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ON OPERATOR EXTENSIONS: THE ALGEBRAIC THEORY APPROACH

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Abstract. The Localized Adjoint Method (LAM) is a new and promising methodology for discretizing partial differential equations, which is based on Herrera's Algebraic Theory of Boundary Value Problems. A large number of numerical applications have already been made. Herera's Algebraic Theory implies a kind of operator extensions of great generality, which can be applied to fully discontinuous trial and test functions, simultaneously. This is in contrast with standard theory of distributions, which can be applied to discontinuous trial functions, only if test functions satisfy a corresponding degree of regularity, or viceversa. This paper is devoted to make a brief presentation of such extensions.

1. Introduction

The Localized Adjoint Method (LAM) is a new and promising methodology for discretizing partial differential equations, which is based on Herrera's Algebraic Theory of Boundary Value Problems [1]–[5]. Applications have successively been made to ordinary differential equations, for which highly accurate algorithms were developed [4], [6]–[8], multidimensional steady state problems [9] and optimal spatial methods for advection-diffusion equations [10]–[17]. More recently, in a pair of articles [18, 19], generalizations of Characteristic Methods that we refer to as Eulerian-Lagrangian Localized Adjoint Method (ELLAM), were provided. Related work has been published separately [20]–[23] and some more specific applications have already been made [24]–[29].

For differential operators, Herrera's Algebraic Theory of Boundary Value Problems imply a kind of operator extensions of great generality, since using it, fully discontinuous trial and test functions can be applied simultaneously. Actually, the operator extensions implied by the Algebraic Theory (the "algebraic extensions"), yield extensions of distributional operators, because the distributional extensions coincide with the algebraic extensions, whenever the former are defined. However, the operator extensions implied by the Algebraic Theory are well defined, in cases for which the distributional definitions are not. This is the case, for example, when trial and test functions are fully discontinuous.

The definition of the algebraic extensions is based on an algebraic structure which systematically occurs in boundary value problems [2, 5]. In the present paper a comparison is made with the distributional approach [30, 31]. It must be mentioned that although in previous work, attention has been mainly devoted to analyze the implications of the theory for single differential equations, the manner of applying it to systems of equations has been explained in [22]. The interested reader may

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find more thorough expositions of the algebraic structure in [2, 5]. A more recent exposition presenting several aspects of the algebraic structure in a more complete manner, is given in [32] and a more systematic derivation of the operator extensions from such algebraic structure, will appear in [33]. A monograph, in which the discussion was restricted to symmetric operators, has already appeared in book form [1].

The operator extensions implied by the author's Algebraic Theory (the algebraic extensions), are introduced in Section 2. In Section 3, a sketch of the proof that the algebraic extension is indeed an extension of the distributional definition, is given. Section 4 is devoted to present simple illustrations of the results produced by the algebraic extensions.

2. Operator extensions

Consider a region Ω and for simplicity, assume the spaces of trial and test functions, defined in Ω, are the same linear space: D (i.e., $D = D_1 = D_2$). Assume further, that functions belonging to D may have jump discontinuities across some internal boundaries whose union will be denoted by Σ. For example, in applications of the theory to finite element methods, the set Σ would be the union of all the interelement boundaries.

To be specific, consider a linear differential operator $\mathcal{L}$ of order m and assume $\{\Omega_1, \ldots, \Omega_E\}$ is a partition of Ω. More precisely, $\{\Omega_1, \ldots, \Omega_E\}$ is a collection of disjoint open regions (the "elements") of Ω, such that Ω is contained in the closure of the union of $\{\Omega_1, \ldots, \Omega_E\}$. Then, one can define $D = H^m(\Omega_1) \oplus \cdots \oplus H^m(\Omega_E)$. In this case $\Sigma = \Omega - (\Omega_1 \cup \ldots \cup \Omega_E)$.

The definition of formal adjoint requires that a differential operator $\mathcal{L}$ and its formal adjoint $\mathcal{L}^*$, satisfy the condition that $w\mathcal{L}u - u\mathcal{L}^*w$ be a divergence; i.e.:

$$w\mathcal{L}u - u\mathcal{L}^*w = \nabla \cdot \{\underline{\mathcal{D}}(u, w)\} \tag{1}$$

for a suitable vector-valued bilinear function $\underline{\mathcal{D}}(u, w)$, which involves derivatives up to order $m - 1$. Integration of (1) over Ω and application of generalized divergence theorem [34], yield:

$$\sum_i \int_{\Omega_i} \{w\mathcal{L}u - u\mathcal{L}^*w\}\, dx = \int_{\partial\Omega} \mathcal{R}_\partial(u, w)\, dx + \int_\Sigma \mathcal{R}_\Sigma(u, w)\, dx, \tag{2}$$

where

$$\mathcal{R}_\partial(u, w) = \underline{\mathcal{D}}(u, w) \cdot \underline{n} \quad \text{and} \quad \mathcal{R}_\Sigma(u, w) = -[\underline{\mathcal{D}}(u, w)] \cdot \underline{n}. \tag{3}$$

Here, as in what follows, the square brackets stand for the "jumps" across Σ of the function contained inside; i.e., limit on the positive side minus limit on the negative one. The positive side of Σ is chosen arbitrarily and then the unit normal vector $\underline{n}$, is taken pointing towards the positive side of Σ. The operators $\mathcal{L}$ and $\mathcal{L}^*$ are understood in a distributional sense, and since they are of order m, both $\int_{\Omega_i} w\mathcal{L}u\, dx$ and $\int_{\Omega_i} u\mathcal{L}^*w\, dx$ are well defined for every $i = 1, \ldots, E$. However, observe that $D \subset H^0(\Omega)$, but the relation $D \subset H^1(\Omega)$ does not hold, so that when $u \in D$ one can only grant that $\mathcal{L}u \in H^{-m}(\Omega)$ and $\mathcal{L}^*w \in H^{-m}(\Omega)$. Thus, $\int_\Omega w\mathcal{L}u\,dx$ and $\int_\Omega u\mathcal{L}^*w\,dx$ are not well defined for every $u \in D$ and $w \in D$. This Section is devoted to present extensions $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^*$ (the algebraic extensions), of $\mathcal{L}$ and $\mathcal{L}^*$, respectively, for which $\int_\Omega w\hat{\mathcal{L}}u\,dx$ and $\int_\Omega u\hat{\mathcal{L}}^*w\,dx$ are well defined for every $u \in D$ and $w \in D$.

In the general theory of partial differential equations, Green's formulas are used extensively [31]. For the construction of such formulas, it is standard to introduce a decomposition of the bilinear function $\mathcal{R}_\partial$ (see, for example, Lions and Magenes [31], Vol. I, pp. 114–115). Indicating, as it is usual, transposes of bilinears forms by means of a star, the general form of such decomposition is:

$$\mathcal{R}_\partial(u,w) \equiv \underline{\mathcal{D}}(u,w)\cdot\underline{n} = \mathcal{B}(u,w) - \mathcal{C}^*(u,w) \tag{4}$$

where $\mathcal{B}(u,w)$ and $\mathcal{C}^*(u,w)$ are two bilinear functions, which involve derivatives up to order $m-1$. When considering initial-boundary value problems, the definitions of these bilinear forms depend on the type of boundary and initial conditions to be prescribed. A basic property required of $\mathcal{B}(u,w)$ is that for any u which satisfies the prescribed boundary and initial conditions, $\mathcal{B}(u,w)$ is a well-defined linear function of w, independent of the particular choice of u. This linear function will be denoted by g_∂ (thus, its value for any given function w, will be $g_\partial(w)$) and the boundary conditions can be specified by requiring that $\mathcal{B}(u,w) = g_\partial(w)$, for every $w \in D$ (or more briefly: $\mathcal{B}(u,\cdot) = g_\partial$). For example, for Dirichlet problem of Laplace Equation, $\mathcal{B}(u,w)$ can be taken to be $u\partial w/\partial n$, on $\partial\Omega$ [19]. Thus, if u_∂ is the prescribed value of u on $\partial\Omega$, one has $\mathcal{B}(u,w) = u_\partial\partial w/\partial n$, for any function u which satisfies the boundary conditions. Thus, $g_\partial(w) = u_\partial\partial w/\partial n$, in this case.

The linear function $\mathcal{C}^*(u,\cdot)$, on the other hand, can not be evaluated in terms of the prescribed boundary values, but it also depends exclusively, on certain boundary values of u (the "complementary boundary values"). Generally, such boundary values can only be evaluated after the initial-boundary value problem has been solved. Taking again the example of Dirichlet problem for Lapalce Equation, $\mathcal{C}^*(u,w) = w\partial u/\partial n$ and the complementary boundary values, correspond to the normal derivative on $\partial\Omega$ [19].

In a similar fashion, convenient formulations of boundary value problems with prescribed jumps, requires constructing Green's formulas in discontinuous fields. This can be done by means of a general decomposition of the bilinear function $\mathcal{R}_\Sigma(u,w)$ that has been introduced by the author [22] (see also [19]) and whose definition is point-wise on Σ. The general theory includes the treatment of differential operators with discontinuous coefficients [4]. However, for simplicity in this article only continuous coefficients will be considered. In this case, such decomposition is easy to obtain and it stems from the algebraic identity:

$$[\underline{\mathcal{D}}(u,w)] = \underline{\mathcal{D}}([u],\dot{w}) + \underline{\mathcal{D}}(\dot{u},[w]) \tag{5}$$

where

$$[u] = u_+ - u_-, \qquad \dot{u} = (u_+ + u_-)/2 \tag{6}$$

The desired decomposition is obtained combining the second of Equs. (3) and (5):

$$\mathcal{R}_\Sigma(u,w) = \mathcal{J}(u,w) - \mathcal{K}^*(u,w) \tag{7}$$

where

$$\mathcal{J}(u,w) = -\underline{\mathcal{D}}([u],w)\cdot\underline{n} \tag{8a}$$

$$\mathcal{K}^*(u,w) = \mathcal{K}(w,u) = \underline{\mathcal{D}}(\dot{u},[w])\cdot\underline{n} \tag{8b}$$

Observe that the expressions for $\mathcal{J}(u,w)$ and $\mathcal{K}^*(u,w)$, involve jumps and averages across Σ, of u, w and their derivatives up to order $m-1$.

An important property of the bilinear functional $\mathcal{J}(u,w)$ is that when the jumps of u and its derivatives up to order $m-1$, are specified, it defines a unique linear function of w, which is independent of the particular choice of the function u, as long as it satisfies the prescribed jump conditions. When considering initial-boundary value problems with prescribed jumps, the linear function defined by the prescribed jumps in this manner, is denoted by j_Σ (thus, its value for any given function w, will be $j_\Sigma(w)$) and the jump conditions at any point of Σ, can be specified by means of the equation: $\mathcal{J}(u,\cdot) = j_\Sigma$ [19]. In problems with prescribed jumps, the linear function $\mathcal{K}^*(u,\cdot)$, plays a role similar to the complementary boundary values $\mathcal{C}^*(u,\cdot)$. It can only be evaluated after the initial-boundary value problem has been solved and certain information about the average of the solution and its normal derivatives on Σ, is known (see Equ. (8b)). Such information, is called the "generalized averages" [2, 4, 19].

Introducing the notation

$$\langle Pu,w\rangle = \sum_i \int_{\Omega_i} w\mathcal{L}u\,dx; \qquad \langle Q^*u,w\rangle = \sum_i \int_{\Omega_i} u\mathcal{L}^*w\,dx \tag{9a}$$

$$\langle Bu,w\rangle = \int_{\partial\Omega} \mathcal{B}(u,w)\,dx; \qquad \langle C^*u,w\rangle = \int_{\partial\Omega} \mathcal{C}(w,u)\,dx \tag{9b}$$

$$\langle Ju,w\rangle = \int_\Sigma \mathcal{J}(u,w)\,dx \quad \text{and} \quad \langle K^*u,w\rangle = \int_\Sigma \mathcal{K}(w,u)\,dx \tag{9c}$$

equation (2), can be written as:

$$\langle Pu,w\rangle - \langle Q^*u,w\rangle = \langle Bu,w\rangle - \langle C^*u,w\rangle + \langle Ju,w\rangle - \langle K^*u,w\rangle \tag{10}$$

This is an identity between bilinear forms and as such, can be written more briefly, after rearranging, as:

$$P - B - J = Q^* - C^* - K^* \tag{11}$$

This is *Green-Herrera formula for operators in discontinuous fields* [2, 5, 19].

It can be shown [33] that the pair of operators $\{J, -K^*\}$ constitutes a *weak decomposition* of $(P-B)-(Q-C)^*$, that B and J are *boundary operators* for P, which are *fully disjoint* and that (11) is indeed a Green's formula, in the *weak sense*. On the other hand, when $\mathcal{J}$ and $\mathcal{K}^*$ are defined by (8), then the pair of bilinear functionals $\{\mathcal{J}, -\mathcal{K}^*\}$, constitutes a strong decomposition, point-wise, of the bilinear functional $\mathcal{R}_\Sigma$ which is defined point-wise, also [33].

The algebraic extension $\hat{\mathcal{L}}$ of the distributional operator $\mathcal{L}$, is defined to be the bilinear functional $P-J$. More precisely, $\hat{\mathcal{L}}$ is defined by:

$$\int_\Omega w\hat{\mathcal{L}}u\,dx \equiv \langle (P-J)u,w\rangle \tag{12}$$

which holds whenever $u \in D$ and $w \in$D. Similarly, the operator extension corresponding to $\hat{\mathcal{L}}^*$ is defined to be the bilinear functional $Q^* - K^*$; i.e.:

$$\int_\Omega u\hat{\mathcal{L}}^* w\, dx \equiv \langle (Q-K)^* u, w\rangle \tag{13}$$

which also holds when both u and w belong to D. Thus, using these operator extensions, Green-Herrera formula (11) can be written as:

$$\int_\Omega w\hat{\mathcal{L}} u\, dx - \int_\Omega u\hat{\mathcal{L}}^* w\, dx \equiv \langle (B - C^*)u, w\rangle \tag{14}$$

for elements $u \in D$ and $w \in D$.

3. Comparison between $\hat{\mathcal{L}}$ and $\mathcal{L}$

Since the definitions for $\int_\Omega w\mathcal{L}u\, dx$ and $\int_\Omega u\mathcal{L}^* w\, dx$, which are standard in the theory of distributions, can not be applied to all possible pairs $\{u, w\}$, such that $u \in D$ and $w \in D$, the algebraic extensions $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^*$ were introduced in the last Section, for which both $\int_\Omega w\hat{\mathcal{L}}u\, dx$ and $\int_\Omega u\hat{\mathcal{L}}^* w\, dx$ are well defined, whenever $u \in D$ and $w \in D$. It can be shown that the operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}^*$, defined by Equs. (12) and (13), are indeed extensions of the distributional operators $\mathcal{L}$ and $\mathcal{L}^*$, respectively, and the main purpose of this Section is to briefly explain a proof of this result. To achieve this goal, it is only necessary to prove that for every $u \in D$ and $w \in D$, the following two implications hold:

$$\int_\Omega w\mathcal{L}u\, dx \text{ is defined } \Rightarrow \int_\Omega w\mathcal{L}u\, dx = \int_\Omega w\hat{\mathcal{L}}u\, dx \tag{15a}$$

and

$$\int_\Omega u\mathcal{L}^* w\, dx \text{ is defined } \Rightarrow \int_\Omega u\mathcal{L}^* w\, dx = \int_\Omega u\hat{\mathcal{L}}^* w\, dx \tag{15b}$$

We only sketch a proof of implications (15) for the case when the order of the operator is 1 (i.e., $\mathcal{L} \equiv A(\underline{x})\partial/\partial x_i + B(\underline{x})$, where i may be $1, \ldots, N$, while the coefficients $A(\underline{x})$ and $B(\underline{x})$ are given functions of $\underline{x}$), since the result for the case when $\mathcal{L}$ is of arbitrary order, can be derived from this case, by induction on the order of the operator (see [33] for details). For this choice of $\mathcal{L}$, one has $\mathcal{L}^* w \equiv -\partial(Aw)/\partial x_i + Bw$ and $\underline{\mathcal{D}}(u, w) \cdot \underline{n} = Auwn_i$, so that

$$\mathcal{J}(u, w) = A[u]\dot{w}n_i \quad \text{and} \quad \mathcal{K}^*(u, w) = -A\dot{u}[w]n_i \tag{16}$$

by virtue of Equ. (8a). Actually, only the implication (15a) will be shown, since the proof of (15b) is similar. When $u \in D \subset H^0(\Omega)$ and $w \in H^1(\Omega)$ or when $u \in H^1(\Omega)$ and $w \in D \subset H^0(\Omega)$, $\int_\Omega w\mathcal{L}u\, dx$ is defined. Consider first the case when $u \in H^1(\Omega)$ and $w \in D \subset H^0(\Omega)$. In this case

$$\int_\Omega w\mathcal{L}u\, dx = \sum_i \int_{\Omega_i} w\mathcal{L}u\, dx \tag{17}$$

since both w and $\mathcal{L}u$ belong to $H^0(\Omega) = L^2(\Omega)$. In addition, when $u \in H^1(\Omega)$, u is continuous and $\mathcal{J}(u,w) \equiv 0$ on Σ, by virtue of Equ. (16). Thus, $Ju = 0$. This proves that

$$\int_\Omega w\hat{\mathcal{L}}u\,dx = \langle (P-J)u, w\rangle = \langle Pu, w\rangle = \sum_i \int_{\Omega_i} w\mathcal{L}u\,dx \tag{18}$$

Comparing Equs. (17) and (18), the desired equality follows.

If $u \in D \subset H^0(\Omega)$ and $w \in H^1(\Omega)$, a standard Green's formula used in the theory of distributions (see p. 115 of Lions and Magenes [31]), yields:

$$\int_\Omega w\mathcal{L}u\,dx = \int_\Omega u\mathcal{L}^*w\,dx + \langle (B-C^*)u, w\rangle = \sum_i \int_{\Omega_i} u\mathcal{L}^*w\,dx + \langle (B-C^*)u, w\rangle \tag{19}$$

The last equality holds because u and $\mathcal{L}^*w$ belong to $H^0(\Omega) = L^2(\Omega)$. On the other hand, using Green-Herrera formula (11), it is seen that

$$\int_\Omega w\hat{\mathcal{L}}u\,dx = \langle (P-J)u, w\rangle = \langle (Q^*-K^*)u, w\rangle + \langle (B-C^*)u, w\rangle \tag{20}$$

However, w is continuous, because $w \in H^1(\Omega)$. Thus, $\mathcal{K}(w,\cdot) \equiv 0$ on Σ, by virtue of Equ. (16), and $Kw = 0$. Hence, $\langle K^*u, w\rangle = \langle Kw, u\rangle = 0$. Using this fact, Equ. (20) reduces to

$$\int_\Omega w\hat{\mathcal{L}}u\,dx = \langle Q^*u, w\rangle + \langle (B-C^*)u, w\rangle = \sum_i \int_{\Omega_i} u\mathcal{L}^*w\,dx + \langle (B-C^*)u, w\rangle \tag{21}$$

Comparing this equation with (19), the desired result follows.

4. Examples

As a first illustration, let us consider the operators $\mathcal{L}$ and $\hat{\mathcal{L}}$, in the case when the distributional operator $\mathcal{L} \equiv d/dx$, the region Ω is the interval $(-1,1)$ of the real line and the partition of Ω is made of two subintervals: $\Omega_1 = (-1,0)$ and $\Omega_2 = (0,1)$. Then $\mathcal{L}^* \equiv -d/dx$, while $\mathcal{D}(u,w) \equiv uw$. Let the function u be defined by: $u = 0$ for $-1 < x < 0$ and $u = 1$ for $0 \le x < 1$. Thus, u is essentially, a Heaviside step function. The test function w will be taken having different degrees of smoothness.

Case A. $w \in H^1(\Omega)$, so that w is continuous.

i) In this case, application of a Green's formul operators (see [31], p.115) yields:

$$\int_{-1}^{1} w\mathcal{L}u\,dx = \int_{-1}^{1} u\mathcal{L}^*w\,dx + (uw)|_{-1}^{1}$$

and evaluating, it is obtained

$$\int_{-1}^{1} w\mathcal{L}u\,dx = -\int_0^1 \frac{dw}{dx}\,dx + w(1) = -w|_0^1 + w(1) = w(0). \tag{22a}$$

This result is standard. In essence, it establishes that du/dx is a Dirac's Delta function when u is a Heaviside step function.

ii) Using the fact that $\mathcal{D}(u, w) \equiv uw$ and applying Equ. (8a), it is seen that

$$\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx = \sum_i \int_{\Omega_i} w\mathcal{L}u\, dx + (\dot{w}[u])_{x=0} = w(0) \tag{22b}$$

since $[u]_{x=0} = 1$, while $\dot{w}(0) = w(0)$, because w is continuous.

Case B. w has a jump discontinuiy at $x = 0$, so that $w \in D$ but $w \notin H^1(\Omega)$.

i) $\int_{-1}^{1} w\mathcal{L}u\, dx$ is not defined.

ii) $\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx$ is well defined and it is still given by (22b), except that $\dot{w}(0) \neq w(0)$, so that

$$\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx = \dot{w}(0) \tag{23}$$

It is recalled that $\dot{w}(0) = (w(0^+) + w(0^-))/2$.

As a second illustration, replace d/dx by d^2/dx^2, in the previous example. Then $\mathcal{L}^* \equiv \mathcal{L}$, while $\mathcal{D}(u, w) \equiv w\dfrac{du}{dx} - u\dfrac{dw}{dx}$ and proceeding as before:

Case A. $w \in H^2(\Omega)$, so that w is continuous, with continuous first order derivative.

i) In this case, as before, application of a Green's formula yields:

$$\int_{-1}^{1} w\mathcal{L}u\, dx = \int_{-1}^{1} u\mathcal{L}^* w\, dx + (wu' - uw')|_{-1}^{1}$$

and evaluating, it is obtained

$$\int_{-1}^{1} w\mathcal{L}u\, dx = \int_{0}^{1} w''\, dx - w'(1) = -w'(0) \tag{24a}$$

This is a standard result. In essence, it establishes that u'' is the derivative of Dirac's Delta function, when u is a Heaviside step function.

ii) Using the fact that $\mathcal{D}(u, w) \equiv wu' - uw'$ and applying Equ. (8a), it is seen that

$$\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx = \sum_i \int_{\Omega_i} w\mathcal{L}u\, dx + (\dot{w}[u'] - \dot{w}'[u])_{x=0} = -w'(0) \tag{24b}$$

where the fact that $\dot{w}'(0) = w'(0)$, because w' is continuous, has been used.

Case B. w' has a jump discontinuity at $x = 0$, so that $w \in D$ but $w \notin H^2(\Omega)$.

i) $\int_{-1}^{1} w\mathcal{L}u\, dx$ is not defined.

ii) $\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx$ is well defined and it is still given by (24b), except that $\dot{w}'(0) \neq w'(0)$, so that

$$\int_{-1}^{1} w\hat{\mathcal{L}}u\, dx = -\dot{w}'(0) \tag{25}$$

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