Computational Modelling of Free and Moving Boundary Problems II

Editors:
L.C. Wrobel
Wessex Institute of Technology
&
C.A. Brebbia
Wessex Institute of Technology

Computational Mechanics Publications
Southampton Boston
1993

Invited Paper

Mechanisms of shock generation in variable bubble point systems

I. Herrera[a], R. Camacho[b], A. Galindo[c]

[a]*Instituto de Geofísica, National University of Mexico (UNAM), Apdo Postal 22-582, Mexico D.F., Mexico*

[b]*PEMEX and UNAM, Mexico City, Mexico*

[c]*Instituto Mexicano del Petroleo and UNAM, Mexico City, Mexico*

ABSTRACT

In a sequence of papers the authors have investigated shock modeling in miscible displacement, specially in connection with variable bubble-point problems in Petroleum Engineering. The conclusions in connection with the mechanisms of shock generation are summarized here. The main conclusion is that there is a clear difference between the mechanisms of shock generation in miscible and immiscible displacements. In particular, Bucley-Leverett Theory which has been the main tool to understand shocks in immiscible displacement, is not applicable to shocks in miscible displacement models. In addition, some of the implications of these results, for the numerical modeling of shocks in miscible displacement are discussed.

1 INTRODUCTION

In a sequence of papers [1,2,3], the authors have investigated the different kinds of shocks that can be generated when modeling petroleum reservoirs and the procedures available for numerically modeling them. This paper is devoted to summarize our conclusions, so far, in connection with shock generation.

When discussing this topic, it is necessary to distinguish between "miscible" and "immiscible" displacement. By immiscible displacement, it is usually understood [4], one in which the different phases involved do not mix at all and there is not

mass exchange between them. This is the case, for example, when water is injected through some wells, in secondary recovery.

On the other hand, if complete mixing or "miscibility" is attained, so that only one phase is formed, the term "miscible" displacement is most frequently applied. This is the case, for example, when CO_2 is injected to form a single fluid phase with the resident hydrocarbon.

Another situation that can occur [1], to which we will refer as "partial mixing", is the case when two or more phases can exchange mass but in which complete mixing does not take place, so that each one of the different phases keeps its own identity. This happens, for example, in reservoirs containing liquid oil and soluble gas, when the bubble point varies and a gas phase is present, at least in part of the region modeled.

The starting point for the understanding of shocks in immiscible displacement, was the classical Buckley-Leverett theory [5-7], which was further enlighten by the work of Cardwell and Sheldon [8,9], who explained clearly the way in which shocks are generated in such processes. The mechanism in this case, is similar to that occuring in the theory of inviscid compressible fluids, in which shocks are generated when characteristics intersect. For a more complete explanation of these points, the reader is referred to [1].

Shocks which occur when the phases are treated as partially miscible, as is the case when a gas front advances into a region occupied by undersaturated liquid oil, are not generated by the crossing of characteristics and have not received as much attention. In this paper it is shown, that such shocks are generated by the sudden transformation of an undersaturated oil particle into a saturated one, when such particle is reached by a saturating phase, as a gas phase. For immiscible displacement, it is generally accepted that shocks can occur only when capillary forces are neglected. On the other hand, for partially miscible displacement, the results summarized in this paper indicate that shocks may occur even when capillary forces are taken into account, in the beta or

black-oil model. This last statement implies an evolution in the understanting of the processes of shock generation, with respect to previous works (see for example [1]).

One of the most successful methods that have been proposed for modeling shocks in immiscible displacement, is "front tracking". This was introduced by Richtmyer [10], and was developed extensively by Glimm, McBryan and coworkers (see, for example [11]). Many descriptions of the method at different states of development have been published (see, for example [12] or [13]). The reader is referred to [1], for a more detailed discussion.

One of our conclusions, which is specially relevant for the numerical modeling of shocks, is that the front-tracking method, which is based on the use of characteristics, is not applicable to partially miscible displacement, as is explained in Section 5. Due to this fact, it is necessary to look for competitive alternatives. For this purpose, the authors have proposed a procedure: the Eulerian-Lagrangian modeling of shocks [3]. This method is being tested at present, and some publications have already been devoted to it [1,2]. However, additional research which is needed, is underway. The interested reader is referred to the above publications, since in this paper we have prefered restricting our attention to the mechanisms of shock generation, exclusively.

To give our developments a firm basis, the results presented in this article are derived from first principles. Thus, Section 2 is devoted to present the basic equations, which constitute the starting point of our developments. Section 3, in which shock generation in immiscible displacement is discussed, the classical Buckley-Leverett theory is revisited. Shock generation in partially miscible displacement is discussed in Section 4. Some of the implications that the results presented have in the modeling of shocks, are examined in Section 5.

2 THE BASIC EQUATIONS

To give to our developments a firm physical and mathematical basis, we start from first principles. In the case of multi-phase systems, each phase α moves with its own particle velocity $\underline{v}^{\alpha}$. Here, $\alpha = 1, \ldots, N$, where N is the total number of components. In any phase there may be several components, but all components contained in the same phase move with the same velocity. The balance equations satisfied by any intensive property ψ^{α} associated with component α, are (see the Appendix):

$$\psi^{\alpha}_{t} + \nabla\cdot(\psi^{\alpha}\underline{v}^{\alpha}) - \nabla\cdot\underline{\tau}^{\alpha} = g^{\alpha} \qquad (1a)$$

and

$$[\psi^{\alpha}(\underline{v}^{\alpha} - \underline{v}_{\Sigma}) - \underline{\tau}^{\alpha}]\cdot\underline{n} = g^{\alpha}_{\Sigma} \qquad (1b)$$

Here, the vector $\underline{\tau}^{\alpha}$, is the **flux** of ψ^{α} across surfaces in space, while the quantities g^{α} and g^{α}_{Σ} represent **external supply** of ψ^{α} [14,15], per unit volume and unit time, in the case of g^{α}, while g^{α}_{Σ} represents **external supply** of ψ^{α} through the discontinuity, per unit area and unit time. In addition, $\underline{v}_{\Sigma}$ stands for the velocity with which the discontinuity moves. In all the applications that follow, the intensive properties are densities (mass per unit of total volume) of each one of the components of the systems to be considered. In them, the fluxes $\underline{\tau}^{\alpha}$, are produced by diffusive processes such as molecular diffusion and dispersion.

Consider a "black oil" or "beta" model [16], which is based on the following assumptions:

a).- There are three phases: water, liquid oil and gas (whose particle velocities will be denoted by $\underline{v}^{w}$, $\underline{v}^{o}$ and $\underline{v}^{g}$, respectively);

b).- Water and oil are immiscible, while gas is soluble only in liquid oil; i.e. the water and gas phases consist of only one component, while the liquid oil is made of two components (dissolved gas and non-volatile oil). This implies that the total number of components are four and that the latter two components move with the same

velocity; and

c).- No physical diffusion is present. This includes both molecular diffusion and that induced by the randomness of the porous medium (dispersion).

It is important to observe that these assumptions do not exclude capillary pressure.

In what follows, the notations $\bar{\rho}_o$ and $\bar{\rho}_{dg}$, for the effective densities of non-volatile oil and dissolved gas, respectively, together with the relation

$$\bar{\rho}_{dg} = \bar{R}_s \bar{\rho}_o \equiv \frac{\rho_{gSTC}}{\rho_{oSTC}} R_s \bar{\rho}_o \qquad (2)$$

will be used. Here, the factor R_s is the "solution gas:oil ratio" [16].

A straight-forward application of Equ. (1a), yields:

$$(\phi S_w \rho_w)_t + \nabla\cdot(\phi \rho_w S_w \underline{v}^w) = 0 \qquad (3)$$

$$(\phi S_o \bar{\rho}_o)_t + \nabla\cdot(\phi \bar{\rho}_o S_o \underline{v}^o) = 0 \qquad (4)$$

$$(\phi S_o \bar{R}_s \bar{\rho}_o)_t + \nabla\cdot(\phi \bar{R}_s \bar{\rho}_o S_o \underline{v}^o) = g^o_{Ig} \qquad (5)$$

$$(\phi S_g \rho_g)_t + \nabla\cdot(\phi \rho_g S_g \underline{v}^g) = g^g_{Io} \qquad (6)$$

as the governing differential equations of the black oil model [16]. Here, g^o_{Ig} is the mass of gas that is dissolved in the liquid oil per unit volume per unit time, while g^g_{Io} is the mass of dissolved oil that goes into the gas phase, per unit volume per unit time, and the extraction terms have been set equal to zero. Clearly

$$g^o_{Ig} + g^g_{Io} = 0 \qquad (7)$$

for mass conservation.

When shocks occur, each one of the four components must satisfy the jump conditions which are implied by mass balance. By virtue of Equ. (1b), they are:

$$[\phi \rho_w S_w (\underline{v}^w - \underline{v}_\Sigma)]\cdot\underline{n} = 0 \qquad (8a)$$

$$[\phi \bar{\rho}_o S_o (\underline{v}^o - \underline{v}_\Sigma)]\cdot\underline{n} = 0 \qquad (8b)$$

$$[\phi \bar{\rho}_o S_o \bar{R}_s (\underline{v}^o - \underline{v}_\Sigma)]\cdot\underline{n} = g^o_{\Sigma g} \qquad (9a)$$

$$[\phi\ \rho_g S_g (\underline{v}^g - \underline{v}_\Sigma)] \cdot \underline{n} = g^g_{\Sigma o} \qquad (9b)$$

In addition, Darcy's Law requires:

$$[p_l] = 0\ ;\ l = w,\ o,\ g \qquad (10)$$

Above, the quantities $g^o_{\Sigma g}$ and $g^g_{\Sigma o}$ stand for the exchange of mass between the gaseous phase and the liquid oil which takes place on Σ. As before, mass conservation requires:

$$g^o_{\Sigma g} + g^g_{\Sigma o} = 0 \qquad (11)$$

When these quantities are different from zero, and this is the case at the gas front when it advances into a region of undersaturated oil, a mass exchange concentrated on the surface Σ, between the gaseous and the liquid oil phases must occur. This is in contrast with the quantities g^o_{Ig} and g^g_{Io} of Equs. (5) and (6), which represent a mass exchange distributed on a volume and not concentrated on a surface.

3 IMMISCIBLE DISPLACEMENT

In previous work [1], an analysis of the different processes of shock generation that can occur in multiphase flow, has been carried out. In that research, two different processes of shock generation have been identified. Namely:

i).- Intersection of characteristics, just like in flow of compressible fluids; and ii).- The sudden transformation of an undersaturated particle into a saturated one.

The first one of these mechanisms takes place in immiscible displacement. Such process is described by the classical Buckley-Leverett Theory. On the other hand, the second one occurs in partially miscible displacement, as for example, when gas invades a region of undersaturated oil.

A.- Immiscible Displacement

In this case, shocks are generated by the first of these mechanisms, exclusively [1]. For the sake of completeness and comparison, Buckley-Leverett Theory is revised briefly in this Section, where, our discussion will be restricted to the case

when only two phases are present and each one of them is made of one component: non-volatil oil and the displacing fluid. No mass exchange between these phases is assumed and capillary pressure is neglected.

The Darcy velocities are defined by:

$$\underline{u}_\alpha = \phi\, S_\alpha\, \underline{v}^\alpha ; \qquad \alpha = o \text{ and } D \text{ (displacing fluid)} \tag{12}$$

Using them, Equs. (3) and (4) can be written as:

$$(\phi\, S_D \rho_D)_t + \nabla\cdot(\rho_D \underline{u}_D) = 0 \tag{13a}$$

$$(\phi\, S_o \rho_o)_t + \nabla\cdot(\rho_o \underline{u}_o) = 0 \tag{13b}$$

The "total Darcy velocity" is defined by

$$\underline{u}_T = \underline{u}_D + \underline{u}_o = \phi\{S_D\, \underline{v}^D + S_o\, \underline{v}^o\} \tag{14}$$

When capillary pressure and gravity forces are neglected, Darcy velocities are given by

$$\underline{u}_l = - \frac{\underline{\underline{k}} k_{rl}}{\mu_l} \nabla p, \quad l=o,D \tag{15}$$

and the total Darcy velocity, as well as the velocity of the displacing fluid, are colinear. Thus,

$$\underline{u}_D = f_D \underline{u}_T, \tag{16}$$

where f_D is a proportionality factor. In view of this equation, one can replace $\underline{u}_D$ by $f_D \underline{u}_T$, in Equ. (13a), to obtain

$$(\phi\, S_D \rho_D)_t + \nabla\cdot(\rho_D f_D \underline{u}_T) = 0 \tag{17}$$

When the fluids are incompressible, the density of the displacing fluid can be cancelled out, in this equation. If in addition, the solid matrix is also incompressible, such equation reduces to

$$(S_D)_t + \phi^{-1}\nabla\cdot(f_D \underline{u}_T) = 0 \tag{18}$$

Equations (14) to (16), together imply

$$f_D(S_D) = \frac{1}{1 + \dfrac{k_{ro}\mu_D}{k_{rD}\mu_o}} \tag{19}$$

When the liquid phases and the solid matrix are incompressible, Equs. (13) together imply that $\nabla\cdot\underline{u}_T=0$. Hence, Equ. (18), can be written as

$$(S_D)_t + \phi^{-1} f'_D \underline{u}_T\cdot\nabla S_D = 0 \tag{20}$$

because f_D is function of S_D, only. Here, f'_D stands for the

derivative of f_D with respect to S_D. This equation is a first order differential equation for S_D and when complemented with suitable boundary conditions, it can be solved uniquely for S_D. Such equation states that the rate of advance of a point that has a certain fixed saturation, equals the total Darcy velocity $\underline{u}_T$, multiplied by the factor $\phi^{-1}f'_D$. Here, no gravity segregation has been taken into account, but it is not difficult to incorporate it (see, for example [12]).

Buckley and Leverett [5,6], were the first to derive the one-dimensional version of Equ. (20). For such case, one has

$$(S_D)_t + \phi^{-1}f'_D\underline{u}_T\partial S_D/\partial x = 0 \qquad (21)$$

Writing q_T for the total rate of flow through a section, the total Darcy velocity can be expressed as $\underline{u}_T=q_T/A$, where A is the cross-sectional area, and Equ. (21) becomes:

$$(S_D)_t + (q_T/A\phi)f'_D\partial S_D/\partial x = 0 \qquad (22)$$

This is the classical Buckley-Leverett equation. If f'_D is non-constant, the space-time curves in which S_D remains constant will intersect, in general, leading to multi-valued solutions which are non-physical. The problem is similar to that occurring in the study of compressible fluids, either supersonic flow or the piston problem, and is solved by introducing discontinuous solutions or shocks. The same is done in multidimensional problems.

Using Equ. (12) and the incompressibility of the liquid phases and the solid matrix, the jump conditions (8), can be written as:

$$[\underline{u}_D] = \phi[S_D]\underline{v}_\Sigma \qquad (23a)$$

$$[\underline{u}_o] = \phi[S_o]\underline{v}_\Sigma \qquad (23b)$$

Adding up these two equations, it is seen that

$$[\underline{u}_T] = 0 \qquad (24)$$

i.e., the total Darcy velocity $\underline{v}_T$, is continuous. Making use of this result and of Equs. (16) and (14), it is seen that

$$[\underline{u}_D] = [f_D]\underline{u}_T = \phi[f_D]\{S_D\,\underline{v}^D + S_o\,\underline{v}^o\} \qquad (25)$$

which when combined with (23a), yields

$$\underline{v}_\Sigma = \frac{[f_D]}{[S_D]} \{S_D \underline{v}^D + S_o \underline{v}^o\} = \phi^{-1} \frac{[f_D]}{[S_D]} \underline{u}_T \qquad (26)$$

This relation was first derived by Sheldon and Cardwell [25], for one-dimensional problems. As has been presented here, it applies to problems in several dimensions, as well.

B.- Shock Formation in Immiscible Displacement

According to the discussion presented in Section 3, for immiscible displacement, in the absence of capillary forces, the points in which the saturations remain constant move with velocity $\phi^{-1} f'_D \underline{u}_T$. Let $x_c(S_D, t)$, be the position at time t, of a point at which the saturation is S_D. Then, such a point satisfies the differential equation

$$\frac{\partial x_c}{\partial t}(S_D, t) = \phi^{-1} f'_D(S_D) \underline{u}_T \qquad (27)$$

The solutions of Equ. (27), define straight lines in the space-time plane, since the slope is constant in each one of them.

Assume, $x_I(S_D)$ is the initial position, at time equal to zero, of a point in which the saturation of the displacing fluid is S_D. Then:

$$x_c(S_D, t) = x_I(S_D) + t\phi^{-1} f'_D(S_D) \underline{u}_T \qquad (28)$$

and the solution of the partial differential equation (21), will be single valued, unless the equation

$$\frac{\partial x_c}{\partial x}(S_D, t) = x'_I(S_D) + t\phi^{-1} f''_D(S_D) \underline{u}_T = 0 \qquad (29)$$

is satisfied for some S_D. Clearing for t, one gets:

$$t = -\frac{\phi x'_I(S_D)}{f''_D(S_D) \underline{u}_T} = -\frac{\phi}{S'_{DI} f''_D(S_D) \underline{u}_T} \qquad (30)$$

where $S_{DI}(x)$ is the initial distribution of S_D and a prime is used to denote the derivative of such function. A shock has to be introduced at the minimal time (t_{sh}) which satisfies (30). Under the assumption that the velocity $\underline{u}_T$ is positive, a t satisfying Equ. (30) would be positive, only if $S'_{DI} f''_D(S_D) < 0$. If this latter condition is fulfilled, t_{sh} is obtained when $|S'_{DI} f''_D(S_D)|$ is maximum, in Equ. (30).

On the other hand, let $x_\Sigma(t)$ be the position of the shock at time t. According to Equ. (26), one has

$$\frac{dx_{\Sigma}}{dt} = \phi^{-1}\frac{[f_D]}{[S_D]}\,\underline{u}_T \qquad (31)$$

In general, the saturation S_D at the shock varies with time. A necessary condition for remaining constant, is that the shock moves with the velocity of a point which keeps fixed the value of S_D; i.e.:

$$v_{\Sigma} = \frac{\partial x_c}{\partial t}(S_D, t) = \phi^{-1} f'_D(S_D)\underline{u}_T \qquad (32)$$

In view of Equ. (26), this condition is

$$f'_D(S_D) = \frac{[f_D]}{[S_D]} \qquad (33)$$

Equ. (33) can be fulfilled during a finite period of time, only if the shock advances into a region of constant S_D. A special case of this situation is when $S_D \equiv 0$ ahead of the shock ($S_{D+}=0$). For this case:

$$f'_D(S_D) = \frac{f_D}{S_D} \qquad (34)$$

since $f_D(0)=0$. A point satisfying Equ. (34), can be obtained drawing a tangent to the curve $f_D(S_D)$ from the origin. This is the graphical construction first suggested by Buckley and Leverett [22]. Such construction is the basis of the simplified method for computing oil recovery, due to Welge [7].

In the more general situation in which S_D is a constant different from zero, ahead of the shock ($S_{D+}\neq 0$), the relation (33) in its more general form, must be fulfilled. It can be written more explicitly, as:

$$f'_D(S_D) = \frac{f_D(S_{D+}) - f_D(S_{D-})}{S_{D+} - S_{D-}} \qquad (35)$$

A point S_{D+} satisfying such condition can be obtained drawing a tangent to the curve $f_D(S_D)$ from the point $(S_{D-}, f_D(S_{D-}))$.

4 PARTIALLY MISCIBLE DISPLACEMENT

In Section 3, we have seen that in immiscible displacement shocks correspond to discontinuities of the saturation fluid S_D and that shocks are generated when space-time lines carrying constant values of S_D, intersect. In the present Section, it will be shown that on the contrary, in miscible displacement, shocks are associated with discontinuities of the solution

gas:oil ratio R_s and that lines carrying constant values of R_s can not intersect. Thus, the intersection of characteristics as a shock generating mechanism must be ruled out, in this case.

In addition, it will be shown that when a particle of undersaturated oil is reached by a gas front, the transition from an undersaturated state to a saturated one, is discontinuous and generates a shock.

A.- Bubble-Point Conservation Principle

In this Section, two phases will be considered: liquid oil and gas. As was explained previously, in a Beta or Black Oil model, diffusion is excluded. Due to this fact, the following general result holds.

The Bubble-Point Conservation Principle.

In the absence of a gas phase, oil particles conserve their bubble- point.

Proof. When the gas phase is not present, the governing differential equations are:

$$(\phi\, S_o \bar{\rho}_o)_t + \nabla\cdot(\phi\, \bar{\rho}_o S_o \underline{v}^o) = 0 \qquad (36)$$

$$(\phi\, S_o \bar{R}_s \bar{\rho}_o)_t + \nabla\cdot(\phi\, \bar{R}_s \bar{\rho}_o S_o \underline{v}^o) = g^o_{Ig} \qquad (37)$$

In the presence of (36), Equ. (37) can be replaced by

$$\phi\, S_o \bar{\rho}_o \{(\bar{R}_s)_t + \underline{v}^o\cdot\nabla \bar{R}_s\} = g^o_{Ig} \qquad (38)$$

Observe that when the gas phase is not present $g^o_{Ig}=0$ and Equ. (38) may be reduced to

$$(R_s)_t + \underline{v}^o\cdot\nabla R_s = 0 \qquad (39)$$

This equation states that in the absence of a gas phase, R_s remains constant on particles moving with the velocity of the oil phase, $\underline{v}^o$. Thus, R_s remains constant on oil particles and the bubble-point is conserved.

B.- Shock Generation

This imposes some limitations on the paths that the value of R_s on an oil particle can describe on the R_s-p plane (see, Fig. 1). The one shown in Fig. 1a, is feasible and corresponds to the well known phenomenon which occurs when a single-phased

mixture of oil and gas, starting at state "n", is depressurized beyond the bubble point, so that free gas becomes available and the liquid oil remains saturated when depressurization is continued. Then the mixture is pressurized again until all the gas present is dissolved and the liquid oil becomes undersaturated, finally reaching state "n+1", in Fig. 1a. Of course, such path is reversible: we can start at state "n+1" and by successive depressurization and pressurization, reach state "n". Observe that the point at which the mixture will leave the saturation curve when it is repressurized, depends on the amount of free gas available. In actual reservoir models, such amount of gas is supplied by the gas phase, which in turn is determined by the relative motion of the gas phase, with respect to the oil.

On the other hand, on an oil particle, the values of R_s cannot follow a trajectory such as the one joining points "n" and "n+1", in Fig. 1b, since it implies that the R_s changes without reaching the bubble point. That is, that R_s changes when the gas phase is absent, so that the bubble-point conservation principle is violated.

It seems that in actual models, when dealing with variable bubble-point problems, it is more efficient to replace Equ. (37) by the "bubble-point conservation principle", instead of integrating both Equs. (37) and (38) simutaneously, as is usually done. When this approach is followed, in the absence of a gas phase all trajectories of particles are horizontal, in the p-R_s plane. Trajectories like the one shown in Fig. 1b, which are usually included (see for example, Figs. 12.6b and c, of [16]), must be excluded when an oil particle is followed.

On the other hand, a trajectory such as the one illustrated in Fig. 1c, is admissible for an oil particle. It corresponds to an oil particle which is initially undersaturated (point "n") so that the gas phase is absent necessarily, and at some point the oil particle is reached by a gas phase (point SH) so that it suddenly becomes saturated and

under further pressurization R_s moves on the saturation curve. Such trajectory has a point of discontinuity at SH and in actual reservoir models, it gives rise to a discontinuous front or shock. This is the mechanism of shock generation which is the main subject of the present paper.

When a gas front advances into a region of undersaturated oil the surface where the properties are discontinuous will be denoted by Σ and will be referred to, as the shock. On Σ, the jump conditions (8b) and (9), must be satisfied. They are:

$$[\phi\ \bar{\rho}_o S_o (\underline{v}^o - \underline{v}_\Sigma)]\cdot\underline{n} = 0 \qquad (40a)$$

$$[\phi\ \bar{\rho}_o S_o \bar{R}_s (\underline{v}^o - \underline{v}_\Sigma)]\cdot\underline{n} = g^o_{\Sigma g} \qquad (40b)$$

$$[\phi\ \rho_g S_g (\underline{v}^g - \underline{v}_\Sigma)]\cdot\underline{n} = g^g_{\Sigma o} \qquad (40c)$$

Notice, in one side of Σ no free gas is available, while the other one is occupied by the advancing gas. To be specific, the unit normal vector "$\underline{n}$" to Σ will be taken pointing towards the side of the advancing gas. Then $S_{g-}=0$, and only the gas properties on the positive side are defined. Thus, in what follows the subindex "plus" will be dropped when referring to the gas properties.

In the developments that follow, a formula for the "jump of a product" that has been used extensively in previous work by Herrera [16-19], will be applied. It is:

$$[rs] = \dot{r}[s] + \dot{s}[r] \qquad (41)$$

where the dot stands for the "average" across the surface of discontinuity. More precisely, for any function "r", one has

$$\dot{r} = (r_+ + r_-)/2 \qquad (42)$$

In the presence of (8b), Equ. (9a) can be written as:

$$\dot{\overline{\phi\ \bar{\rho}_o S_o (\underline{v}^o - \underline{v}_\Sigma)[\bar{R}_s]}}\cdot\underline{n} = g^o_{\Sigma g} \qquad (43)$$

However, the product $\phi\bar{\rho}_o S_o (\underline{v}^o - \underline{v}_\Sigma)$ is continuous, by virtue of Equ. (6b). Thus

$$\dot{\overline{\phi\ \bar{\rho}_o S_o (\underline{v}^o - \underline{v}_\Sigma)}} = \phi_+ \bar{\rho}_{o+} S_{o+} (\underline{v}^o_+ - \underline{v}_\Sigma) \qquad (44)$$

and (43) becomes

$$\phi_+ \bar{\rho}_{o+} S_{o+} (\underline{v}^o_+ - \underline{v}_\Sigma)[\bar{R}_s]\cdot\underline{n} = g^o_{\Sigma g} \qquad (45)$$

Furthermore, adding Equs. (45) and (9b), it is obtained:

Let us define the "retardation factor ε" by:

$$(\underline{v}_\Sigma - \underline{v}^o_+)\cdot\underline{n} = \varepsilon(\underline{v}^g - \underline{v}^o_+)\cdot\underline{n} \qquad (47)$$

Observe that $\underline{v}_\Sigma - \underline{v}^o_+$, is the relative velocity of the advancing gas front with respect to the oil, while $\underline{v}^g - \underline{v}^o_+$ is the relative velocity of the particles in the gas phase also with respect to the oil. The retardation factor ε, which as will be seen is always positive and less than one, expresses how smaller the relative velocity of the gas front is in comparison to that of the gas.

Noticing that

$$(\underline{v}^g - \underline{v}_\Sigma) = (\underline{v}^g - \underline{v}^o_+) - (\underline{v}_\Sigma - \underline{v}^o_+) \qquad (48)$$

and clearing for ε in Equ. (12), it is seen that

$$\varepsilon = \frac{1}{1 + [R_s]\dfrac{\rho_{o+} S_{o+}}{\rho_g S_g}} \qquad (49)$$

The whole system of jump conditions can now be replaced by Equs. (8), together with (47), where ε is given by Equ. (49). Observe that, as stated, $0<\varepsilon\leq 1$, since $R_{s+} \geq R_{s-}$. This shows that the relative velocity of the gas front with respect to the oil, is not equal to the relative velocity of the gas, but it is reduced by a "retardation factor" ε.

5.- NUMERICAL IMPLICATIONS

One of the most effective procedures for dealing with shocks in Petroleum Engineering, is the Front-Tracking Method [10-13]. Such method is based on the use of characteristics, since a Riemann problem is solved at each time step (see [1], for example). However, the shock which occurs in partially miscible displacement and which have been described in Section 4 of this paper, is not generated by the crossing of characteristics. Thus, the Front-Tracking Method is not

applicable. In such situations alternative procedures must be applied. A procedure, that has been proposed by the authors is Eulerian-Lagrangian, modeling of shocks (see [1,2]). This method is being the subject of research at present whose results will be reported elsewhere.

REFERENCES

1.- Herrera, I., A. Galindo and R. Camacho., Shock Modelling in Petroleum Engineering, Chapter 7 of the book Computational Methods for Moving Boundary Problems in Heat and Fluid Flow, L.C. Wrobel Ed, Computational Mechanics Publications, 1993.

2.- Herrera, I., A. Galindo and R. Camacho, "*Shock Modelling in Variable Bubble Point Problems of Petroleum Engineering*", Computational Modelling of Free and Moving Boundary Problems, **Vol. 1: Fluid Flow**, Eds. L.C. Wrobel and C.A. Brebbia, Computational Mechanics Publication, pp. 399-415, 1991.

3.- A. Galindo., Herrera, I., R.G. Camacho and L. Chargoy., "Eulerian-Lagrangian Approach to the Modeling of Shocks in Petroleum Reservoirs", 7th IMACS International Conference on Computer Methods for Partial Differential Equations, Brunswickm N,J., June 1992

4.- Ewing, R.E., "*Problems Arising in the Modeling of Processes for Hydrocarbon Recovery*", **The Mathematics of Reservoir Simulation**, Ed. R.E. Ewing, Frontiers in Applied Mathematics Vol. 1, SIAM, Philadelphia, 3-34, 1983.

5.- Leverett, M.C.,"Capillary Behaviour in Porous Solids", Trans. AIME, 1941, 142, 152.

6.- Buckley, S.E. and Leverett, M.C.: "Mechanics of Fluid displacement in sands". Trans., AIME (1942) 146, 107.

7.- Welge, H.J.,"A Simplified Method for Computing Oil Recovery by Gas or Water Drive", Petroleum Trans. AIME, 1952, 195, pp 91-98.

8.- Cardwell, W.T.,"The Meaning of the Triple Value in Noncapillary Buckley-Leverett Theory", T.P. 808, Trans. AIME, 1959, 216, pp 271-276.

9.- Sheldon, J. W., Cardwell, W. T., "One-Dimensional, Incompressible, Noncapillary, Two-Phase Fluid in a Porous Medium", Petroleum Trans. AIME, 1959, 216, pp 290-296.

10.- Richtmyer, R.D., *Difference Methods for Initial Value*

Problems, Interscience Publ., New York, 1957.

11.- Glimm, J., E. Isaacson, D. Marchesin and O. McBryan, *Front Tracking for Hyperbolic Systems,* Adv.Appl.Math. **2**, 91-119, 1981.

12.- Colella, P., P. Concus and J. Sethian, *Some Numerical Methods for Discontinuous Flows in Porous Media*, in **The Mathematics of Reservoir Simulation**, R.E. Ewing, Ed., Frontiers in Applied Mathematics Vol. 1, SIAM, Philadelphia, 161-186, 1983.

13.- Bratvedt F., K. Bratvedt, C.F. Buchholz, T. Gimse, H. Holden and N.H. Riseboro, *Front Tracking for Groundwater Simulations*, Computational Methods in Water Resources IX, Vol. I: Numerical Methods in Water Resources, T.F. Russell et al. Eds., CM Publications, Elsevier Applied Science, 97-104, 1992.

14.- Allen, B.M., Herrera I. and Pinder G.F., "Numerical Modeling in Science and Engineering", John Wiley & Sons, New York, 1988.

15.- Herrera, I. and Allen, M.B., "Modelación Computacional de Sistemas en Ciencias e Ingeniería", Comunicación Técnica No 9, Serie Docencia e Investigación, Instituto de Geofísica, UNAM 1986.

16.- Aziz, K., and Settari, A., Petroleum Reservoir Simulation, Ed. Applied Science Publishers Ltd, London, 1979

APPENDIX

The reader can find a convenient presentation of the "Balance Equations of Continuum Mechanics", in [14]. They have been written in Equs. (1). However, our notation differs slightly from the notation of [14] and also Equ.(1b) is in a form slightly more general than that presented in [14]. Thus, this Appendix is devoted to explain those differences.

In the notation here followed, which was introduced in [15], the mass M^{α} of any component α, in a region Ω is written as

$$M^{\alpha} = \int_{\Omega} \psi^{\alpha}(\underline{x}, t)\, d\underline{x} \qquad (A.1)$$

In particular, the masses of water, non-volatile oil, disolved gas and gas in the gas phase, are:

$$M^{w}=\int_{\Omega}\phi S_{w}\rho_{w}\, d\underline{x};\ \bar{M}^{o}=\int_{\Omega}\phi S_{o}\bar{\rho}_{o}\, d\underline{x};\ \bar{M}^{dg}=\int_{\Omega}\phi S_{o}\bar{\rho}_{dg}\, d\underline{x};\ M^{g}=\int_{\Omega}\phi S_{g}\rho_{g}\, d\underline{x} \qquad (A.2)$$

This permits identifying

$$\psi^{w} = \phi S_{w}\rho_{w};\ \bar{\psi}^{o} = \phi S_{o}\bar{\rho}_{o};\ \bar{\psi}^{dg} = \phi S_{o}\bar{\rho}_{dg};\ \psi^{g} = \phi S_{g}\rho_{g} \qquad (A.3)$$

Then (3) to (6) follow by sustitution of (A.3) into (1a), and (8) and (9) by sustitution of (A.3) into (1b).

In addition, Equ. (1b) is more general than Equ. (1.3-6) of [14], because we have included the term g_{Σ}^{α} which accounts for the possibility of having mass supply through a discontinuity surface. As was explained at the end of Section 2, this is essential in order to be able to model a gas front which advances into a region occupied by undersaturated oil.

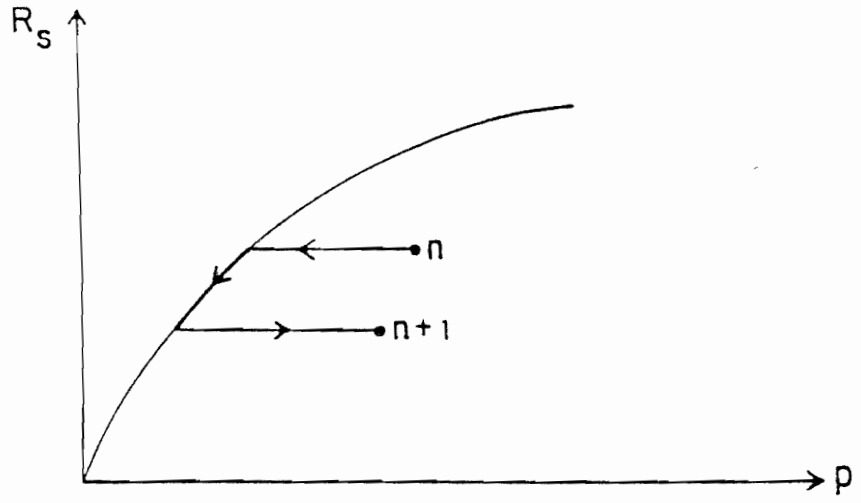

Fig. 1a

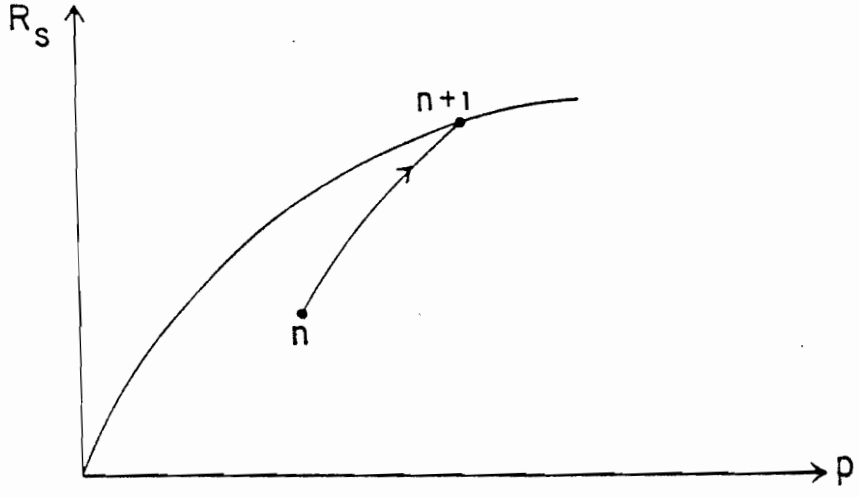

Fig. 1b

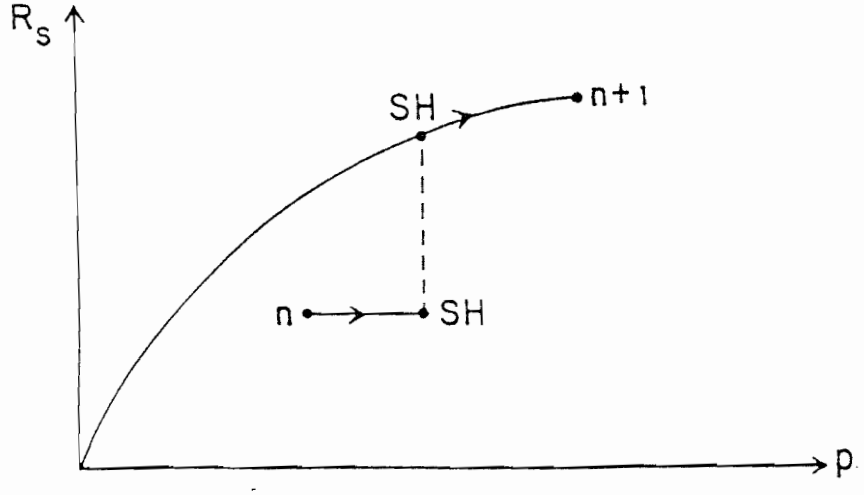

Fig. 1c

Figure 1 Paths in the $R_s - p$ plane