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Trefftz–Herrera domain decomposition

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The author's algebraic theory of boundary value problems has permitted systematizing Trefftz method and expanding its scope. The concept of TH-completeness has played a key role for such developments. This paper is devoted to revise the present state of these matters. Starting from the basic concepts of the algebraic theory, Green–Herrera formulas are presented and Localized Adjoint Method (LAM) derived. Then the classical Trefftz method is shown to be a particular case of LAM. This leads to a natural generalization of Trefftz method and a special class of domain decomposition methods: Trefftz–Herrera domain decomposition.

1 INTRODUCTION

By a boundary method, it is usually understood as a procedure for solving partial differential equations and/or systems of such equations, in which a subregion or the entire region, is left out of the numerical treatment, by use of available analytical solutions (or, more generally, previously computed solutions). Boundary methods reduce the dimensions involved in the problems, leading to considerable economy of work and constitute a very convenient manner for treating adequately unbounded regions. Generally, the dimensionality of the problem is reduced by one, but even when part of the region is treated by finite elements, the size of the discretized domain is reduced.[1,2]

There are two main approaches for the formulation of boundary methods; one is based on boundary integral equations and the other one, on the use of complete systems of solutions. The author has studied extensively a version of the method based on the use of complete systems of solutions, known as Trefftz method.[3–5] Although Trefftz's original formulation was linked to a variational principle, this is not required. What is peculiar of Trefftz method, is that solutions of the homogenous differential equation — more generally, adjoint differential equation — are used as weighting functions.

The method has been used in many fields. For example, applications to Laplace's equation are given by Mikhlin,[6] to the biharmonic equation by Rektorys[7] and to elasticity by Kupradze.[8] Also, many scattered contributions to the method can be found in the literature. Special mention is made here of work by Amerio, Fichera, Kupradze, Picone and Vekua.[9–13] Colton has constructed families of solutions which are complete for parabolic equations.[14,15]

Some years ago, the author started a systematic research of Trefftz method oriented to clarify the theoretical foundations required for using complete systems of solutions in a reliable manner, and expand the versatility of the method, making it applicable to any problem which is governed by partial differential equations and/or systems of such equations which are linear.

For symmetric systems, the results obtained were presented in several reports[5,16–30] and later integrated in book form.[4] They include: (a) a criterium of completeness (introduced in Ref. 16 and called Trefftz–Herrera, or TH-completeness); (b) approximating procedures and conditions for their convergence;[5,17–19] (c) formulation of variational principles;[20–25] and (d) development of complete systems of solutions.[16,26–28] In addition, the algebraic frame-work in which the theory has been constructed (the Algebraic Theory of Boundary Value Problems), has been used for developing biorthogonal systems of solutions.[29,30] Free-boundary problems, which are non-linear even when the governing differential equations are linear, were also treated–using Trefftz method.[31,32] Numerical procedures for fitting the boundary conditions were also discussed.[33]

Function theoretic methods supply general results for developing analytically complete systems of solutions.[34] The work by Vekua in 1948,[13] by Bergman in 1961[35] and by Gilbert in 1969,[34] and 1974,[36] on this subject was followed by many others (see, for example Refs 37–40). The author's algebraic theory of boundary problems permitted applying the results of function theoretical methods to specific problems; in particular the concept of TH-completeness has been quite relevant. According to Begehr & Gilbert, in their recent survey of function theoretic methods (Ref. 41, p. 115):

The function theoretic approach which was pioneered by Bergman[35] and Vekua[13] and then further developed by Colton,[37–39] Gilbert,[34,36] Kracht–Kreyszig,[40] Lanckau[42] and others, may now be effectively applied because of results of the formulation by Herrera[5] as an effective means to solving boundary value problems.

In addition, they present many applications of the TH-completeness concept. A MACSYMA program is given which illustrates how the TH-complete families generated by integral operators may be used to solve boundary value problems. They also combine Transmutation Theory of Carroll[43,44] with the idea of TH-complete families to solve boundary value problems and show how the farfield for a stratified ocean of finite depth may be computed.

Many engineering applications have been made (see, for example Refs 45–48). A method specifically designed to deal with elastic diffraction problems was presented in Ref. 19 and later applied to answer questions of seismic engineering and seismology.[49,53]

Work done after the results for symmetric operators appeared, permitted extending the algebraic theory to non-symmetric operators, leading to a generalized version of Trefftz method,[54–57] presently known as Localized Adjoint method,[58–61] in which discontinuous trial and test functions can be used simultaneously. Although, in actual numerical applications only in special cases the use of discontinuous trial and test functions is advisable, the frame-work for systematic analysis supplied by the author's algebraic theory is quite useful to elucidate many questions about the performance of weighting and base functions. Indeed, since the base functions can be appropriately thought as interpolators (or extrapolators) of the actual information contained in approximate solutions, and this latter information is determined by the weighting functions that are applied, such analysis is required for matching effectively both of them.[62]

Applications of this approach have been made successively to ordinary differential equations, for which highly accurate algorithms were developed,[56,63] multidimensional steady-state problems,[64] and optimal spatial methods for advection diffusion equations.[65–67]

A very successful application of LAM, to problems of transport was presented in a couple of papers.[59,60] The resulting methology is known as Eulerian–Lagrangian Localized Adjoint Method (ELLAM) and was developed by the ELLAM group (M. A. Celia, R. E. Ewing, T. F. Russel and the author), which was formed for this purpose. Many applications of ELLAM have been made[68–74] and a recent account of the subject was presented in Ref. 61.

On the other hand, in recent years domain decomposition methods have received much attention, as a tool for solving partial differential equations. This is mainly due to the development of parallel machines, since such methods are efficient for parallelizing numerical algorithms. In addition, they can be used to design adaptive algorithms which capture steep fronts that appear in many problems, such as modeling of transport. Domain decomposition methods are also used to simplify problems with complicated geometries or match regions with different physical parameters or different types of differential equations. A wealth of literature on the subject has appeared in recent years (see for example Refs 75–83).

The author's generalized version of Trefftz method in which discontinuous trial and test functions are admitted, leads in a direct manner to domain decomposition procedures. Such methodology was advanced in a previous publication,[64] although only recently research on the procedure, as an approach to domain decomposition methods, was initiated.[84–87]

The present paper aims to explain briefly the different topics that have been mentioned. Due to the abundance of material it was difficult to make an exhaustive survey; thus, it is not intended. Instead, we dwell on the different matters unevenly, making the explanations more extensive in subjects which are more recent and less published material is available.

Section 2, is devoted to present the abstract frame-work of the theory. Although much of it has been published elsewhere, the way it is presented is new. Section 3, presents Green–Herrera formulas which are used, in Section 4, to develop Localized Adjoint Method. In Section 5, the classical Trefftz method is revisited and Section 6, is devoted to apply the author's version of Trefftz method to develop domain decomposition procedures. It is worth mentioning that a non-standard method of collocation — TH-collocation — which possesses some attractive features in comparison with standard collocation, is also presented in this last section.

2 ALGEBRAIC THEORY OF BOUNDARY VALUE PROBLEMS

In this Section some of the most basic concepts and results — although proofs are generally not supplied — of the author's algebraic theory of boundary value problems, are presented. Some of these concepts and results were introduced in Refs 54 and 57. They also imply a kind of operator extensions whose connection with the theory of distributions was discussed in Ref. 88.

The discussions refer to linear operators of the type $P : D_1 \to D_2^*$, whose domain is a the linear space D_1, and whose values are linear functionals on D_2 (i.e. elements of D_2^*: the algebraic dual of the linear space D_2), as well as to operators whose domain is D_2 and with values in D_1^*, such as the transpose $P^* : D_2 \to D_1^*$ of P.

The notation $\langle Pu, v\rangle$ is used to denote the value of the

functional Pu at $v \in D_2$. Clearly, $\langle Pu, v\rangle$ is bilinear and this defines a one-to-one correspondence between operators $P : D_1 \to D_2^*$ and bilinear forms $\langle Pu, v\rangle$, on $D_1 \oplus D_2$.

Definition 2.1. — Boundary operators.
B is a boundary operator for P, iff

$$\langle Pu, v\rangle = 0, \qquad \forall v \in N_{B^*} \Longrightarrow Pu = 0 \tag{1}$$

Definition 2.2. — Disjoint operators.
A pair of operators $\{R_1, R_2\}$ of the same kind is said to be disjoint when R_1 is a boundary operator for R_2, while R_2 is a boundary operator for R_1.

A system of operators $\{R_1, R_2, \ldots, R_n\}$ of the same kind is said to be disjoint when each pair $\{R_i, R_j\}$, with $i \neq j$, is disjoint.

Definition 2.3. — Completely disjoint.
A pair of operators $\{R_1, R_2\}$ of the same kind are said to be completely (or fully) disjoint when, in addition to being disjoint, the pair $\{R_1^*, R_2^*\}$ is also disjoint.

A system of operators $\{R_1, R_2, \ldots, R_n\}$ of the same kind is said to be fully disjoint when each pair $\{R_i, R_j\}$, with $i \neq j$, is fully disjoint.

Definition 2.4. — Decomposition $\{R_1, R_2\}$ of R.
A pair of operators $\{R_1, R_2\}$ is said to be a decomposition of R, when they are completely disjoint and

$$R = R_1 + R_2 \tag{2}$$

A system of operators $\{R_1, R_2, \ldots, R_n\}$ is said to be a decomposition of R, when they are fully disjoint and

$$R = R_1 + \ldots + R_n \tag{3}$$

Proposition 2.1. — Assume the pair $\{R_1, R_2\}$ decomposes R, then

$$N_R = N_{R_1} \cap N_{R_2} \tag{4}$$

REMARK. — If $\{R_1, R_2\}$ decomposes R, then $\{R_1^*, R_2^*\}$ decomposes R^*.

Definition 2.5. — Formal adjoints.
Two operators $P : D_1 \to D_2^*$ and $Q : D_2 \to D_1^*$ are formal adjoints when $S \equiv P - Q^*$ is a boundary operator for P, while $S^* \equiv P^* - Q$ is a boundary operator for Q.

Definition 2.6. — Green's formula.
The equation

$$P - B = Q^* - C^* \tag{5}$$

is said to be a Green's formula for the pair $\{P, Q\}$, when P and Q are formal adjoints and the pair $\{B, -C^*\}$ decomposes $S = P - Q^*$, while $\{B^*, -C\}$ decomposes $S^* = P^* - Q$.

Theorem 2.1. — Assume eq (5) is satisfied and it is a Green's formula for the pair $\{P, Q\}$. Let $\{B_1, B_2\}$ and $\{C_1, C_2\}$, be decompositions of B and C, respectively. Then, the equation

$$(P - B_1) - B_2 = (Q - C_1)^* - C_2^* \tag{6}$$

is a Green's formula for the pair $\{(P - B_1), (Q - C_1)\}$.

Definition 2.7. – The (abstract) boundary value problem.
Let B be a boundary operator for P. Given $f \in D_2^*$ and $g \in D_2^*$, the boundary problem consists in finding $u \in D_1$ such that

$$Pu = f \quad \text{and} \quad Bu = g \tag{7}$$

Theorem 2.2. — Variational formulation in terms of the data.
$u \in D_1$, is solution of the boundary problem, iff

$$(P - B)u = f - g \tag{8}$$

Theorem 2.3. — Variational formulation in terms of the sought information.
When $P - B = Q^* - C^*$ is a Green's formula, $u \in D_1$, is solution of the boundary problem, iff

$$(Q^* - C^*)u = f - g \tag{9}$$

Definition 2.8. — The subspaces I_P and I_Q.
Let $P - B = Q^* - C^*$, be a Green's formula and write $S = B - C^*$. Then the subspace $I_P \subset D_1$ is defined by

$$I_P = N_P + N_S \tag{10a}$$

and the subspace $I_Q \subset D_2$ by

$$I_Q = N_Q + N_{S^*} \tag{10b}$$

Definition 2.9. — TH-completeness.
Given a Green's formula: $P - B = Q^* - C^*$, a subset $\mathscr{W} \subset I_Q \subset D_2$, is said to be TH-complete (for P), if and only if, for any $U \in D_1$ and $V \in D_1$, one has

$$\langle BU, w\rangle - \langle C^* V, w\rangle = 0 \qquad \forall w \in \mathscr{W} \Longrightarrow \exists u \in I_P,$$
$$\text{such that } Bu = BU \text{ and } C^* u = C^* V \tag{11}$$

REMARK. — Observe that actually, one can take $u \in N_P$.

Theorem 2.4. — Let $P - B = Q^* - C^*$ be a Green's formula and $\mathscr{W} \subset I_Q$ be TH-complete for P. Assume there exists a solution $u \in D_1$, of the boundary value problem. Then, an element $\hat{u} \in D_1$ satisfies

$$C^* \hat{u} = C^* u' \tag{12}$$

for some solution $u' \in D_1$, of the boundary problem, if and only if,

$$\langle C^* \hat{u}, w\rangle = \langle C^* u, w\rangle, \qquad \forall w \in \mathscr{W}$$

When $\mathscr{W} \subset N_Q$, this equation can be replaced by

$$\langle C^* \hat{u}, w\rangle = -\langle f - g, w\rangle, \qquad \forall w \in \mathscr{W} \tag{14}$$

Proof. — When u and u' are solutions of the boundary value problem, one has

$$B(u-u')=0 \quad \text{and} \quad P(u-u')=0 \tag{15}$$

Hence $u-u' \in I_P$ and $B(u-u')=0$. Thus

$$-\langle C^*(u-u'), w\rangle = \langle S(u-u'), w\rangle = 0 \qquad \forall w \in \mathscr{W} \tag{16}$$

and if $C^*\hat{u} = C^*u$, then $C^*\hat{u} = C^*u'$.

Conversely, if

$$\langle C^*\hat{u}, w\rangle = \langle C^*u, w\rangle, \quad \forall w \in \mathscr{W} \tag{17}$$

then

$$\langle BV, w\rangle - \langle C^*(\hat{u}-u), w\rangle = 0, \qquad \forall w \in \mathscr{W} \tag{18}$$

with $V=0$. Then, by virtue of the definition of TH-completeness, there exists $r \in N_P$ such that

$$Br = 0 \quad \text{and} \quad C^*r = C^*(\hat{u}-u) \tag{19}$$

Define $u' = u + r$. Then, $Pu' = Pu = f$ and $Bu' = Bu = g$; i.e. $u' \in D_1$ is the solution of the boundary value problem.

Finally, when $u \in D_1$ is the solution of the boundary value problem and $\mathscr{W} \subset N_Q$, we have

$$\begin{aligned} -\langle C^*u, w\rangle &= \langle (Q-C)w, u\rangle = \langle (Q^*-C^*)u, w\rangle \\ &= \langle (P-B)u, w\rangle = \langle f-g, w\rangle \end{aligned} \tag{20}$$

which shows that the two previous equations are equivalent.

3 GREEN–HERRERA FORMULAS

Consider a region Ω and the linear spaces D_1 and D_2 of trial and test functions, respectively, defined in Ω. Assume further that functions belonging to D_1 and D_2 may have jump discontinuities across some internal boundaries whose union will be denoted by Σ (Fig. 1).

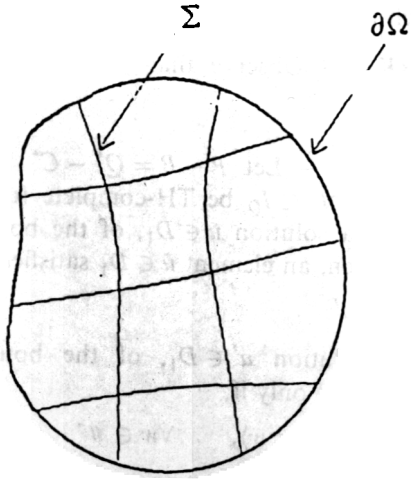


Fig. 1. The region Ω, its boundary $\partial\Omega$ and internal boundaries Σ.

For example, in applications of the theory to finite element methods, the set Σ could be the union of all the interelement boundaries. In this setting the general boundary value problem to be considered is one with prescribed jumps, across Σ.

The differential equation is

$$\mathscr{L}u = f_\Omega, \qquad \text{in } \Omega \tag{21}$$

where Ω may be a purely spatial region or, more generally, a space–time region. Certain boundary and jump conditions are specified on the boundary $\partial\Omega$ and on the internal boundaries Σ, respectively. When Ω is a space–time region, such conditions generally include initial conditions.[59] In the literature on mathematical modeling of macroscopic physical systems, there are a variety of examples of initial–boundary value problems with prescribed jumps. To mention just one, problems of elastic wave diffraction can be formulated as such.[19,89] The jump conditions to be satisfied across Σ by the sought solution, depend on the specific application and on the differential operator considered. For example, for elliptic problems of second order, continuity of the sought solution and its normal derivative, is usually required. When the partial differential equations mimic-continuous systems — as is frequently the case — the jump conditions can be derived, very systematically, from the balance equations of Continuum Mechanics.[90]

The definition of formal adjoint requires that a differential operator $\mathscr{L}$ and its formal adjoint $\mathscr{L}^*$, satisfy the condition that $w\mathscr{L}u - u\mathscr{L}^*w$ be a divergence; i.e.

$$w\mathscr{L}u - u\mathscr{L}^*w = \nabla\cdot\{\underline{\mathscr{Q}}(u,w)\}$$

for a suitable vector-valued bilinear function $\underline{\mathscr{Q}}(u,w)$. Integration of eqn (22) over Ω and application of the generalized divergence theorem,[90] yield:

$$\begin{aligned} \int_\Omega \{w\mathscr{L}u - u\mathscr{L}^*w\}\,\mathrm{d}x &= \int_{\partial\Omega} \underline{\mathscr{Q}}(u,w)\cdot\mathbf{n}\,\mathrm{d}x \\ &\quad - \int_\Sigma [\underline{\mathscr{Q}}(u,w)]\cdot\mathbf{n}\,\mathrm{d}x \end{aligned}$$

Here, the square brackets stand for the 'jumps' across Σ of the function contained inside, i.e. limit on the positive side minus limit on the negative one. Here, as in what follows, the positive side of Σ is chosen arbitrarily and then the unit normal vector $\mathbf{n}$ is taken pointing towards it. Integrals over the region Ω, are understood as sums of integrals over the individual regions Ω_i. For example, if $\{\Omega_1,\ldots,\Omega_N\}$ is a partition of the region Ω, then:

$$\int_\Omega w\mathscr{L}u\,\mathrm{d}x = \sum_i^N \int_i w\mathscr{L}u\,\mathrm{d}x$$

Thus, such integrals are well defined even differential operator is not defined on Σ.

In the general theory of partial differential equations, Green's formulas are used extensively. For the construction of such formulas it is standard to introduce a decomposition of the bilinear function $\mathscr{R}_\partial$ (see, for example, Lions & Magenes[91]). Indicating, transposes of bilinears forms by means of a star, the general form of such decomposition is:

$$\mathscr{R}_\partial(u,w) \equiv \underline{\mathscr{Q}}(u,w)\cdot\mathbf{n} = \mathscr{B}(u,w) - \mathscr{C}^*(u,w) \tag{25}$$

where $\mathscr{B}(u,w)$ and $\mathscr{C}(w,u) = \mathscr{C}^*(u,w)$ are two bilinear functions. When considering initial–boundary value problems, the definitions of these bilinear forms depend on the type of boundary and initial conditions to be prescribed. A basic property required of $\mathscr{B}(u,w)$ is that for any u that satisfies the prescribed boundary and initial conditions, $\mathscr{B}(u,w)$ is a well-defined linear function of w, independent of the particular choice of u. This linear function will be denoted by g_∂ (thus, its value for any given function w will be $g_\partial(w)$), and the boundary conditions can be specified by requiring that $\mathscr{B}(u,w) = g_\partial(w)$, for every $w \in D_2$ (or more briefly: $\mathscr{B}(u,\cdot) = g_\partial$). The linear function $\mathscr{C}^*(u,\cdot)$, on the other hand, can not be evaluated in terms of the prescribed boundary values, but it also depends exclusively on certain boundary values of u (the 'complementary boundary values'). Generally, such boundary values are determined only after the initial–boundary value problem has been solved.

In a similar fashion, convenient formulations of boundary value problems with prescribed jumps requires constructing Green's formulas in discontinuous fields. This can be done by introducing a general decomposition of the bilinear function

$$\mathscr{R}_\Sigma(u,w) \equiv -[\underline{\mathscr{Q}}(u,w)]\cdot\mathbf{n} = \mathscr{J}(u,w) - \mathscr{K}^*(u,w) \tag{26}$$

whose definition, on Σ, is point-wise. For continuous coefficients

$$\mathscr{J}(u,w) \equiv -\underline{\mathscr{Q}}([u],\dot{w})\cdot\mathbf{n} \tag{27a}$$

$$\mathscr{K}^*(u,w) \equiv \mathscr{K}(w,u) = \underline{\mathscr{Q}}(\dot{u},[w])\cdot\mathbf{n} \tag{27b}$$

The decomposition of eqns (26) and (27), stems from the algebraic identity:

$$[\underline{\mathscr{Q}}(u,w)] = \underline{\mathscr{Q}}([u],\dot{w}) + \underline{\mathscr{Q}}(\dot{u},[w]) \tag{28}$$

where

$$[u] = u_+ - u_-, \qquad \dot{u} = (u_+ - u_-)/2 \tag{29}$$

The general theory includes the treatment of differential operators with discontinuous coefficients[54,56] and for the examples to be considered in the sequel, formulas will be given that are valid also in that case.

An important property of the bilinear function $\mathscr{J}(u,w)$ is that, when the jump of u is specified, it defines a unique linear function of w, which is independent of the particular choice of u. When considering initial–boundary value problems with prescribed jumps, the linear function defined by the prescribed jumps in this manner will be denoted by j_Σ (thus, its value for any given function w will be $j_\Sigma(w)$) and the jump conditions at any point of Σ can be specified by means of the equation $\mathscr{J}(u,\cdot) = j_\Sigma$. In problems with prescribed jumps, the linear function $\mathscr{K}^*(u,\cdot)$, plays a role similar to that of the complementary boundary values $\mathscr{C}^*(u,\cdot)$. It can only be evaluated after the initial–boundary value problem has been solved and certain information about the average of the solution and its derivatives on Σ is known. Such information refers to the averages of the function and its derivatives across the jump. Observe that $\mathscr{B}(u,\cdot)$ and $\mathscr{J}(u,\cdot)$, by the way they have been defined, are part of the 'data of the problem' when jumps are prescribed. On the other hand, $\mathscr{C}^*(u,\cdot)$ and $\mathscr{K}^*(u,\cdot)$ are 'sought information', since they are not known beforehand.

As an illustration, the most general elliptic operator of second order can be written as:

$$\mathscr{L}u \equiv -\nabla\cdot(\underline{\underline{\mathbf{a}}}\cdot\nabla u) + \underline{\mathbf{b}}\cdot\nabla u + (\mathbf{c} + \tfrac{1}{2}\nabla\cdot\underline{\mathbf{b}})u$$

In such case

$$\mathscr{L}^*w \equiv -\nabla\cdot(\underline{\underline{\mathbf{a}}}\cdot(\nabla w) - \underline{\mathbf{b}}\cdot\nabla w + (\mathbf{c} - \tfrac{1}{2}\nabla\cdot\underline{\mathbf{b}})w \tag{30b}$$

and

$$\mathscr{Q}(u,w) = \underline{\underline{\mathbf{a}}}\cdot(u\nabla w - w\nabla u) + \underline{\mathbf{b}}uw$$

When the boundary conditions are of Dirichlet type one can define:

$$\begin{aligned}\mathscr{B}(u,w) &= (\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w + \mathbf{b}_n w)u, \text{ and}\\ \mathscr{C}^*(u,w) &= w(\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u)\end{aligned} \tag{32}$$

The complementary boundary values $\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u$, can be interpreted as diffussive flux. However, depending on the specific application considered, other physical interpretations are feasible.

According to eqns (29), one has:

$$\mathscr{J}(u,w) = -\overline{(\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w + \mathbf{b}_n w)}\,[u] + \dot{w}[\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u] \tag{33a}$$

and

$$\mathscr{K}^*(u,w) = [\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla w + \mathbf{b}_n w]\dot{u} - [w]\,\overline{\mathbf{n}\cdot\underline{\underline{\mathbf{a}}}\cdot\nabla u} \tag{33b}$$

In eqns (33), a bar is used to make it clear that the dot on top refers to the whole expression covered by the bar. In these equations $\mathbf{b}_n = \underline{\mathbf{b}}\cdot\mathbf{n}$. It can be shown that in the form eqns (33) have been written, they are valid even if the coefficients of $\mathscr{L}$ are discontinuous across Σ.

Going back to the general developments, it is seen that in view of eqns (23), (25) and (28), one can write:

$$\int_\Omega w\mathscr{L}u\mathrm{d}x - \int_{\partial\Omega}\mathscr{B}(u,w)\mathrm{d}x - \int_\Sigma \mathscr{J}(u,w)\mathrm{d}x = \int_\Omega u\mathscr{L}^* w\mathrm{d}x - \int_{\partial\Omega}\mathscr{C}^*(u,w)\mathrm{d}x - \int_\Sigma \mathscr{K}^*(u,w)\mathrm{d}x \quad (34)$$

which, in turn, can be written as:

$$\langle Pu,w\rangle - \langle Bu,w\rangle - \langle Ju,w\rangle = \langle Q^*u,w\rangle - \langle C^*u,w\rangle - \langle K^*u,w\rangle \quad (35)$$

with the notation

$$\langle Pu,w\rangle = \int_\Omega w\mathscr{L}u\mathrm{d}x; \qquad \langle Q^*u,w\rangle = \int_\Omega u\mathscr{L}^* w\mathrm{d}x \quad (36a)$$

$$\langle Bu,w\rangle = \int_{\partial\Omega}\mathscr{B}(u,w)\mathrm{d}x; \quad (36b)$$

$$\langle C^*u,w\rangle = \int_{\partial\Omega}\mathscr{C}(w,u)\mathrm{d}x$$

$$\langle Ju,w\rangle = \int_\Sigma \mathscr{J}(u,w)\mathrm{d}x \quad \text{and} \quad (36c)$$

$$\langle K^*u,w\rangle = \int_\Sigma \mathscr{K}(w,u)\mathrm{d}x$$

Equation (35) is an identity between bilinear forms; a more brief expression for it is:

$$P - B - J = Q^* - C^* - K^* \quad (37)$$

This is *Green–Herrera formula for operators in discontinuous fields.*[59]

4 LOCALIZED ADJOINT METHOD

The boundary value problem with prescribed jumps can be formulated point-wise by means of the eqn (21) together with

$$\mathscr{B}(u,\cdot) = g_\partial(\cdot) \quad \text{and} \quad \mathscr{J}(u,\cdot) = j_\Sigma(\cdot \quad (38)$$

Here u is any function satisfying the prescribed boundary and jump conditions, while g_∂ and j_Σ are linear functions with the property that

$$\mathscr{B}(u,w) = g_\partial(w) \quad \text{and} \quad \mathscr{J}(u,w) = j_\Sigma(w); \quad \forall w \in D_2 \quad (39)$$

Thus for example, when $\mathscr{L}$ is the elliptic operator of eqns (30), and the boundary and jump conditions are

$$u = u_\partial, \text{ on } \partial\Omega; \quad [u] = j^0, \ [\underline{\underline{a}}\cdot\nabla u]\cdot\mathbf{n} = j^1, \quad \text{on } \Sigma$$

Here, u_∂, j^0 and j^1 are prescribed functions — defined on $\partial\Omega$, the first one and on Σ the last two. In agreement with previous discussions, a physical interpretation of j^1 is the jump of diffusive flux. Then, by virtue of eqns (32)* and (33), one has

$$g_\partial(w) = (\mathbf{n}\cdot\underline{\underline{a}}\cdot\nabla w)u_\partial \quad \text{and}$$
$$j_\Sigma(w) = -\overline{(\mathbf{n}\cdot\underline{\underline{a}}\cdot\nabla w + \mathbf{b}_n w)}\, j^0 + w j^1 \quad (41)$$

In order to associate a variational formulation with this problem, define the linear functionals f, g, $j \epsilon D_2'$ by means of:

$$\langle f,w\rangle = \int_\Omega w f_\Omega \mathrm{d}x; \qquad \langle g,w\rangle = \int_{\partial\Omega} g_\partial(w)\mathrm{d}x;$$
$$\langle j,w\rangle = \int_\Sigma j_\Sigma(w)\mathrm{d}x \quad (42)$$

Then a variational formulation of the initial–boundary value problem with prescribed jumps, using the linear functionals defined before, is:

$$Pu = f; \qquad Bu = g; \qquad Ju = j \quad (43)$$

In what follows, it will be assumed that the operators B and J, defined in Section 3 (Eqns 36.b and 36.c), are *boundary operators* for P, which are *fully disjoint*, in the sense of Section 2. This depends on the choice of the subspaces D_1 and D_2. Consider, for example, the very important case when $\mathscr{L}$ is the elliptic operator of eqn (30a) and Ω is a bounded domain of R^2 with a piecewise smooth boundary $\partial\Omega$, such as the polygonal domains discussed by Bramble, Pasciak & Schatz,[92,93] for instance, and each of the subregions Ω_i of the decomposition of Ω, also have piecewise smooth boundaries — they may be, for example, triangles or quadrilaterals. If the coefficients are sufficiently smooth, for example $C^\infty(\Omega)$, take $D_1 = D_2 = D$, and define the linear space D as follows. For every $u \in D$, let $u(\Omega_i)$ be its restriction to Ω_i; then, the linear space of functions D, is defined by the condition that $u(\Omega_i) \in H^s(\Omega_i)$ for every $u \in D$ and every Ω_i of the decomposition. For such D, it is sufficient to chose $s \geqslant 1$. When $s \geqslant 2$, it is straight forward to verify that B and J are boundary operators for P which are fully disjoint. If $1 \leqslant s < 2$, this is more subtle and will be explained elsewhere.[94]

When these conditions are satisfied, the system of eqns (43) is equivalent to the single variational equation

$$\langle (P - B - J)u, w\rangle = \langle f - g - j, w\rangle \qquad \forall w \epsilon D_2 \quad (44)$$

This is said to be 'the variational formulation in terms of the data of the problem', because Pu, Bu and Ju are prescribed. Making use of Green–Herrera formula (37), the variational formulation (44) is transformed into

$$\langle (Q^* - C^* - K^*)u, w\rangle = \langle f - g - j, w\rangle \qquad \forall w \epsilon D_2 \quad (45)$$

This is said to be 'the variational formulation in terms of the sought information', because Q^*u, C^*u and K^*u are not prescribed. Notice that the variational formulations, (44) and (45), are equivalent since (37) is an identity. The

linear functionals Q^*u, C^*u and K^*u supply information about the sought solution at points in the interior of the region Ω, the complementary boundary values at $\partial\Omega$, and the generalized averages of the solution at Σ, respectively, as can be verified by inspection of eqns (36).

Localized Adjoint Methods (LAM) are based on the following observations. When the method of weighted residuals, with a system $\{w^1, \ldots, w^N\} \subset D_2$ of weighting or test functions is applied, an approximate solution $\hat{u} \epsilon D_2$ satisfies:

$$\langle (P - B - J)\hat{u}, w^\alpha \rangle = \langle f - g - j, w^\alpha \rangle, \qquad \alpha = \quad , N \tag{46}$$

or equivalently, using the variational principle in terms of the sought information

$$\langle (Q^* - C^* - K^*)\hat{u}, w^\alpha \rangle = \langle f - g - j, w^\alpha \rangle \qquad \alpha = 1, \ldots, N \tag{47}$$

Since the exact solution satisfies (45) it must be that

$$\langle (Q^* - C^* - K^*)\hat{u}, w^\alpha \rangle = \langle (Q^* - C^* - K^*)u, w^\alpha \rangle \qquad \alpha = 1, \ldots, N \tag{48}$$

Either in this or in the form:

$$\langle (Q^* - C^* - K^*)(\hat{u} - u), w^\alpha \rangle = 0, \qquad \alpha = 1, \ldots, N \tag{49}$$

these equations can be used to analyze the information about the exact solution that is contained in an approximate one.[54,57,59]

5 THE CLASSICAL TREFFTZ METHOD

Specialized weighting functions can be developed that concentrate the information in a specified manner. The information contained in the sought solution has been classified in three groups: the values of the solution in the interior of the subregions Ω_i, which are given by Q^*u; the complementary boundary values at the external boundary $\partial\Omega$, which are given by C^*u; and the average values, of the solution and its derivatives, in the interelement boundaries Σ, which are given by K^*.

The classical Trefftz method corresponds to the case when the information is concentrated on the complementary boundary values which are defined at the outer boundary $\partial\Omega$ of the region Ω. By inspection of eqns (45), it is seen that this requires that the system of test functions $\{w^\alpha\} \equiv \mathscr{W} \subset D_2$, have the property

$$\langle Q^*u, w^\alpha \rangle \equiv \langle Qw^\alpha, u \rangle = 0 \quad \text{and} \quad \langle K^*u, w^\alpha \rangle \equiv \langle Kw^\alpha, u \rangle = 0 \tag{50}$$

Thus

$$Qw^\alpha = 0, \quad \text{and} \quad Kw^\alpha = 0 \tag{51}$$

since eqns (50), must be satisfied for any function u.

When these conditions are satisfied eqns (45), for the approximate solution $\hat{u}$, reduce to

$$-\langle C^*\hat{u}, w^\alpha \rangle = \langle f - g - j, w^\alpha \rangle, \qquad \forall w^\alpha \in \mathscr{W} \tag{52}$$

which are equivalent to

$$\langle C^*\hat{u}, w^\alpha \rangle = \langle C^*u, w^\alpha \rangle, \qquad \forall w^\alpha \in \mathscr{W} \tag{53}$$

since the exact solution u also satisfies

$$-\langle C^*u, w^\alpha \rangle = \langle f - g - j, w^\alpha \rangle, \qquad \forall w^\alpha \in \mathscr{W} \tag{54}$$

As it has been seen in Section 2 of this chapter, when the system of functions $\{w^\alpha\} \equiv \mathscr{W}$ is TH-complete, then any function $\hat{u}$ satisfies the system of eqns (52), if and only if

$$C^*\hat{u} = C^*u \tag{55}$$

In words: *when the system of weighting (or test) functions $\mathscr{W}$, is TH-complete, a function $\hat{u}$ is an approximate solution* — in the sense of the weighted residuals method, defined with precision before — *if and only if its complementary boundary values on $\partial\Omega$, are those of the exact solution.* This result exhibits clearly that with such weighting functions the information has been effectively concentrated on the boundary $\partial\Omega$ of Ω.

The condition $Qw^\alpha = 0$, is tantamount to requiring

$$\mathscr{L}^*w^\alpha = 0, \qquad \text{in } \Omega \tag{56}$$

by virtue of the second of eqns (36). Thus, in order to concentrate the information on the boundaries, the test functions must be solutions of the homogeneous version of the adjoint differential equation. Many applications of Trefftz method have been concerned with self-adjoint operators, in which case eqn (56), may be replaced by

$$\mathscr{L}u^\alpha = 0 \qquad \text{in } \Omega \tag{57}$$

The second of eqns (51) i.e. $Kw^\alpha = 0$–, impose some smoothness conditions on the test functions. If the differential operator is the general elliptic operator of eqn (30), then such conditions can be seen to be

$$[w] = 0, \qquad \text{and} \qquad [\mathbf{n} \cdot \underline{\mathbf{a}} \cdot \nabla w + \mathbf{b}_n w] = 0 \tag{58}$$

That is, continuity of the function and of the total flux. When the coefficients are continuous these equations are equivalent to continuity of the function and its first partial derivative.

For applications of Trefftz method, it is frequently more convenient to define $D = D_1 = D_2$ as a Sobolev space $H^s(\Omega)$. When the operator is elliptic of second order, $s = 1/2$ is convenient.[4,5,41] Under very general conditions the null subspace of $P : N_P \subset D$, is a TH-complete system.[4,5,41] However, for the representation of solutions it has greater interest to have denumerable subsets $\mathscr{W} \subset N_P$ which are TH-complete. Begehr & Gilbert,[41] have presented a thorough and updated exposition of analytical methods for developing TH-complete systems. They include general equations related to Laplace operator, such as

$$\Delta u - q(\underline{x})u = f_\Omega, \qquad \Omega \subset R^n \tag{59}$$

Table 1. TH-complete systems in two dimensions

Bounded Ω	Ω Exterior of a bounded region
Laplace equation $\{1, r^n \cos n\theta, r^n \sin n\theta\}$	$\{\ln r, r^{-n} \cos n\theta, r^{-n} \sin n\theta\}$
Reduced wave equation $\Delta u + u = 0$ $\{J_0(r), J_n(r) \cos n\theta, J_n(r) \sin n\theta\}$ $n = 1, 2, \ldots$	$\{H_0^{(1)}(r), H_n^{(1)}(r) \cos n\theta, H_n^{(1)}(r) \sin n\theta\}$

Table 2. TH-complete systems in three dimensions

Bounded Ω	Ω = Exterior of a bounded region
Laplace equation $\{r^n P_n^q(\cos\theta) e^{jq\varphi}\}$	$\{r^{-n-1} P_n^q(\cos\theta) e^{jq\varphi}\}$
Helmoholtz or reduced wave equation $\{j_n(r) P_n^q(\cos(\theta) e^{jq\varphi}\}$ $n = 0, 1, 2, \ldots; -n \leqslant q \leqslant n$	$\{h_n^{(1)}(r) P_n^q(\cos\theta) e^{jq\varphi}\}$

Examples of such systems are given in Tables 1 and 2. For the Helmholtz or reduced wave equation the author has shown that a system of plane waves, which have a very simple structure, is TH-complete in any bounded and simply connected region.[19]

In these tables $J_n(r)$ and $H_n^{(1)}$ are Bessell and Hankel functions of the first class.[95,96] P_n^q is the associated Legendre function, while j_n and h_n^1 are the spherical Bessel and Hankel functions.[95] We recall, in addition, that the TH-complete systems given in Tables 1 and 2 for Laplace equation in a bounded region are harmonic polynomials expressed in polar and spherical coordinates

Equations associated with the biharmonic (Δ^2) differential operator, also have great interest and are not included among the elliptic second order operators, that have been discussed here, since they are of higher order. To incorporate this equation in the general frame-work we have developed, recall the identity

$$w\Delta^2 u - u\Delta^2 w \equiv \nabla \cdot \{w\nabla\Delta u - \nabla w \Delta u + \Delta w \nabla u - u\nabla\Delta w\} \tag{60}$$

which exhibits the biharmonic operator as a symmetric one, for which

$$\mathcal{Q}(u, w) = w\nabla\Delta u - \nabla w\Delta u + \Delta w\nabla u - u\nabla\Delta w \tag{61}$$

The author has presented TH-complete systems for this operator,[27,28] and Begehr & Gilbert discussed the matter further.[41]

6 TREFFTZ APPROACH TO DOMAIN DECOMPOSITION AND TH-COLLOCATION

In recent years domain decomposition methods have received much attention, as a tool for solving partial differential equations. This is mainly due to the development of parallel machines, since such methods are efficient for parallelizing numerical algorithms. In addition, they can be used to design adaptive algorithms which capture steep fronts that appear in many problems, such as modeling of transport. Domain decomposition methods are also used to simplify problems with complicated geometries or match regions with different physical parameters or different type of differential equations. A wealth of literature on the subjected has appeared in recent years (see for example Refs 75–83).

A basic feature of domain decomposition methods is that the region Ω, in which the boundary value problem is formulated, is decomposed into subregions $\{\Omega_1, \ldots, \Omega_E\}$. Then the global problem, in Ω, is obtained by solving local problems in each of the subregions, exclusively. The generalized version of Trefftz method, proposed by the author,[54–57,62,63] in which discontinuous trial and test functions are admitted, leads in a direct manner to domain decomposition procedures.

This can already be seen in one dimensional problems.[56,63] Thus, consider the most general ordinary differential equation of second order which is linear. A physical situation that this equation mimics is transport in the presence of advection, diffusion and linear sources, and a notation related with such processes will be adopted. The general equation to be considered is:

$$\mathcal{L}u \equiv -\frac{d}{dx}\left(D\frac{du}{dx} - Vu\right) + Ru = f_\Omega, \quad \text{in} \quad \Omega \equiv [0, l] \tag{62a}$$

The developments will be carried out in a way that possible discontinuities of the coefficients across Σ, will be accommodated. The function u will be assumed to be continuous

$$[u] = 0, \qquad \text{on } \Sigma, \tag{62b}$$

and the smoothness condition implied by conservation of mass:[90]

$$\left[Vu - D\frac{\partial u}{\partial x}\right] = 0, \qquad \text{on } \Sigma \tag{62c}$$

to be satisfied.

A partition $\{0 = x_0, x_1, \ldots, x_{E-1}, x_E = l\}$ is introduced. In this case the interelement boundary Σ, is a finite set of points; namely, $\Sigma = \{x_1, \ldots, x_{E-1}\}$. Trial and test functions will be assumed to be sufficiently differentiable in the interior of each of the subintervals of the partition, so that the differential operator is defined there and the jump discontinuities can only occur at internal nodes. Observe that the normal vector $\mathbf{n} = 1$ at l, and $\mathbf{n} = -1$ at 0. On Σ, the choice $\mathbf{n} = 1$ is convenient, because in this manner the positive side of Σ is the side that is determined by the sense of the x-axis. Suitable boundary conditions are assumed to be satisfied, at 0 and l, in order to have a well defined boundary value problem. The boundary conditions can be Dirichlet, Neumann, or Robin,[56] but they are left unspecified, since the following developments accommodate all of them.

The formal adjoint of the operator $\mathscr{L}$, as defined by (62a), is

$$\mathscr{L}^* w \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\left(D\frac{\mathrm{d}w}{\mathrm{d}x}\right) - V\frac{\mathrm{d}w}{\mathrm{d}x} + Rw \tag{63}$$

Therefore

$$w\mathscr{L}u - u\mathscr{L}^* w \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left\{u\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right) - wD\frac{\mathrm{d}u}{\mathrm{d}x}\right\} \tag{64}$$

and

$$\mathscr{D}(u, w) \equiv u\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right) - wD\frac{\mathrm{d}u}{\mathrm{d}x} \tag{65}$$

Application of eqns (27) yields

$$\mathscr{J}(u, w) = -[u]\,\overline{\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right)} + \dot{w}\left[D\frac{\mathrm{d}u}{\mathrm{d}x}\right] \tag{66a}$$

$$\mathscr{K}^*(u, w) = \mathscr{K}(w, u) = \dot{u}\left[D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right] - [w]\,\overline{D\frac{\mathrm{d}u}{\mathrm{d}x}} \tag{66b}$$

Therefore

$$\langle Ju, w\rangle = \sum_{j=1}^{E-1} \mathscr{J}(u, w)_j$$

$$= -\sum_{j=1}^{E-1}\left\{[u]\,\overline{\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right)} - \dot{w}\left[D\frac{\mathrm{d}u}{\mathrm{d}x}\right]\right\}_j \tag{67a}$$

$$\langle K^* u, w\rangle = \sum_{j=1}^{E-1} \mathscr{K}^*(u, w)_j$$

$$= -\sum_{j=1}^{E-1}\left\{\dot{u}\left[D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right] - [w]\,\overline{D\frac{\mathrm{d}u}{\mathrm{d}x}}\right\}_j \tag{67b}$$

For Dirichlet problems, one can define

$$\mathscr{B}(u, w) = u\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right)\cdot \mathbf{n};$$
$$\mathscr{C}(w, u) = wD\frac{\mathrm{d}u}{\mathrm{d}x}\cdot \underline{\mathrm{n}} \tag{68a}$$

For Neuman problems, $\mathrm{d}u/\mathrm{d}x$ is datum and

$$\mathscr{B}(u, w) = -wD\frac{\mathrm{d}u}{\mathrm{d}x}\cdot \underline{\mathrm{n}};$$
$$\mathscr{C}(w, u) = -u\left(D\frac{\mathrm{d}w}{\mathrm{d}x} + Vw\right)\cdot \underline{\mathrm{n}} \tag{68b}$$

In the most general form of Robin's boundary conditions, a linear combination of the derivative and the value of the solution are prescribed, and this general case was treated in Ref. 56. One, which is specially important is when the total flux $D\mathrm{d}u/\mathrm{d}x - Vu$ is prescribed. Then

$$\mathscr{B}(u, w) = -w\left(D\frac{\mathrm{d}u}{\mathrm{d}x} - Vu\right)\cdot \mathbf{n};$$
$$\mathscr{C}(w, u) = -uD\frac{\mathrm{d}w}{\mathrm{d}x}\cdot \mathbf{n} \tag{68c}$$

Domain decomposition methods are classified into overlapping and non-overlapping, depending on whether the domains of the decomposition have a non-void or a void intersection, respectively (see, for example Ref. 75). Both kinds of procedures can be formulated using the author's version of Trefftz method. It all depends on the conditions satisfied by the weighting functions.

A non-overlapping domain decomposition procedure

When the test functions $\{w^1, \ldots, w^E\}$ are required to satisfy $\mathscr{L}^* w^i \equiv 0$, in $\Omega_i \equiv (x_{i-1}, x_i)$ and $\mathscr{C}(w, \cdot) = 0$, at 0 and l where $i = 1, \ldots, E$, one can construct the weighting functions independently, in each one of the non-overlapping regions Ω_i (see Refs 56 and 63). There are two linearly independent solutions in each Ω_i, for $i = 2, \ldots, E-1$. However, due to the boundary conditions $\mathscr{C}(w, \cdot) = 0$, at 0 and l, there is only one in the subregions Ω_1 and Ω_E. This, yields $2(E-1)$ linearly independent test functions altogether.

On the other hand, for this choice of weighting functions, eqn (47) reduces to

$$-\langle K^* \hat{u}, w^\alpha\rangle = \langle f - g - j, w^\alpha\rangle, \qquad \alpha = 1, \ldots, 2(E-1) \tag{69}$$

By inspection of eqns (67), taking $\dot{u}$ and $\overline{D\mathrm{d}u/\mathrm{d}x}$ as unknowns, it is seen that there are two unknowns

associated with each internal node; hence, a total of $2(E-1)$ unknowns.[56,63]

A rigorous discussion of the conditions under which the resulting system of eqns (69) possesses a unique solution requires the use of the concept of TH-completeness of Section 2. However, in the present case the situation is specially simple, since the system of weighting functions is finite. When considering problems in more than one dimension matters get more complicated, since TH-complete systems are of infinite dimension.

As a last remark, it must be mentioned that the system of eqns (69), which is $2(E-1)\times 2(E-1)$, has a block bidiagonal structure, the blocks being 2×2.[56,63]

An overlapping domain decomposition procedure

Take the partition $\{0=x_0, x_1,\ldots,x_{E-1}, x_E=l\}$, as before, and define the collection $\{\Omega_1,\ldots,\Omega_{E-1}\}$ of subregions, by:

$$\Omega_i=(x_{i-1},x_{i+1}) \tag{70}$$

Then, the subregions are overlapping. Next, consider a collection of functions $\{u^1,u^2,\ldots,u^{E-1}\}$ such that, for every $i=1,\ldots,E-1$, satisfies $\mathscr{L}u^i=f_\Omega$ (eqn 62a), in Ω_i. In addition, when $i=1$ and $i=E-1$, u^i satisfies the left and right boundary conditions, respectively. Let $u(x)$, be the exact solution of the boundary value problem in $[0,l]$, which is assumed to exist and be unique. Then the following result, which will be the basis of the overlapping domain decomposition procedure is easy to see:

The conditions

$$u^{i+1}(x_i)=u^i(x_i),\quad \text{for}\quad i=1,\ldots,E-1 \tag{71}$$

are satisfied if and only if

$$u(x_i)=u^i(x_i),\quad \text{for}\qquad ,E- \tag{72}$$

To derive an algorithm, based on this result and using Trefftz method, it is convenient to develop a system of weighting functions $\{w^1,\ldots,w^{E-1}\}$, such that w^i, for each $i=1,\ldots,E-1$, has support in Ω_i and satisfies

$$\mathscr{L}^*w^i\equiv 0,\quad \text{in } \Omega_i;\quad [w^i]=0,\quad \text{at } x_i;$$
$$\mathscr{C}(w,\cdot)=0,\quad \text{at } 0 \text{ and } l \tag{73a}$$

The continuity condition $[w^i]=0$, implies that

$$w^i(x_{i-1})=w^i(x_{i+1})=0 \tag{73b}$$

since the support of w^i is contained in Ω_i. There is only one linearly independent solution, in every subregion Ω_i, which fulfills conditions (73). At the same time, for this choice of test functions, eqn (66b) reduces to

$$\mathscr{K}^*(u,w)=\mathscr{K}(w,u)=u\left[D\frac{\mathrm{d}w}{\mathrm{d}x}\right] \tag{74}$$

This implies that at each interior node there is only one unknown: namely the value of the solution. The resulting system of equations is $(E-1)\times(E-1)$ and tridiagonal. See Refs 56 and 63 for details.

TH-Collocation

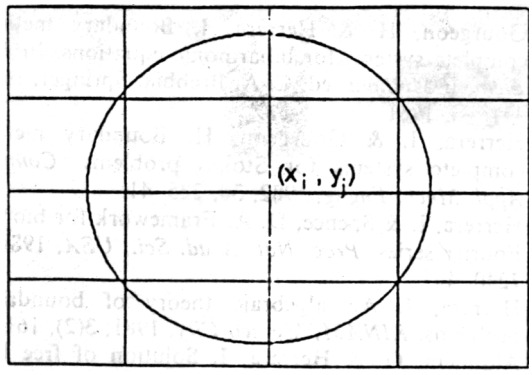
The test functions can be constructed by any means. Analytical means were used in Ref. 97, but such procedures have many restrictions. Numerical methods, on the other hand, are very general; in particular, collocation is very effective. A possibility is to use Gaussian collocation on polynomials to satisfy the differential equation in an approximate manner, as was done in Ref. 63. This leads to an alternative to standard collocation — TH-collocation —, based on the use of TH-complete systems of weighting functions.

Collocation is a well established method for solving partial differential equations (see, for example Ref. 98); the usual procedure — standard collocation — is a direct constructive method; that is, approximate solutions are constructed, piece by piece, and then they are put together imposing suitable smoothness conditions at the junctions. Gaussian collocation on polynomials is also used frequently[98] and, for second order differential equations, continuity of the function and its first derivative is required at the junctions.

On the other hand, when TH-collocation is applied, the procedure is indirect. Indeed, collocation is used to construct weighting functions which satisfy, in an approximate manner, the homogeneous-adjoint differential equation. The resulting method possesses some attractive features. Thus, for example, as was mentioned before, when applying standard collocation, continuity of the function and its first derivative is required. In TH-collocation, on the other hand, continuity conditions are relaxed, since the test functions for the overlapping version, are continuous but with discontinuous first derivative. Another important point is that, in standard collocation, the resulting system of equations involves both, the function and its derivative, while in TH-collocation the system of equations involves the function only. Due to these facts the structure and size of the matrices of the systems of equations to be solved are simpler and smaller. Finally, in TH-collocation, when the elliptic operator is symmetric and the test functions are also used as base functions, the matrix is positive definite, which is not the case for standard collocation.

TH-Collocation in several dimensions

TH-collocation has also been applied in several dimensions,[84-87] in the case when $\mathscr{L}$ is the general elliptic operator of eqn (30a). Here, a brief description of the procedure for two dimensional problems is presented. With slight modifications the same can be done in three dimensions.



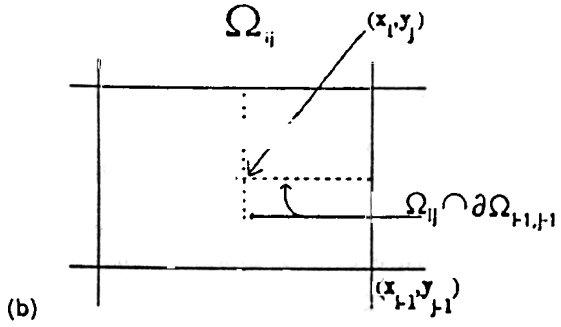


Fig. 2. Overlapping of $\Omega_{i-1,j-1}$ with $\Omega_{i,j}$

For simplicity, consider a rectangular region decomposed into rectangles (Fig. 2a). With each internal node (x_i, y_j), associate a subregion (Ω_{ij}) made of four rectangles, which surround such node. For nodes located on the boundary, only the rectangles lying inside Ω, are included in Ω_{ij}. The subregions so defined, are overlapping. For the boundary value problem, of Section 3, defined by eqn (30a) and suitable boundary conditions, it is not difficult to establish the property that is formulated next.

Assume a unique solution, u, *of the boundary value problem exists and, for every* Ω_{ij}, *let* u^{ij} *be a function defined in* Ω_{ij}, *which satisfies the differential eqn (21). Then* $u^{ij} = u$, *in* Ω_{ij}, $\forall i, j$, *if and only if,* $u^{ij} = u^{kl}$ *on* $\Omega_{ij} \cap \partial\Omega_{kl}$ *(see Fig. 2b),* $\forall ij$ *and* $\forall kl$, *and in addition, the boundary conditions are satisfied by* u^{ij}, *when* $\Omega_{ij} \cap \partial\Omega$ *is not void. Here,* $\partial\Omega_{kl}$ *stands for the boundary of* Ω_{kl} *and* $\partial\Omega$ *is the boundary of* Ω.

Clearly, the set $\Omega_{ij} \cap \partial\Omega_{kl}$, is that part of the boundary of Ω_{kl}, that lies inside Ω_{ij}.

Numerical procedures based in this kind of formulation, require only the evaluation of the function. The derivatives of the sought solution need not be evaluated, which as mentioned before, is an advantage over standard collocation, for which one has to solve for both, the function and its derivatives. Thus, the test functions, w^{α}, in addition to satisfying $\mathscr{L}^* w^{\alpha} = 0$, locally, must be continuous; $[w^{\alpha}] = 0$, on the edges of the rectangles; in particular, functions whose support is

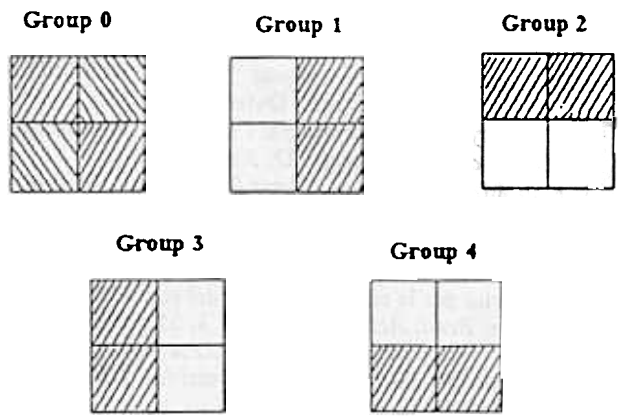


Fig. 3. The five groups of weighting functions, according to their supports.

Ω_{ij}, must vanish on $\partial\Omega_{ij}$. A possibility, that was applied in Refs 85 and 86, is to use polynomials of the same degree (G), in both x and y. In general, G can be any positive integer. Independently of the specific value of G, groups of $2G - 1$ test functions can be associated with each node. One of the test functions of each group can be taken to be equal to one at the corresponding node $(x_{ij}$, say) and the support of such function is the whole subregion Ω_{ij} (see Refs 85 and 86, for details). The remaining functions of the group have smaller support and can be divided into four subgroups according to their support (Fig. 3). The structure of the matrix is block nine-diagonal, blocks being $(2G - 1) \times (2G - 1)$. In addition, the classification of the test functions into five subgroups, is independent of the degree of the polynomials.

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