Finite Elements in Fluids

New trends and applications

Part II

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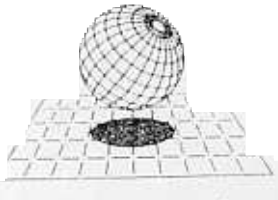

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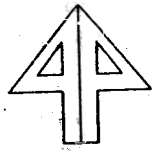

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FINITE ELEMENTS IN FLUIDS
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INNOVATIVE DISCRETIZATION METHODOLOGIES BASED ON LAM

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ABSTRACT. In this paper, a strategy for discretizing partial differential equations or systems of such equations, is proposed. It is based in Localized Adjoint Method (LAM), but has greater generality and flexibility. It consists in identifying firstly the information about the exact solution that is contained in the approximate one and then processing it, in the most efficient manner that is possible, without prejudging its shape. The facts presented, indicate that it has clear advantages over standard procedures and must be further studied and tested in a wider class of problems.

1. INTRODUCTION

The Localized Adjoint Method (LAM) is a methodology for discretizing partial differential equations which was introduced by the author [1]. This procedure is based on Herrera's Algebraic Theory of Boundary Value Problems [2–6]. Applications have successively been made to ordinary differential equations, for which highly accurate algorithms were developed [6–8], multidimensional steady state problems [9] and optimal spatial methods for advection-diffusion equations [10–12]. The generalizations of Characteristic Methods known as Eulerian-Lagrangian Localized Adjoint Methods (ELLAM), have been specially successful in their different implementations [13–20] and many specific applications have been made [21–30].

On the other hand, more recently the author [20] has been proposing a strategy for discretizing partial differential equations or systems of such equations, which consists in identifying firstly the information about the exact solution contained in the approximate one and then processing it, in the most efficient manner that is possible, without prejudging its shape. This differs from finite element approaches, since choosing a set of basis functions is a manner of assigning a shape to the sought solution.

The procedure proposed here has been applied already to some specific problems and the results have been quite satisfactory. In particular, an Eulerian-Lagrangian Method of Cells (CELLAM) was developed in [20], using this point of view and the results were quite satisfactory. Previous attempts for developing such a method were hindered by many numerical difficulties [15–17]. However, such difficulties were not encountered when the new strategy was adopted [18–20]. More specifically, in [17] approximations were derived using a system of basis functions, while in [20] they were derived using the strategy recomended here. Apparently, the numerical difficulties were not encountered in the second approach, because

the strategy proposed by the author, allows deriving more precise approximations. Indeed, assuming the shape of the solution unduly restricts the accuracy of the algorithms that can be derived, as was corroborated in [20].

This paper is devoted to explain this novel strategy and the manner in which it can be applied to general differential equations. Section 2 presents the motivation for proposing this procedure. A brief description of a method to analyze the information that is contained in approximate solutions, is given in Section 3. Such method is based on the application of Green-Herrera formula for functions with jump discontinuities, which allows the localization of the adjoint, when it is applied to partial differential equations or systems of such equations [1, 14]. As an illustration, a comparative analysis of the procedures used in [17] and [20], is carried out in Section 4. It is concluded, that the results obtained indicate that the strategy proposed here, which has wide applicability, may give good results, if it is applied to many other problems.

2. Background

In previous work [14, 20], it has been pointed out that in the construction of approximate solutions two important processes occur:

i) Gathering information about the sought solution; and

ii) Interpolating or, more generally, processing such information.

These two processes are distinct, although in many numerical methods they are not differentiated clearly. The information about the exact solution that is gathered, is determined mainly by the weighting functions used. Since this information does not determine uniquely the sought solution, some processing of it is required, in order to fill the gaps of information and exhibit at the end, a unique approximate solution.

Different methods of solution follow different strategies for carrying out this process of extending the information that is available (interpolation, extrapolation or both). For example, in finite element methods, some basis functions are chosen and the approximate solution is assumed to be a superposition of such functions. In this case, the information about the exact solution which is gathered by the weighting functions, is interpolated in a manner which is determined by the family of basis functions chosen.

Clearly, it is disadvantageous to carry out the process of extending the information blindly, not knowing what is the actual information that is available. However, this is what is usually done. On the contrary, it is advantageous to make use of the insight gained when the available information has been identified, since the selection of the best procedure for extending such information, is strongly dependent on the information that is at hand.

Due to these facts, in recent works the author [14, 20] has advocated an approach for developing numerical methods, in which the processes i) and ii) are clearly separated, and the information about the sought solution that is at hand, is identified and then the procedure for extending it, is defined using that information.

The procedure that was applied in [20] for deriving the algorithms presented there, is an illustration of such methodology. Using Herrera's Algebraic Theory of Boundary Value Problems [2-6], which permits localizing the adjoint,

the information contained in the approximate solution is identified and analyzed [1, 14, 20]. This is what should be properly called Localized Adjoint Method. Then, depending on the information that is identified, interpolation procedures suitable for handling it efficiently, are selected and applied. In most cases the introduction of basis functions is not required and, even more, their use is frequently inconvenient, as for the problem treated in this paper. These points will be discussed further in Section 4, where an example is presented.

Another point which is important, is that the use of the algebraic theory for the analysis of the information, has clear advantages over other options, such as the standard theory of distributions, because of two reasons at least: the use of the algebraic theory permits the localization of the adjoint, and the simultaneous use of discontinuous trial and test functions is feasible.

3. A METHOD FOR ANALYZING THE INFORMATION

As mentioned previously, a convenient manner of carrying out the analysis of the information contained in approximate solutions, is by application of Herrera's Algebraic Theory of Boundary Value Problems [2–6]. A special feature is that the analysis is carried out using exclusively simple inner products which are defined locally. The theory implies a kind of operator extension which differs from the theory of distributions [31].

The main result required is a Green formula for functions with jump discontinuities (Green-Herrera formula), which is explained in this Section, in a brief manner. In its original form was presented in [3, 5, 6], but the interested reader may find detailed and updated expositions of Green-Herrera formula, for partial differential equations and systems of such equations, in [1] and [14]. In some specific applications, it is feasible to derive the results which are needed in an ad-hoc manner (see, for example [13]). However, the frame-work of the general theory supplies the guidelines that permit understanding thoroughly the methodology of analysis, which is applicable to a great variety of problems, including systems of differential equations [14].

Consider functions defined in a region Ω (possibly space-time), which may have jump discontinuities across some internal boundary Σ. In applications to finite element methods, Σ could be the union of all the inter-element boundaries. The general boundary value problem treated by the theory is one with prescribed jumps, across Σ. The differential equation is

$$\mathcal{L}u = f_\Omega, \quad \text{in} \quad \Omega \tag{3.1}$$

Certain boundary and jump conditions are specified on the boundary $\partial\Omega$ of Ω and on Σ, respectively. When Ω is a space-time region, initial conditions may be incorporated as part of the boundary conditions, since they apply in part of $\partial\Omega$, in this case.

Given a differential operator $\mathcal{L}$ and its adjoint $\mathcal{L}^*$, Green-Herrera formula is:

$$\int_\Omega w\mathcal{L}u\,dx - \int_{\partial\Omega} \mathcal{B}(u,w)\,dx - \int_\Sigma \mathcal{J}(u,w)\,dx = \int_\Omega u\mathcal{L}^*w\,dx - \int_{\partial\Omega} \mathcal{C}^*(u,w)\,dx - \int_\Sigma \mathcal{K}^*(u,w)\,dx \tag{3.2}$$

where $\mathcal{B}$ and $\mathcal{J}$ are bilinear functions of u and w, defined point-wise, while $\mathcal{C}^*$ and $\mathcal{K}^*$ are the transposes of $\mathcal{C}$ and $\mathcal{K}$, respectively, which are also bilinear in u and w.

Equation (3.2) can be obtained applying successive integration by parts. However, a more systematic way, which permits exhibiting the frame-work that can be used in a very general class of problems, is presented here. It starts from the definition of formal adjoint, which requires that $\mathcal{L}$ and $\mathcal{L}^*$ satisfy:

$$w\mathcal{L}u - u\mathcal{L}^*w = \nabla \cdot \{\underline{\mathcal{D}}(u,w)\} \tag{3.3}$$

for a suitable vector-valued bilinear function $\underline{\mathcal{D}}(u,w)$. Integration over Ω of Eq. (3.3) and application of generalized divergence theorem [32], yield:

$$\int_\Omega \{w\mathcal{L}u - u\mathcal{L}^*w\}\,dx = \int_{\partial\Omega} \mathcal{R}_\partial(u,w)\,dx + \int_\Sigma \mathcal{R}_\Sigma(u,w)\,dx \tag{3.4}$$

where the bilinear functions $\mathcal{R}_\partial$ and $\mathcal{R}_\Sigma$, are defined on $\partial\Omega$ and Σ, respectively, by:

$$\mathcal{R}_\partial\,(u,w) = \underline{\mathcal{D}}(u,w)\cdot\underline{n} \quad \text{and} \quad \mathcal{R}_\Sigma(u,w) = -[\underline{\mathcal{D}}(u,w)]\cdot\underline{n} \tag{3.5}$$

Here, the square brackets stand for the "jumps" across Σ of the function contained inside, i.e., limit on the positive side minus limit on the negative side. The positive side of Σ is chosen arbitrarily and then the unit normal vector $\underline{n}$ is taken pointing towards the positive side of Σ. Observe that generally $\mathcal{L}u$ will not be defined on Σ, since u and its derivatives may be discontinuous. Thus, in the theory, it is understood that integrals over Ω are carried out excluding Σ. Consequently, differential operators must be always understood in an elementary sense and not in a distributional sense.

In the general theory of partial differential equations, Green's formulas are used extensively. For the construction of such formulas it is standard to introduce a decomposition of the bilinear function $\mathcal{R}_\partial$ [33]. Indicating transposes of bilinear forms by means of a star, the general form of such decomposition is:

$$\mathcal{R}_\partial(u,w) \equiv \underline{\mathcal{D}}(u,w)\cdot\underline{n} = \mathcal{B}(u,w) - \mathcal{C}^*(u,w) \tag{3.6}$$

where $\mathcal{B}(u,w)$ and $\mathcal{C}(w,u) = \mathcal{C}^*(u,w)$ are two bilinear functions.

When considering initial-boundary value problems, the definitions of these bilinear forms depend on the type of boundary and initial conditions to be prescribed. They are chosen satisfying the requirement that for any u that fulfills the prescribed boundary and initial conditions, $\mathcal{B}(u,w)$ is a well-defined linear function of w, independent of the particular choice of u. This linear function will be denoted by g_∂ (thus, its value for any given function w, will be $g_\partial(w)$), and the boundary conditions can be specified by requiring that $\mathcal{B}(u,w) = g_\partial(w)$, for every test function w.

The linear function $\mathcal{C}^*(u,\cdot)$, on the other hand, can not be evaluated in terms of the prescribed boundary values, but it also depends exclusively on certain boundary values of u (the "complementary boundary values"). Generally, such boundary values can only be evaluated after the initial-boundary value problem has been solved.

In a similar fashion, convenient formulations of boundary value problems with prescribed jumps requires constructing Green's formulas in discontinuous

fields. Green-Herrera formula is obtained introducing the general decomposition:

$$\mathcal{R}_\Sigma(u,w) = \mathcal{J}(u,w) - \mathcal{K}^*(u,w) \tag{3.7}$$

of the bilinear function $\mathcal{R}_\Sigma(u,w)$. This has been done in general, for differential operators whose coefficients may be discontinuous [3]. However, the decomposition is specially easy to obtain when the coefficients are continuous, since for this case it stems from the algebraic identity:

$$[\underline{\mathcal{D}}(u,w)] = \underline{\mathcal{D}}([u],\dot{w}) + \underline{\mathcal{D}}(\dot{u},[w]) \tag{3.8}$$

if $\mathcal{J}$ and $\mathcal{K}^*$, are defined by

$$\mathcal{J}(u,w) = \underline{\mathcal{D}}([u],\dot{w})\cdot\underline{n} \tag{3.9a}$$

$$\mathcal{K}^*(u,w) = \mathcal{K}(w,u) = \underline{\mathcal{D}}(\dot{u},[w])\cdot\underline{n} \tag{3.9b}$$

Here

$$[u] = u_+ \quad u_-, \qquad \dot{u} = (u_+ + u_-)/2 \tag{3.10}$$

An important property of the bilinear function $\mathcal{J}(u,w)$ is that, when the jump of u is specified, it defines a unique linear function of w, which is independent of the particular choice of u. When considering initial-boundary value problems with prescribed jumps, the linear function defined by the prescribed jumps in this manner will be denoted by j_Σ (thus, its value for any given function w will be $j_\Sigma(w)$) and the jump conditions at any point of Σ can be specified by means of the equation $\mathcal{J}(u,w) = j_\Sigma(w)$. If the sought solution is required to be smooth, one would usually have $j_\Sigma(w) \equiv 0$.

In problems with prescribed jumps, the linear functional $\mathcal{K}^*(u,\cdot)$, plays a role similar to that of the complementary boundary values $C^*(u,\cdot)$. It can only be evaluated after the initial-boundary value problem has been solved and certain information about the average of the solution and its derivatives on Σ is known. Such information is called the "generalized averages".

A weak formulation of the boundary value problem with prescribed jumps is:

$$\int_\Omega w\mathcal{L}u\,dx - \int_{\partial\Omega}\mathcal{B}(u,w)\,dx - \int_\Sigma \mathcal{J}(u,w)\,dx = \int_\Omega w f_\Omega\,dx - \int_{\partial\Omega} g_\partial(w)\,dx \quad \int_\Sigma j_\Sigma(w)\,dx \tag{3.11a}$$

However, this weak formulation is equivalent to

$$\int_\Omega u\mathcal{L}^*w\,dx - \int_{\partial\Omega} C^*(u,w)\,dx - \int_\Sigma \mathcal{K}^*(u,w)\,dx = \int_\Omega w f_\Omega\,dx - \int_{\partial\Omega} g_\partial(w)\,dx - \int_\Sigma j_\Sigma(w)\,dx \tag{3.11b}$$

by virtue of Eq. (3.2). Eq. (3.11a) is the *variational formulation in terms of the data of the problem*, while (3.11b), is the *variational formulation in terms of the sought information.*

The analysis of the information contained in approximate solutions, is based in the following observations. When the method of weighted residuals is applied, using a system of weighting functions $\{w^1, \ldots, w^N\}$, an approximate solution satisfies

$$\int_\Omega \hat{u}\mathcal{L}^* w^\alpha \, dx - \int_{\partial\Omega} C^*(\hat{u}, w^\alpha)\, dx - \int_\Sigma \mathcal{K}^*(\hat{u}, w^\alpha)\, dx =$$

$$\int_\Omega w^\alpha f_\Omega \, dx - \int_{\partial\Omega} g_\partial(w^\alpha)\, dx - \int_\Sigma j_\Sigma(w^\alpha)\, dx, \qquad \alpha = 1, \ldots, N \quad (3.12)$$

Taking into account that Eq. (3.12) is also satisfied by the exact solution, it is seen that

$$\int_\Omega (\hat{u} - u)\mathcal{L}^* w^\alpha \, dx - \int_{\partial\Omega} C^*(\hat{u} - u, w^\alpha)\, dx - \int_\Sigma \mathcal{K}^*(\hat{u} - u, w^\alpha)\, dx = 0 \quad (3.13)$$

Equation (3.13) is the basis for analyzing the information contained in approximate solutions. The term $\int_\Omega (\hat{u} - u)\mathcal{L}^* w^\alpha \, dx$ gives the information about the exact solution u, in the interior of the region of definition of the problem Ω, the term $\int_{\partial\Omega} C^*(\hat{u} - u, w^\alpha)\, dx$ gives the information about the complementary boundary values on $\partial\Omega$ and the term $\int_\Sigma \mathcal{K}^*(\hat{u} - u, w^\alpha)\, dx$ gives the information about the generalized averages in the interelement boundaries Σ.

4. An example

In this Section we illustrate by means of an example, some of the advantages of the procedure presented in this paper.

In [17] and [20], the advection-diffusion equation in conservative form, was considered:

$$\mathcal{L}u \equiv \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x} - Vu\right) = f_\Omega(x, t); \qquad V \geq 0 \quad (4.1)$$

in the interval $\Omega_x = [0, l]$ and $t \geq 0$, subject to initial conditions

$$u(x, 0) = u_0(x) \quad (4.2)$$

and suitable boundary conditions.

For constructing a solution of this problem step by step in time, the time interval is decomposed into subintervals. Let $[t^n, t^{n+1}]$, be one such subinterval (Fig. 1), then assuming that the solution at t_n has already been obtained, a procedure for constructing it at t_{n+1} is developed. The region of definition of such problem is $\Omega = \Omega_x \times [t_n, t_{n+1}]$ and the initial conditions of Equ. (4.2), have to be replaced by

$$u(x, t_n) = u^n(x) \quad (4.3)$$

where the function $u^n(x)$ corresponds to the values of the solution at time t_n, which by assumption have been previously obtained.

In addition, a partition $\{x_1, x_{3/2}, x_{5/2}, \ldots, x_{E-1/2}, x_E\}$ into E subintervals of the interval $[0, \ell]$ is introduced, for which $x_1 = 0$ and $x_E = \ell$. The points

$$x_i = (x_{i-1/2} + x_{i+1/2})/2, \qquad i = 2, \ldots, E - \quad (4.4)$$

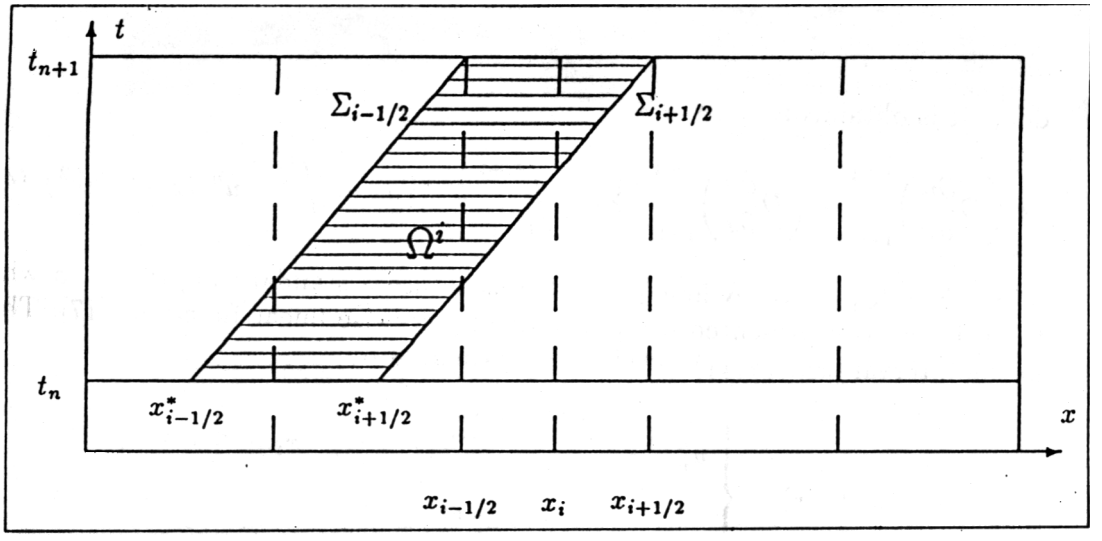

FIGURE 1.

being the "cell centers". Here, we assume that the partition is "uniform"; i.e, $h_i = x_{i+1/2} - x_{i-1/2} = h$, is independent of i, for interior nodes. Based on such partition, a partition $\{\Omega_1, \ldots, \Omega_E\}$ of the region $\Omega_x \times [t^n, t^{n+1}]$, is defined. A typical one of those subregions (Ω_i), in the case when Ω_i does not intersect the lateral boundaries, is limited by the two space-time characteristic curves $\Sigma_{i-1/2}$ and $\Sigma_{i+1/2}$, which pass through the points $x_{i-1/2}$ and $x_{i+1/2}$ at time t_{n+1}, respectively, as shown in Fig. 1. The velocity of propagation V_Σ, of each one of these lines satisfies $V_\Sigma = V$, because they are assumed here to be characteristic curves. In addition, it is assumed that discontinuities may occur on these lines exclusively, so that

$$\Sigma = \bigcup_{i=1}^{E-1} \Sigma_{i+1/2}.$$

The weighting functions to be used are constant functions; namely, the characteristic functions of each one of the subregions of this partition. For Ω_i, it is

$$w^i(x,t) = \begin{cases} 1 & \text{if } (x,t) \in \Omega_i, \\ 0 & \text{if } (x,t) \notin \Omega_i \end{cases} \tag{4.5}$$

For the special case in which the sought solution is smooth ($j_\Sigma \equiv 0$) and Ω_i does not intersect the lateral boundaries of Ω, Eq. (3.11b) is [20]:

$$\int_{x_{i-1}}^{x_{i+1}} u^{n+1}\,dx + \int_{t_n}^{t_{n+1}} \left(D\frac{\partial u}{\partial x}\right)_{\Sigma_{i+1/2}} dt - \int_{t_n}^{t_{n+1}} \left(D\frac{\partial u}{\partial x}\right)_{\Sigma_{i-1/2}} dt = \int_{x^*_{i-1}}^{x^*_{i+}} u^n\,dx \tag{4.6}$$

All the terms which supply information about the sought solution are in the left-hand side of Eq. (4.6). The first integral supply information about u, at time t_{n+1}, while the other two integrals supply information about the first-order spatial derivatives, along the characteristic curves, $\Sigma_{i+1/2}$ and $\Sigma_{i-1/2}$.

In what follows, we discuss two possible strategies for processing this information. In both of them the integrals with respect to time in Eq. (4.6), are evaluated using a backward-Euler (fully implicit) approximation. When this is

done, it is obtained:

$$\left\{\left(D\frac{\partial u}{\partial x}\right)_{i+1}^{n+1} \quad \left(D\frac{\partial u}{\partial x}\right)_{i-1/2}^{n+1}\right\} k + \int_{x_{i-1}}^{x_{i+1}} u^{n+1}\,dx \quad \int_{x_{i-1}^{*}}^{x_{i+1}^{*}} u^{n}\,dx + O(hk^2) \tag{4.7}$$

where $k = t_{n+1} - t_n$. A first possibility is to use a standard approach, in which the solution is represented using a basis of piecewise linear functions [17]. Then, in the interval (x_{i-1}, x_{i+1}), u^{n+1} is approximated by

$$\hat{u}^{n+1}(x) = \begin{cases} u_i^{n+1} + \dfrac{u_i^{n+1} - u_{i-1}^{n+1}}{h}(x - x_i) & x_{i-1} \le x \le x \\ u_i^{n+1} + \dfrac{u_{i+1}^{n+1} - u_i^{n+}}{h}(x - x_i) & x_i \le x \le x_{i+1} \end{cases}$$

Evaluation of the derivative of u^{n+1} by means of (4.8) and substitution in the first term of Eq. (4.7), yield

$$\left\{\left(\frac{\partial u^{n+1}}{\partial x}\right)_{i+1/2} \quad \left(\frac{\partial u^{n+1}}{\partial x}\right)_{i-1/2}\right\} k = \frac{u_{i+1} + u_{i-1} - 2u_i}{h} k$$

The corresponding substitution in the next integral of Eq. (4.7) and the exact evaluation of the resulting expression, yield

$$\int_{x_{i-1}}^{x_{i+1}} u^{n+1}\,dx \quad \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} + 6u_i^{n+1}}{8} h \tag{4.10}$$

The other strategy does not use any representation for the sought solution [20]. It consists in concentrating all the information on u_i^{n+1}; i.e., the value of the solution at the cell centers, at time t_{n+1}. This we do, evaluating the terms occurring in left-hand side of Eq. (4.7), as accurately as possible, using the values of u at the cell centers, exclusively. An additional condition that we impose, in order to obtain a tri-diagonal matrix, is that only three cell centers be used. Then

$$\left\{\left(\frac{\partial u^{n+1}}{\partial x}\right)_{i+1/2} \quad \left(\frac{\partial u^{n+1}}{\partial x}\right)_{i-1/2}\right\} k = \frac{u_{i+1} + u_{i-1} - 2u_i}{h} k + O(h^3 k) \tag{4.11}$$

and

$$\int_{x_{i-1}}^{x_{i+} u^{n+1}\,dx \quad \frac{u_{i+1}^{n+1} + u_{i-1}^{n+1} + 22u_i^{n+1}}{24} h + O(h^5) \tag{4.12}$$

Equation (4.12) can be derived in many different ways. For example, the Taylor series expansion of u^{n+1} around the cell center x_i, may be integrated in the interval $(x_{i-1/2}, x_{i+1/2})$. When this is done, to obtain (4.12), the relation

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1} \quad \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2} + O(h^2) \tag{4.13}$$

Comparing the results obtained with each strategy, it is seen that Eqs. (4.9) and (4.11) coincide, while (4.10) and (4.12) differ. Thus, depending on the strategy that we have followed, we have obtained two different approximations and a natural question to ask, is if the difference between them is relevant. If this were the case, which one must be preferred.

When deriving Eq. (4.12), we were able to obtain the order of precision of the approximation. On the other hand, in Eq. (4.10), the order of accuracy is not given. However, this latter order of precision can be derived from Eq. (4.12). Ideed, using the identity

$$\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}+22u_i^{n+1}}{24}=\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}+6u_i^{n+1}}{8}-\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}\quad 2u_i^{n+1}}{12} \tag{4.14}$$

Eq. (4.12) can be written as

$$\int_{x_{i-1}}^{x_{i+}} u^{n+1}\,dx=\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}+6u_i^{n+1}}{8}h-\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}-2u_i^{n+1}}{12}h+O(h^5) \tag{4.15}$$

Using Eq. (4.13), it is seen that

$$\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}-2u_i^{n+1}}{12}h=6\left(\frac{\partial^2 u}{\partial x^2}\right)^{n+1}h^3+O(h^5) \tag{4.16}$$

Therefore

$$\int_{x_{i-1}}^{x_{i+1}} u^{n+1}\,dx=\frac{u_{i+1}^{n+1}+u_{i-1}^{n+1}+6u_i^{n+1}}{8}h+O(h^3) \tag{4.17}$$

Our conclusion is that by having followed an strategy in which all the information was concentrated in the cell centers, the order of accuracy of the aproximation was improved by two orders of magnitude.

As mentioned previously, when trying to derive an Eulerian-Lagrangian FVELLAM in [17], many numerical difficulties were encountered. However, such difficulties did not show up when deriving CELLAM in [20]. This is so, in spite of the fact that both methods were derived using constant test functions. Thus, apparently the explanation of this difference is that the approximation (4.10) was used in [17], while we used approximation (4.12), which is two orders more accurate, in [20].

5. Conclusions

The results of this paper indicate that the strategy proposed here, for discretizing partial differential equations and systems of such equations, has clear advantages over standard procedures and must be further studied and tested in a wider class of problems.

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