T-complete systems are difficult to construct because the condition $\mathcal{L}^*\varphi^\alpha = 0$ can be satisfied exactly for simple equations only. Thus, the approach developed in this paper is based on relaxing that condition and constructing approximations to T-complete systems. In general, the system $\{\varphi^\alpha\}$ of test functions will be taken so that $\mathcal{L}^*\varphi^\alpha = 0$ at a finite number of collocation points. In this case (see Section VI)

$$\sum_{e=1}^{E} \int_{x_{e-1}}^{x_e} u\mathcal{L}^*\varphi^\alpha \, dx = 0(h^r), \tag{27}$$

where "h" is the norm of the partition and $r > 0$ is some integer. The algorithms constructed in this manner do not yield exact values at the nodes any longer, but they can be made to be of any desired order of accuracy.

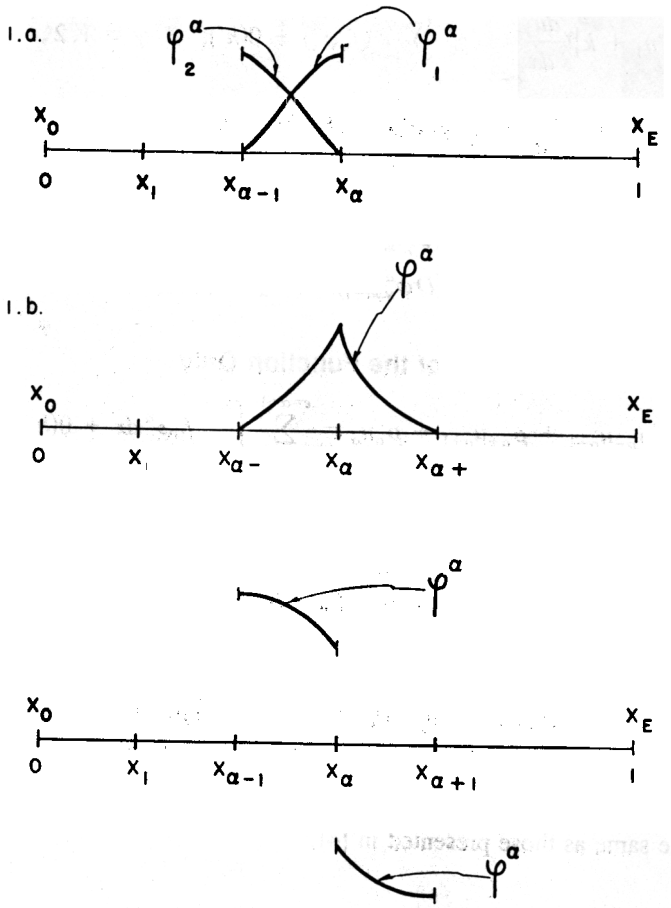


FIG. 1. The test functions. For Algorithms 1, 2, and 3 correspond to a, b, and c, respectively.

A procedure for constructing weighting functions which satisfy (27) for arbitrary r will be discussed in Section V and the relation between r and the order of accuracy of the algorithm will be explained in Section VI.

When these weighting functions are applied in (16), one gets modified versions of Eq. (25). Indeed, the results reported by Herrera and co-workers [4] are applicable with minor changes. Depending on the algorithm considered, they are as follows:

Algorithm 1—The Value of the Function and Its Derivative

For this case

$$k_{\alpha-}^{0\gamma}u_{\alpha-1} + k_\alpha^{0\gamma}u_\alpha + k_{\alpha-}^{1\gamma}\frac{du_{\alpha-1}}{dx} + k_\alpha^{1\gamma}\frac{du_\alpha}{dx} = -\langle f, \varphi_\gamma^\alpha\rangle + 0(h^r);$$

$$\alpha = 2, \ldots, E-1: \qquad \gamma = \quad 2$$

$$k_1^{0\gamma}u_1 + k_1^{1\gamma}\frac{du_1}{dx} = \langle g_0, \overset{1}{\underset{\gamma}{}}\rangle - \langle f, \varphi_\gamma^1\rangle + 0(h^r); \qquad \gamma = 1,2, \tag{29a}$$

$$k_{E-}^{0\gamma}u_{E-1} + k_{E-}^{1\gamma}\frac{du_{E-1}}{dx} = \langle g, \varphi_\gamma^E\rangle - \langle f, \varphi_\gamma^E\rangle + 0(h^r); \qquad \gamma = 1,2,$$

where

$$k_{\alpha-}^{0\gamma} = -[\![q(\varphi_\gamma^\alpha)]\!]_{\alpha-1}; \qquad k_\alpha^{0\gamma} = -[\![q(\varphi_\gamma^\alpha)]\!]_\alpha$$
$$k_{\alpha-}^{1\gamma} = [\![D\varphi_\gamma^\alpha]\!]_{\alpha-1}; \qquad k_\alpha^{1\gamma} = [\![\varphi_\gamma^\alpha]\!]_\alpha.$$

Algorithm 2—The Value of the Function Only

$$\rho_{\alpha-}u_{\alpha-1} + \rho_{\alpha+}u_{\alpha+1} + \rho_\alpha u_\alpha = \sum_{e=\alpha}^{e=\alpha+1}\int_{x_{e-1}}^{x_e} f_\Omega\varphi^\alpha\,dx + 0(h^r) \tag{31}$$

$$\rho_{1+}u_2 + \rho_1 u_1 = \int_0^{x_2} f_\Omega\varphi^\alpha\,dx - \langle g, \varphi^1\rangle + 0(h^r),$$

$$\rho_{(E-1)-}u_{E-2} + \rho_{E-1}u_{E-1} = \int_{x_E}^1 f_\Omega\varphi^\alpha\,dx - \langle g, \varphi^{E-1}\rangle + 0(h^r)$$

Here

$$\rho_{\alpha-} = -[\![q(\varphi^\alpha)]\!]_{\alpha-1}; \qquad \rho_{\alpha+} = [\![q(\varphi^\alpha)]\!]_{\alpha+1} \tag{33a}$$

$$\rho_\alpha = -[\![q(\varphi^\alpha)]\!]_\alpha. \tag{33b}$$

The results for Algorithms 3 and 4 are not given here, but they are essentially the same as those presented in [4].

V. CONSTRUCTION OF WEIGHTING FUNCTIONS

The weighting functions were constructed satisfying the adjoint equation at collocation points. There were n collocation points at each subinterval $(x_{\alpha-1}, x_\alpha)$ of the partition. Thus, for every $\alpha = 1,\ldots,E$, by construction they satisfy

$$\mathcal{L}^*\varphi^\alpha = 0, \quad \text{at} \quad x_{\alpha 1},\ldots,x_{\alpha n} \tag{34}$$

The accuracy of the resulting algorithm is enhanced if the collocation points are Gaussian at each subinterval, and this will be assumed in what follows. Polynomials of sufficiently high degree were used in order to accommodate all the conditions to be satisfied.

Algorithm 1

For each $\alpha = 1,\ldots,E$, two linearly independent functions $\{\varphi_1^\alpha, \varphi_2^\alpha\}$, associated with every subinterval $(x_{\alpha-1}, x_\alpha)$, must be constructed. They satisfy Eq. (34). This can be achieved using polynomials of degree $G = n + 1$.

The actual construction is quite simple. For φ_1^α, which vanishes identically outside the interval $(x_{\alpha-1}, x_\alpha)$, impose the additional condition

$$\varphi_1^\alpha(x_{\alpha-1}) = 0, \tag{35}$$

taking $p_1^\alpha = 1$, write

$$\varphi_1^\alpha(x) = \sum_{j=1}^{G} p_j^\alpha \xi^j \tag{36}$$

where $\xi = x - x_{\alpha-1}$. Then (Here, to simplify the presentation, D is taken to be identically 1.):

$$\mathcal{L}^*\varphi_1^\alpha = \sum_{j=1}^{G} \{j(j-1) + cj\xi + d\xi^2\}\xi^{j-2} p_j^\alpha \tag{37}$$

where $c = -2a$ and $d = b - da/dx$. Applying Eq. (24) at the collocation points and using $p_1^\alpha = 1$, it is obtained

$$\sum_{J=2}^{G} \{j(j-1) + cj\xi + d\xi^2\}\xi^{j-2} p_j^\alpha = -c\xi - d\xi^2; \qquad \text{at } n \text{ collocation points.} \tag{38}$$

The system of equations defined by (38) is $n \times n$ and can be solved for the n coefficients $p_2^\alpha, \ldots, p_G^\alpha$, since $G = n + 1$.

A similar construction applies to φ_2^α, for which one can take

$$\varphi_2^\alpha(x_\alpha) = 0; \qquad \varphi_2^\alpha(x) = \sum_{j=1}^{G} p_j^\alpha \xi^j \tag{39}$$

with $\xi = x - x_\alpha$. Once the system $\{\varphi_1^\alpha, \varphi_2^\alpha\}$ has been constructed for every $\alpha = 1, \ldots E$, one can apply formulas (28) and (29) to obtain the desired finite difference algorithms.

Algorithms 2–4

For each $\alpha = 1, \ldots, E - 1$, a single function φ^α, associated with the subinterval $(x_{\alpha-1}, x_{\alpha+1})$, must be constructed. It satisfies (34) together with the boundary conditions and the continuity conditions (26). This can be achieved using polynomials of degree $G = n + 1$ and the whole procedure for constructing the test functions is quite similar to that for Algorithm 1.

The test functions used for Algorithms 2–4 belong to the linear space spanned by those used for Algorithm 1. Due to this fact, Algorithms 2–4 can be derived from Algorithm 1 by algebraic manipulations.

VI. AN ERROR ESTIMATE

An error estimate for the finite difference algorithms obtained using the method presented in this paper has been established and the detailed derivation is carried out elsewhere [14]. Here, only a summary of results is given.

The systems of weighting functions used in Algorithms 2–4 are subsystems of the linear space spanned by those used in Algorithm 1. Using this fact, it can be seen that the error estimate for Algorithm 1 also applies to the other ones. Thus, attention will be restricted to this case.

As mentioned previously, a convenient procedure to produce the weighting functions is by collocation. When Eqs. 34 are satisfied and the collocation points are Gaussian, a standard argument yields [8]

$$\int_0^1 u' \mathcal{L}^* \varphi_\gamma^\alpha \, dx = 0(h^{2n+1}); \qquad \gamma = 1, 2. \tag{40}$$

A far more elaborated argument whose details are given in [14], shows that when (27) holds, then

$$|u'_\alpha - u_\alpha| = 0(h^{r-2}) \quad \text{and} \quad \left| \frac{du'_\alpha}{dx} - \frac{du_\alpha}{dx} \right| = 0(h^{r-2}). \tag{41}$$

Equation 40 corresponds to (27) with $r = 2n + 1$, so that the error estimate is

$$|u'_\alpha - u_\alpha| = 0(h^{2n-1}); \qquad \left| \frac{du'}{dx} - \frac{du_\alpha}{dx} \right| = 0(h^{2n-1}) \tag{42}$$

for the nodal values, when n collocation points Gaussianly distributed are used at each subinterval.

VII. NUMERICAL RESULTS

To demonstrate the applicability of the superconvergent algorithms and to verify the theoretical results, a completely general computing program was developed and implemented in an HP 9000 computer. The test functions $\varphi_\gamma^\alpha(x)$ satisfy the equation

$$\mathcal{L}^* \varphi_\gamma^\alpha(x) = 0 \quad \text{at} \quad x_{\alpha-1} < x = x_{\alpha 1}, \ldots, x_{\alpha n} < x_\alpha \tag{43}$$

where n is the number of collocation points at every subinterval $(x_{\alpha-1}, x_\alpha)$. When the functions $\varphi_\gamma^\alpha(x)$ are polynomials of degree G, it is necessary that $G = n + 1$.

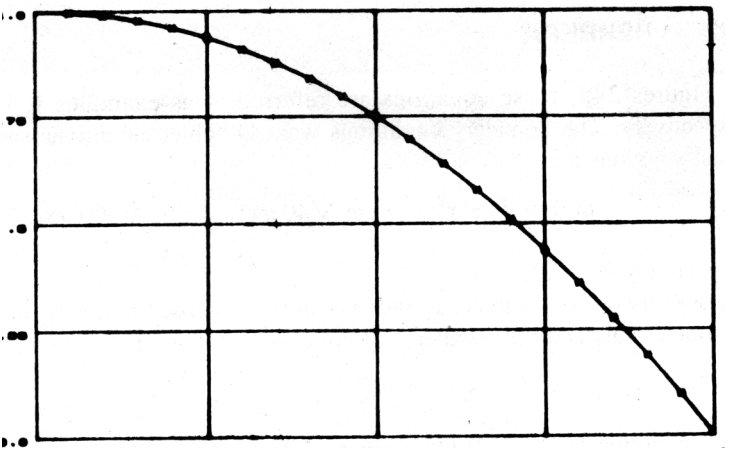
Collocation equations (43), together auxilliary or continuity conditions lead to a system of $(n + 1) \times (n + 1)$; i.e., a system of GXG, for the coefficients of $\varphi_\gamma^\alpha(x)$. Once these coefficients have been obtained, it is easy using Eqs. (28)–(33) to construct the finite difference algorithms.

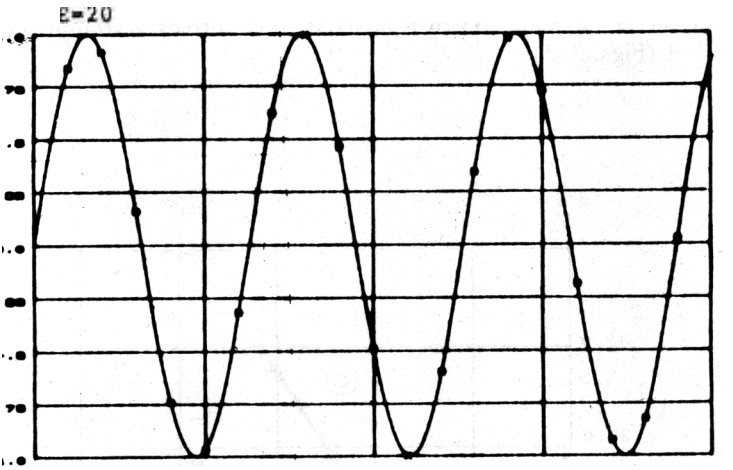
This procedure is very easy to program and turns out to be quite versatile. The computer program was tested applying it to three equations

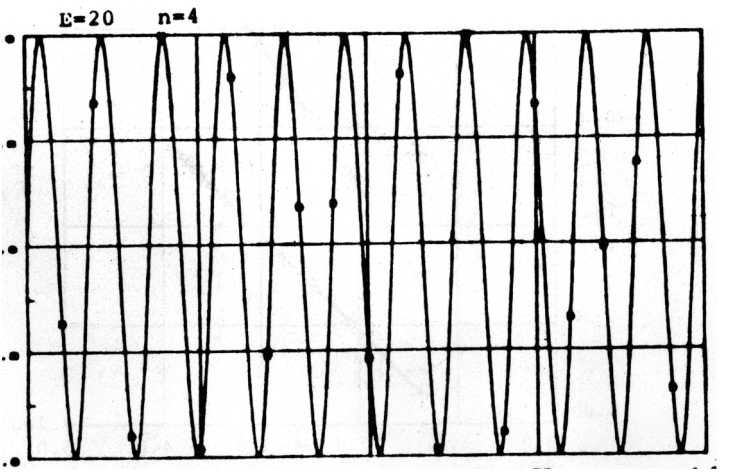
$$\frac{d^2u}{dx^2} - \frac{2x}{1 + x^2} \frac{du}{dx} + \frac{2}{1 + x^2} u = 0, \tag{44a}$$

$$\frac{d^2u}{dx^2} + 40\pi^2 u = 0, \tag{44b}$$

$$\frac{d^2u}{dx^2} + 500\pi^2 u = 0. \tag{44c}$$







examples

In Figures 2–8, these equations are referred to as examples I, II, and III, respectively. The boundary conditions were Dirichlet conditions such that the exact solutions are

$$u(x) = 1 - x^2; \qquad \sin\sqrt{40}\,\pi x; \qquad \sin\sqrt{500}\,\pi x \tag{45}$$

respectively (Fig. 2).

For every one of the three algorithms, one can choose arbitrarily the number E of subintervals and the number n of collocation points. Let

$$\varepsilon = \max_{\alpha} |u'_\alpha - u_\alpha|; \qquad \alpha = 1, \ldots, E - 1, \tag{46}$$

then according to Eq. (42) the asymptotic behavior is

$$-\log \varepsilon \geq (2n - 1) \log E - M \tag{47}$$

where M is constant. When n is fixed, this defines a straight line of slope $2n - 1$ (Figs. 3–5).

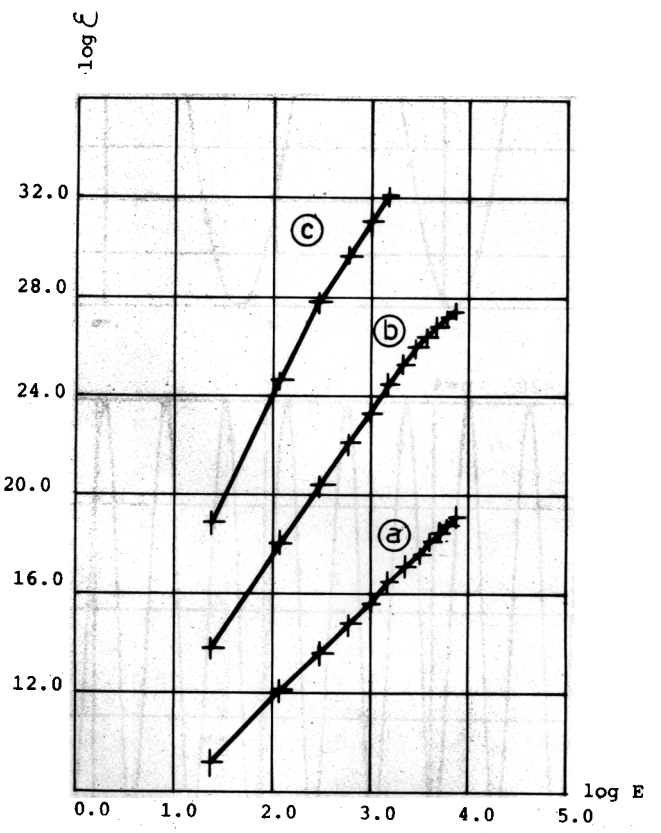


FIG. 3. Numerical results for Eq. (I) (fixing n). (a) $n = 2$ slope (th = 3 act = 4). (b) $n = 3$ slope (th = 5.0 act = 5.9). (c) $n = 4$ slope (th = 7 act = 7.5).

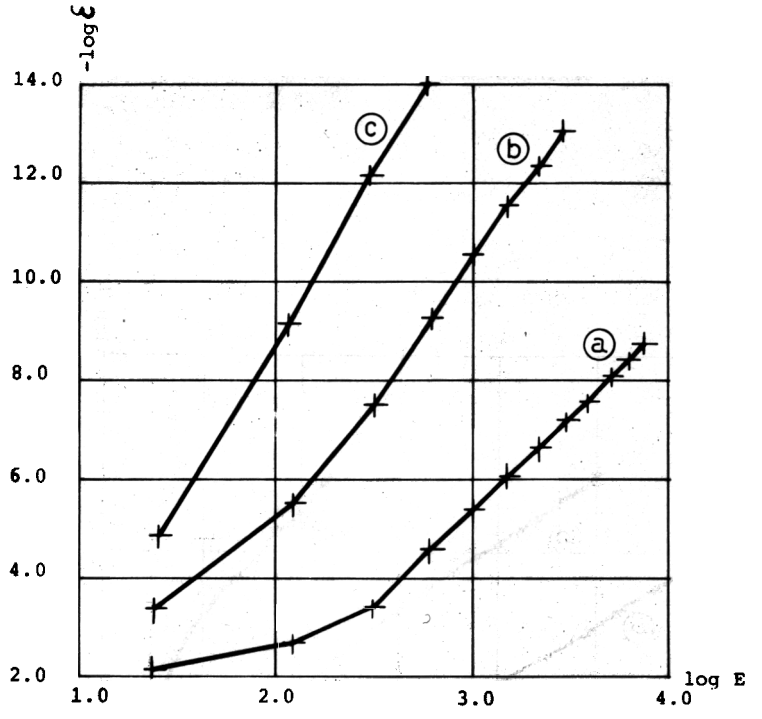


FIG. 4. Numerical results for Eq. (II) (fixing n). (a) $n = 2$ slope (th = 3.0 act = 3.6). (b) $n = 3$ slope (th = 5.0 act = 5.5). (c) $n = 4$ slope (th = 7.0 act = 7.4).

In Figures 3–5, the graphs labeled a, b, c, and d correspond to $n = 2, 3, 4,$ and 5, respectively. The theoretical predictions for the asymptotic slope agree well with the actual results, although they are slightly conservative. For example, when $n = 2$, this formula gives the values 3, and the corresponding graphs in the figures are 4, 3.6, and 3.5. For $n = 4$, the theoretical slope is 7, while the actual is 7.5, 7.4, and 7.1.

The procedure is also convergent if one keeps E fixed and increases the number n of collocation points. In this case, the numerical prediction approaches the exact values at a finite number $(E - 1)$ nodes. The error behavior for this case is illustrated in Figures 6–8 for the three examples considered. Here, the curves $a, b, c,$ etc. correspond to $E = 4, 8, \ldots, 28$.

As mentioned, the method converges when n is kept fixed and $E \to \infty$, and also when E is kept fixed and $n \to \infty$. Thus, one can refer to an "E" version and an "n" version of the "Algebraic Approach." This situation is similar to Babuska's p and h-p versions of the finite element method [15].

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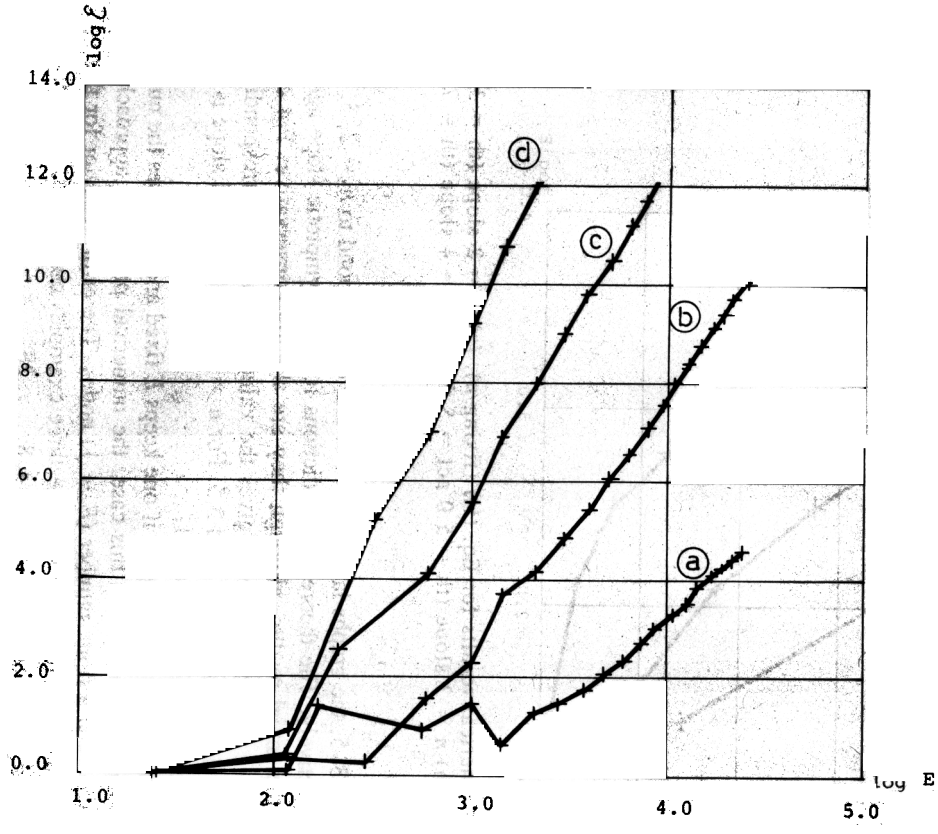
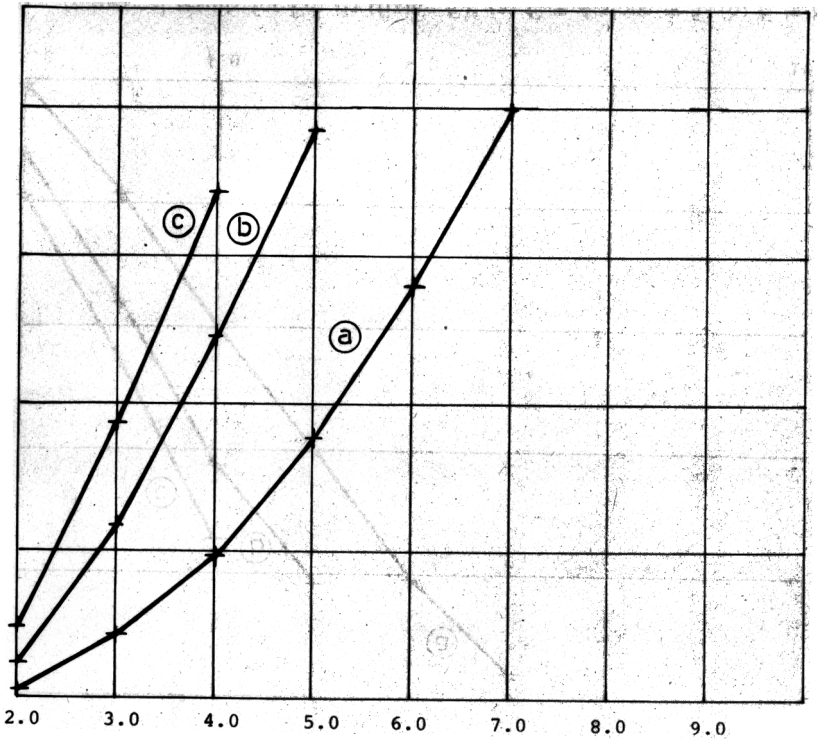


FIG. 5. Numerical results for Eq. (III) (fixing n). (a) $n = 2$ slope (th = 3.0 act = 3.5). (b) $n = 3$ slope (th = 5.0 act = 5.2). (c) $n = 4$ slope (th = 7.0 act = 7.1). (d) $n = 5$ slope (th = 9.0 act = 9.0).

FIG. 6. Numerical results for Eq. (I) (fixing E). (a) $E = 4$ slope = 5. (b) $E = 8$ slope = 6. (c) $E = 12$ slope = 7.



slope Numerical (c) E slope (fixing slope

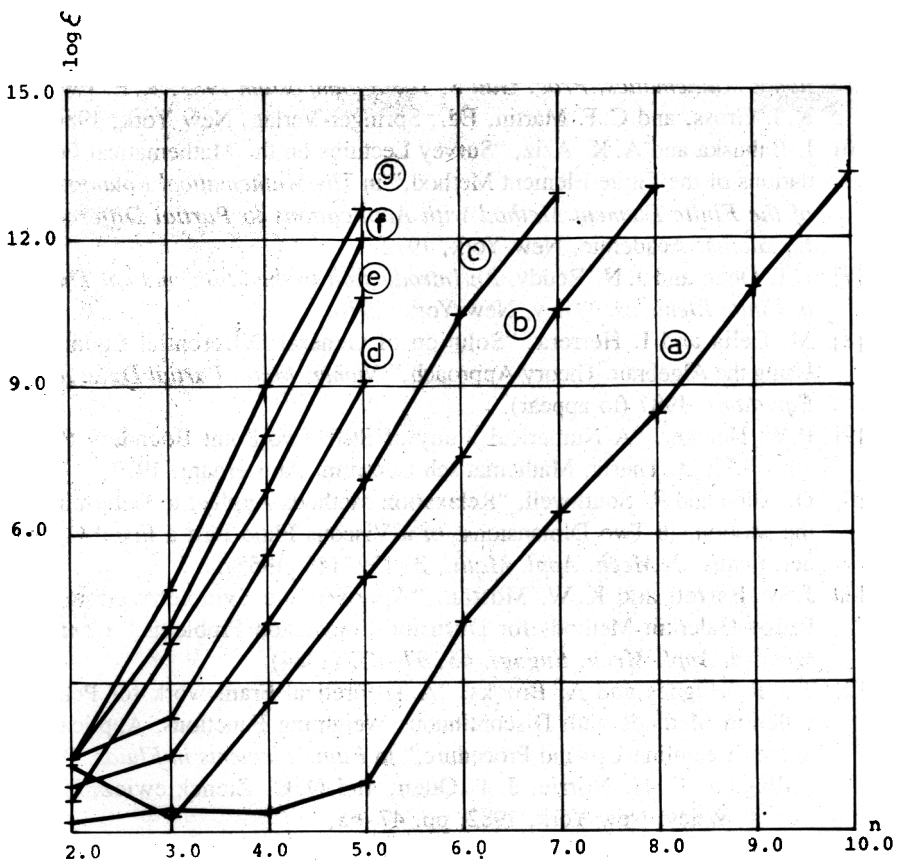


FIG. 8. Numerical results for Eq. (III) (fixing E). (a) $E = 8$ slope = 2.5. (b) $E = 12$ slope = 2.6. (c) $E = 16$ slope = 3.5. (d) $E = 20$ slope = 3.4. (e) $E = 24$ slope = 3.4. (f) $E = 28$ slope = 3.5. (g) $E = 32$ slope = 3.7.

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