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A non-overlapping TH-domain decomposition

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In a previous paper a class of domain decomposition procedures, which may be viewed as a generalization of the Trefftz method, was announced: TH-domain decompositions. Their essential feature is that local solutions are not matched directly, but instead are used as specialized test functions with the property that, when they are applied as weighting functions, the information is concentrated on the internal boundaries of the subdomains. In this manner, the solution on the whole domain is constructed. In this paper such a method, which has already been used in some specific applications, is presented. © 1997 Elsevier Science Limited

Key words: domain decomposition methods, numerical methods, partial differential equations.

1 INTRODUCTION

Domain-decomposition methods for the numerical solution of partial differential equations have received much attention in recent years.[1] At present, this is mainly due to the fact that they constitute a very effective manner of parallelizing numerical models for continuous systems. Parallel computing is already a very important resource used by supercomputers and it is expected to be even more important in the future.

There are additional reasons for the interest in domain-decomposition methods, such as domains of irregular shape can be decomposed into more regular subdomains, and regions of relative nonuniformity of the differential operator or roughness of the solutions can be isolated into different subdomains. This paper presents a brief description of domain-decomposition methods and discusses new options for their formulation.

A theoretical approach to domain decomposition deals with the systems that are obtained after the differential equations have been discretized — the *discretized approach* — but it is also possible to formulate domain-decomposition procedures treating the differential equations before discretization — the *continuous approach.* In this paper the latter approach will be applied because it is more elegant and has the advantage of permitting the use of the known properties of partial differential equations in a direct manner. In addition, it is always possible afterwards to give discretized versions of the results so obtained.

In the exposition that follows the different methods will be derived from a unified perspective. This is based on the experience and clarity that has been gained through the considerable amount of work that has been done in recent years. This way of presenting matters has clear expository advantages, but it does not correspond to the way in which, historically, the methods were developed. In domain-decomposition methods, the region in which the problem is formulated is split into several, usually many, subregions. Given a differential equation, consider the set of its solutions at each one of such subregions. Then, the main objective of domain-decomposition methods is to select a solution at each subregion in such a way that suitable matching conditions are satisfied and the sought solution, over the whole region, is constructed in this manner.

Generally, the methods may be classified into two broad categories, depending on whether the subregions are disjoint — non-overlapping methods — or they have non-void intersections — overlapping methods. The main difference between these two kinds of procedures is the way in which the matching conditions are applied. For elliptic equations of second order, for example, the non-overlapping procedures require the solution, together with its normal derivative across the common boundaries of the non-overlapping subregions,

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to be continuous. On the other hand, due to the manner in which the matching conditions are imposed when using overlapping procedures, for the same kind of equations the smoothness of approximate solutions is relaxed. This occurs since only the function itself is required to be continuous, but its normal derivatives are generally discontinuous.

Only elliptic differential equations of second order,[2] will be considered in some detail in what follows. In this case, independent of the kind of domain decomposition that is applied overlapping or non-overlapping, knowing the restriction of the solution on the boundaries of the subregions, the *internal boundaries*, determines uniquely the solution at the interior of the subregions. In addition, the process of extending such restriction from the boundaries to the interior involves solving local problems only. Due to this fact, domain-decomposition procedures frequently aim to find such a restriction and the main goal of the 'domain-decomposition problem' is obtaining the solution at the boundaries of the subregions. A feature that characterizes that restriction is the following: when it is extended to the interior of the subregions, in the manner stated before, the resulting normal derivatives are continuous across the internal boundaries. The procedures that are usually applied in the continuous approach are based on this property.[3–5]

However, there is another approach to determining the solution of the *domain-decomposition problem*, which has recently been proposed by Herrera:[6] TH-domain decomposition. It is based on a generalization of the Trefftz method (Trefftz–Herrera method[7]). A very successful application of this point of view is the localized adjoint method.[8,9] Essentially, it is based on the fact that when local solutions, in the subregions of the domain decomposition, of the adjoint differential equation are applied as weighting functions — in the method of weighted residuals — all the information about the sought solution contained in approximate ones, refers to the solution values at the internal boundaries.[10–13] In particular, if the system of weighting functions is TH-complete,[6,14] the values of the approximate solution at the internal boundaries are exactly equal to those of the solution of the boundary value problem. An important property is that TH-domain decomposition is applicable to differential equations associated with symmetric and non-symmetric operators, and the systems of such equations.

Time-dependent problems of the parabolic type, when time discretization is applied, give rise to elliptic problems at each time step and the discussion presented in this paper is also relevant for them. Some special methods for parabolic equations explicitly profit from the locality of the fundamental responses,[15] but even if the methodology is not specialized in this manner, when iterative procedures are used, rapid convergence is usually achieved.[16]

2 DOMAIN-DECOMPOSITION FORMULATION

The general ideas of the method are outlined next.

Consider the boundary value problem (BVP) defined in Ω (Fig. 1), associated with a general elliptic differential operator which consists in satisfying

$$\mathscr{L}u \equiv -\nabla\cdot(\underline{\underline{a}}\cdot\nabla u) + \nabla\cdot(\underline{b}u) + cu = f_\Omega \qquad \text{in } \Omega \tag{1}$$

subject to Dirichlet boundary conditions:

$$u(\underline{x}) = u_\partial(\underline{x}) \qquad \text{in } \partial\Omega \tag{2}$$

Generally, the matrix $\underline{\underline{a}}$ is assumed to be positive definite.

A partition $\{\Omega_1,\ldots,\Omega_N\}$ of Ω into subdomains will be considered and the internal boundary, separating the subdomains from each other, will be denoted by Σ (Fig. 1). In addition, define $\partial_i\Omega = \partial\Omega\cap\partial\Omega_i$. Of course, $\partial_i\Omega$ is void when the closure of Ω_i lies in the interior of Ω; however, the relation

$$\partial\Omega_i \subset \Sigma\cup\partial_i\Omega \tag{3}$$

is always satisfied.

The goal of domain-decomposition methods is to construct the solution (u) of the BVP, by solving exclusively boundary value problems in each Ω_i (Fig. 1) separately. To this end, for each $i = 1,\ldots,N$, define the function u_{Pi}, in Ω_i, as the unique solution of the boundary value problem

$$\mathscr{L}u_{Pi} = f_\Omega \qquad \text{in } \Omega_i \tag{4}$$

subject to the boundary conditions

$$u_{Pi}(\underline{x}) = u_\partial(\underline{x}) \quad \text{in } \partial_i\Omega; \qquad u_{Pi}(\underline{x}) = V(\underline{x}) \quad \text{in } \Sigma \tag{5}$$

where $V(\underline{x})$ is a function defined on Σ, suitably chosen. In particular, $V(\underline{x})$ can be chosen to be identically zero on Σ, and for simplicity we assume this in what follows.

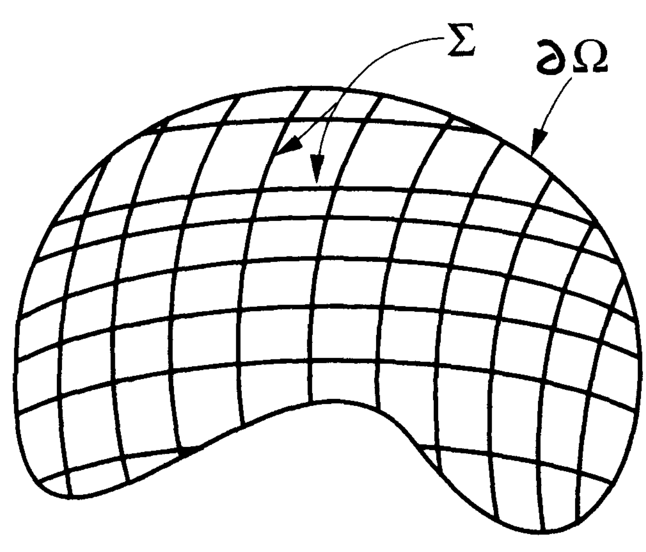

Fig. 1. Domain decomposition subregions.

Having defined $u_{Pi}(\underline{x})$ for $i = 1, \dots, N$, the function $u_P(\underline{x})$ is defined in Ω by

$$u_P(\underline{x}) = u_{Pi}(\underline{x}) \qquad \underline{x} \in \Omega_i \qquad i = 1, \dots, N \tag{6}$$

Clearly, the function $u_P(\underline{x})$ so defined, is continuous in Ω, but generally, the normal derivative of u_P is discontinuous across Σ, and the following proposition is straightforward.[5,17]

Proposition 1
Let the function $u(\underline{x})$ be the solution of the BVP in Ω. Then, the function $u_H(\underline{x})$, defined in Ω by $u_H \equiv u - u_P$, is the unique solution of the boundary value problem with prescribed jumps (BVPJ) on Σ, defined by the following conditions:

The differential equation $\mathscr{L}u_H = 0$ is satisfied in Ω_i $(i = 1, \dots, N)$, separately, $u_H(\underline{x}) = 0$ for every $\underline{x} \in \partial\Omega$, u_H is continuous across Σ and its first-order partial derivatives have jump discontinuities across Σ, which satisfy the jump condition

$$[\underline{\underline{a}} \cdot \nabla u_H] \cdot \underline{n} = -[\underline{\underline{a}} \cdot \nabla u_P] \cdot \underline{n} \qquad \text{on } \Sigma' \tag{7}$$

Here, the square-brackets stand for the value of the 'jump' across Σ (value on the 'positive' side minus value on the 'negative' one) and $\underline{n}$ is taken pointing towards the positive side.

Conclusion
Let the boundary value problem, with prescribed jumps, defined by $\mathscr{L}u_H = 0$, $u_H = 0$ on $\partial\Omega$ and the jump conditions of eqn (7) for the normal derivative, be formulated in a space of functions which are continuous across Σ. Then such a problem can be reduced to a search problem for a function defined on Σ.

This is because when the restriction to Σ, of the function $u_H(\underline{x})$ introduced in Proposition 1, is known, the function u_H itself can be constructed, at each Ω_i $(i = 1, \dots, N)$, by solving a local Dirichlet problem in Ω, since $\partial\Omega_1 \subset \Sigma \cup \partial_1\Omega \subset \Sigma \cup \partial\Omega$ and $u_H = 0$ on $\partial\Omega$.

3 THE ALGEBRAIC FORMULATION AND TH-DOMAIN DECOMPOSITION

Consider two linear spaces, D_1 and D_2, of functions defined on a region Ω. Members of D_1 and D_2, referred to as trial and test functions, respectively, may have jump discontinuities across some internal boundaries Σ. Linear operators P, B, $J: D_1 \to D_2^*$, where D_2^* is the algebraic dual of D_2 (i.e. the space of linear functionals defined on D_2) will be considered in what follows. As it can be seen, there is a one-to-one correspondence between such operators and bilinear functionals defined on $D_1 \oplus D_2$.

In Herrera's algebraic theory of boundary value problems, such problems are formulated by means of two variational formulations:[8–10] one in 'terms of the data'

$$\langle (P - B - J)u, w \rangle = \langle f - g - j, w \rangle \qquad \forall w \in D_2 \tag{8a}$$

and the other one in terms of the 'sought information':

$$\langle (Q^* - C^* - K^*)u, w \rangle = \langle f - g - j, w \rangle \qquad \forall w \in D_2 \tag{8b}$$

where $f, g, j \in D_2^*$ are linear functionals defined in terms of the data of the problem. Both variational formulations are equivalent because the operators Q^*, C^*, $K^*: D_1 \to D_2^*$, which are defined so that they satisfy the 'Green–Herrera formula':[9]

$$P - B - J = Q^* - C^* - K^* \tag{9}$$

In the more general case of higher order equations or systems of equations — application of theory to systems of equations has been explained in[9] — the operator J is associated with the jumps of the functions and their derivatives across internal boundaries, and it is additive in the jump of the function itself (J^0), the jump of the first derivative (J^1), etc., so that it can be written as $J = J^0 + J^1 + \dots + J^n$, where n is the highest order of the derivatives occurring in J. Similarly, K^* which is associated with the averages across the internal boundaries, can be written as

$$K^* = K^{0^*} + K^{1^*} + \dots + K^{n^*} \tag{10}$$

For second-order equations, to which attention will be restricted, the operators J and K will be written as sums: $J = J^0 + J^1$ and $K = K^0 + K^1$, and the linear subspaces $D_{1H} \subset D_1$ and $D_{2H} \subset D_2$ will be characterized as follows: $\hat{u} \in D_{1H}$ and $w \in D_{2H}$, if and only if

$$(P - B - J^0)\hat{u} = 0 \qquad \text{and} \qquad (Q - C - K^1)w = 0 \tag{11}$$

respectively. When $w \in D_{2H}$, eqn (8b) reduces to

$$-\langle K^{0^*} u, w \rangle = \langle f - g - j, w \rangle \tag{12}$$

when a system $\mathscr{W} \subset D_{2H}$, is TH-complete, the fact that a trial function $\hat{u} \in D_1$ satisfies eqn (12) for every $w \in \mathscr{W}$, implies that $\exists u \in D_1$ solution of the boundary value problem, for which

$$K^{0^*}\hat{u} = K^{0^*}u \tag{13}$$

4 VARIATIONAL FORMULATION AND POSITIVENESS

In what follows, the most general elliptic operator of second order will be considered, written in conservative form:

$$\mathscr{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot \underline{b}u + cu \tag{14a}$$

Its formal adjoint is:

$$\mathscr{L}^* w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw \tag{14b}$$

In particular $\underline{\underline{a}}$ is assumed to be coercive (thus, positive definite). As described in Section 2, the domain Ω is decomposed into a system of subdomains Ω_i, $i = 1, \ldots, N$ (Fig. 1), and the union of the common boundaries between the subdomains is denoted by Σ. The boundary value problem with prescribed jumps, is defined by

$$\mathscr{L}u = f_\Omega \qquad \text{in } \Omega \tag{15}$$

subject to suitable boundary and jump conditions on $\partial\Omega$ and Σ, respectively.

The variational formulations of Section 3 can be applied to this problem if the corresponding operators are defined by:[9]

$$\langle Pu, w\rangle = \int_\Omega w\mathscr{L}u\,\mathrm{d}x; \qquad \langle Q^*u, w\rangle = \int_\Omega u\mathscr{L}^*w\,\mathrm{d}x \tag{16a}$$

$$\langle Bu, w\rangle = \int_{\partial\Omega} \mathscr{B}(u, w)\,\mathrm{d}x; \qquad \langle C^*u, w\rangle = \int_{\partial\Omega} \mathscr{C}(w, u)\,\mathrm{d}x \tag{16b}$$

$$\langle J^0u, w\rangle = \int_\Sigma \mathscr{J}^0(u, w)\,\mathrm{d}x; \qquad \langle K^0w, u\rangle = \int_\Sigma \mathscr{K}^0(w, u)\,\mathrm{d}x \tag{16c}$$

$$\langle J^1u, w\rangle = \int_\Sigma \mathscr{J}^1(u, w)\,\mathrm{d}x \quad \text{and} \quad \langle K^1w, u\rangle = \int_\Sigma \mathscr{K}^1(w, u)\,\mathrm{d}x \tag{16d}$$

where

$$\mathscr{J}^0(u, w) = -\underline{n}\cdot\overline{(\underline{\underline{a}}\cdot\nabla w + \underline{b}w)}^{\bullet}[u]; \qquad \mathscr{J}^1(u, w) = \dot{w}[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla u] \tag{17a}$$

and

$$\mathscr{K}^0(w, u) = \underline{n}\cdot[\underline{\underline{a}}\cdot\nabla w + \underline{b}w]\dot{u}; \qquad \mathscr{K}^1(w, u) = -[w]\overline{\underline{n}\cdot\underline{\underline{a}}\cdot\nabla u}^{\bullet} \tag{17b}$$

The square brackets and the dots stand for the 'jump' and the 'average' across the discontinuity, of the corresponding functions. Thus, for example,

$$[u] = u_+ - u_-, \qquad \dot{u} = (u_+ + u_-)/2 \tag{18}$$

Also, the bar on top means that the dot refers to the whole expression covered by it. In addition, it will be understood that eqn (10) holds with $n = 1$, i.e.

$$J = J^0 + J^1, \qquad K = K^0 + K^1 \tag{19}$$

On the other hand, $\mathscr{B}(u, w)$ and $\mathscr{C}(w, u) = \mathscr{C}^*(u, w)$ are two bilinear functions whose definitions depend on the type of boundary conditions to be prescribed.[9] When the boundary conditions are of Dirichlet type — to which the following discussion is restricted — one can define

$$\mathscr{B}(u, w) = u(\underline{\underline{a}}\cdot\nabla w + \underline{b}w)\cdot\underline{n}, \qquad \mathscr{C}^*(u, w) = w(\underline{\underline{a}}\cdot\nabla u)\cdot\underline{n} \tag{20}$$

A possible physical interpretation of the 'complementary boundary values', $\underline{n}\cdot\underline{\underline{a}}\cdot\nabla u$, is diffusive flux across the boundary.

Assume, in particular, that the boundary and jump conditions are

$$u = u_\partial \quad \text{on } \partial\Omega; \qquad [u] = 0, \quad [\underline{\underline{a}}\cdot\nabla u]\cdot\underline{n} = 0 \quad \text{on } \Sigma \tag{21}$$

This corresponds to a Dirichlet problem in which the sought solution, together with diffusive flux, are continuous across Σ. When $\underline{\underline{a}}$, as a function of position, is continuous, these conditions are fulfilled, if and only if, u and its first-order derivatives are continuous across Σ. Then the functionals f, g and j, introduced in Section 3, are:

$$\langle f, w\rangle = \int_\Omega wf_\Omega\,\mathrm{d}x; \qquad \langle g, w\rangle = \int_{\partial\Omega} u_\partial(\underline{\underline{a}}\cdot\nabla w + \underline{b}w)\cdot\underline{n}\,\mathrm{d}x; \qquad \forall w \in D_2 \tag{22}$$

while $j \equiv 0$; i.e. $\langle j, w\rangle = 0$, $\forall w \in D_2$.

The variational formulation in terms of the 'sought information' (eqn 8b), yields information about the sought solution in the interior of the subdomains (the term $\langle Q^*u, w\rangle$), the complementary boundary values on $\partial\Omega$ (the term $\langle C^*u, w\rangle$), the normal derivative on Σ (the term $\langle K^{1^*}u, w\rangle$), and the value of the solution itself on Σ (the term $\langle K^{0^*}u, w\rangle$). In order to concentrate the information on the value of the sought solution on Σ, exclusively, weighting functions which eliminate the information everywhere else will be chosen. This requires dropping the terms $\langle Q^*u, w\rangle$, $\langle C^*u, w\rangle$ and $\langle K^{1^*}u, w\rangle$. This is achieved if the weighting functions satisfy the condition $(Q - C - K^1)w = 0$, which is equivalent to

$$\mathscr{L}^*w = 0, \qquad \mathscr{C}(w, \cdot) = 0 \text{ and } \mathscr{K}^1(w, \cdot) = 0 \tag{23}$$

or more explicitly

$$-\nabla\cdot(\underline{\underline{a}}\cdot\nabla w) - \underline{b}\cdot\nabla w + cw = 0 \qquad \text{in each } \Omega_i \tag{24a}$$

and

$$w = 0 \quad \text{on } \partial\Omega; \quad \text{and} \quad [w] = 0 \quad \text{on } \Sigma \tag{24b}$$

Thus, for the case under consideration, the subspace $D_{2H} \subset D_2$ of Section 3, is the subspace of functions which are continuous, vanish on the boundary and satisfy the homogeneous differential equation, in each Ω_i separately. For such test functions the variational equation in terms of the sought information reduces to eqn (12), i.e.

$$-\langle K^{0^*}u, w\rangle = \langle f-g-j, w\rangle \tag{25}$$

If, as it will be assumed in what follows, $\mathscr{W} \subset D_{2H}$ is a TH-complete system, then the fact that a trial function $\hat{u} \in D_1$ satisfies eqn (25) for every $w \in \mathscr{W}$, implies that

$$K^{0^*}\hat{u} = K^{0^*}u \tag{26}$$

If we restrict attention to continuous trial functions $\hat{u}$, then from eqns (16), (17) and (22), it follows that when test functions w_H are taken from D_{2H}

$$\mathscr{K}^{0^*}(u_H, w_H) = [a_n \partial w_H/\partial n]u_H \tag{27}$$

where $a_n \equiv \underline{n}\cdot\underline{\underline{a}}\cdot\underline{n}$, and a more explicit statement of the variational principle is

$$\begin{aligned}-\langle K^{0^*}\hat{u}, w_H\rangle &\equiv -\int_\Sigma [a_n \partial w_H/\partial n]\hat{u}\,\mathrm{d}\underline{x}\\ &= \int_\Omega w_H f_\Omega\,\mathrm{d}\underline{x}\\ &\quad -\int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial w_H/\partial n + \underline{\mathrm{b}}\cdot\underline{n}w_H)\,\mathrm{d}\underline{x}\\ &\quad \forall w_H \in \mathscr{W}\end{aligned} \tag{28}$$

The case when the differential operator is symmetric ($\underline{\mathrm{b}} \equiv 0$), has special interest. Then, it is convenient taking $D_1 = D_2 = D$ and, to simplify the notation, write $D_H = D_{2H}$. Also, observe that due to the symmetry of the differential operator:

$$J^0 = K^1, \qquad J^1 = K^0 \tag{29}$$

Given $\hat{u}_H \in D_H$ and $w_H \in D_H$, the identity

$$\begin{aligned}-\langle K^{0^*}\hat{u}_H, w_H\rangle &\equiv -\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial w_H/\partial n]\,\mathrm{d}\underline{x}\\ &= \int_\Omega (\nabla\hat{u}_H\cdot\underline{\underline{\mathrm{a}}}\cdot\nabla w_H + \mathrm{c}\hat{u}_H w_H)\,\mathrm{d}\underline{x}\end{aligned} \tag{30a}$$

holds; it exhibits the symmetry of the expressions involved. In addition, when $\mathrm{c} \geq 0$ one has

$$\begin{aligned}-\langle K^{0^*}\hat{u}_H, \hat{u}_H\rangle &\equiv -\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial \hat{u}_H/\partial n]\,\mathrm{d}\underline{x}\\ &= \int_\Omega (\nabla\hat{u}_H\cdot\underline{\underline{\mathrm{a}}}\cdot\nabla \hat{u}_H + c\hat{u}_H^2)\,\mathrm{d}\underline{x} \geq 0\end{aligned} \tag{30b}$$

and the equality applies, if and only if $\hat{u}_H \equiv 0$ in Ω. Thus, the restriction $\hat{K}^0$ ($\hat{K}^0: D_H \to D_H^*$) of the operator K^0 to D_H, is positive definite. Observe that eqn (30b) exhibits K^0 as a Poincaré-Steklov operator.[18]

In view of the identity (30a), there are two equivalent expressions for the variational principle (28). The first one:

$$\begin{aligned}-\langle K^0\hat{u}_H, w_H\rangle &\equiv -\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial w_H/\partial n]\,\mathrm{d}\underline{x}\\ &= \int_\Omega w_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial w_H/\partial n)\,\mathrm{d}\underline{x}\\ &\quad \forall w \in \mathscr{W}\end{aligned} \tag{31a}$$

characterizes directly the values of u on Σ, and it can be applied using any trial function $\hat{u}_H \in D$, not necessarily belonging to D_H. The second one:

$$\begin{aligned}-\langle K^0\hat{u}_H, w_H\rangle &\equiv \int_\Omega (\nabla\hat{u}_H\cdot\underline{\underline{\mathrm{a}}}\cdot\nabla w_H + \mathrm{c}\hat{u}_H w_H)\,\mathrm{d}\underline{x}\\ &= \int_\Omega w_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial w_H/\partial n)\,\mathrm{d}\underline{x}\\ &\quad \forall w \in \mathscr{W}\end{aligned} \tag{31b}$$

characterizes any function $\hat{u}_H \in D_H$ which takes, on Σ, the values of the solution u, of the boundary value problem. Observe that in order to supply this equation, the trial function $\hat{u}_H$ must belong to D_H.

In view of the symmetry of $\langle K^{0^*}\hat{u}_H, w_H\rangle$ and eqn (29), the variational principle (31a), can also be written as:

$$\begin{aligned}-\langle J^1\hat{u}_H, w_H\rangle &\equiv -\int_\Sigma w_H[\mathrm{a}_n \partial \hat{u}_H/\partial n]\,\mathrm{d}\underline{x}\\ &= \int_\Omega w_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial w_H/\partial n)\,\mathrm{d}\underline{x}\\ &\quad \forall w \in \mathscr{W}\end{aligned} \tag{32}$$

This equation characterizes the jumps of a function $\hat{u}_H \in D_H$, such that $\hat{u}_H = u$, on Σ. The following discussion clarifies this statement. The right-hand side of eqn (32) involves integrals over Ω and $\partial\Omega$, and for later developments it will be useful to express them as integrals over Σ, as was done for the left-hand side of eqn (31a). Let u_P and u_H be defined as in Section 2, i.e. $\mathscr{L}u_P = f_\Omega$ in Ω_i $(i = 1, \ldots, N)$, $u_P = u_\partial$ on $\partial\Omega$, $u_P = 0$ on Σ and $u_P + u_H = u$, where the function u is a solution of the boundary value problem. In view of the definition of u_P, it is clear that $u_H = u$ on Σ. Also, $u_H \in D_H$ and it can be seen that

$$J^1 u_H = -J^1 u_P, \qquad K^{0^*}u_H = K^{0^*}u \tag{33}$$

Hence

$$\begin{aligned}\langle J^1 u_P, w_H\rangle &= -\langle J^1 u_H, w_H\rangle = -\langle K^0 u_H, w_H\rangle\\ &= -\langle K^{0^*}u_H, w_H\rangle = -\langle K^{0^*}u, w_H\rangle\end{aligned} \tag{34}$$

Using eqn (25), this permits writing the variational principle of eqn (32) in the form:

$$-\langle J^1\hat{u}_H, w_H\rangle = \langle J^1 u_P, w_H\rangle \qquad \forall w \in \mathscr{W} \tag{35}$$

or more explicitly:

$$-\int_\Sigma w_H[\mathrm{a}_n \partial \hat{u}_H/\partial n]\,\mathrm{d}\underline{x} = \int_\Sigma w_H[\mathrm{a}_n \partial u_P/\partial n]\,\mathrm{d}\underline{x} \qquad \forall w \in \mathscr{W} \tag{36}$$

It can be shown that this equation is satisfied, if and only if

$$[\mathrm{a}_n \partial \hat{u}_H/\partial n] = -[\mathrm{a}_n \partial u_P/\partial n] \qquad \text{on } \Sigma \tag{37a}$$

For continuous coefficients this reduces to

$$[\partial \hat{u}_H/\partial n] = -[\partial u_P/\partial n] \qquad \text{on } \Sigma \tag{37b}$$

Finally, from the above discussion it is easy to see that eqn (36) is satisfied by a function $\hat{u}_H \in D_H$, if and only if, $\hat{u}_H = u$ on Σ.

The following remarks are also relevant. Equation (36) implies that

$$\int_\Omega w_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial w_H/\partial n)\,\mathrm{d}\underline{x} = \int_\Sigma w_H[\mathrm{a}_n \partial u_P/\partial n]\,\mathrm{d}\underline{x} \qquad \forall w \in \mathscr{W} \tag{38}$$

and the variational principles of eqn (31), can be replaced by

$$-\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial w_H/\partial n]\,\mathrm{d}\underline{x} = \int_\Sigma w_H[\mathrm{a}_n \partial u_P/\partial n]\,\mathrm{d}\underline{x} \qquad \forall w \in \mathscr{W} \tag{39a}$$

and

$$\int_\Omega (\nabla \hat{u}_H \cdot \underline{\underline{a}} \cdot \nabla w_H + \mathrm{c}\hat{u}_H w_H)\,\mathrm{d}\underline{x} = \int_\Sigma w_H[\mathrm{a}_n \partial u_P/\partial n]\,\mathrm{d}\underline{x} \qquad \forall w \in \mathscr{W} \tag{39b}$$

Equation (39a) has special interest because it permits expressing the variational principle in a form that involves surface integrals over Σ, exclusively.

Before leaving this section, it must be mentioned that a minimum principle can be formulated, because the operator $\hat{K}: D_H \to D_H$, is positive definite. There are two expressions for the functional associated with this principle:

$$\mathscr{F}_1(\hat{u}_H) \equiv -\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial \hat{u}_H/\partial n]\cdot \underline{n}\,\mathrm{d}\underline{x} - 2\left\{\int_\Omega \hat{u}_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial \hat{u}_H/\partial n)\,\mathrm{d}\underline{x}\right\} \tag{40a}$$

and

$$\mathscr{F}_2(\hat{u}_H) \equiv \int_\Omega \{\nabla \hat{u}_H \cdot \underline{\underline{a}} \cdot \nabla \hat{u}_H + \mathrm{c}\hat{u}_H^2\}\,\mathrm{d}\underline{x} - 2\left\{\int_\Omega \hat{u}_H f_\Omega\,\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial(\mathrm{a}_n \partial \hat{u}_H/\partial n)\cdot \underline{n}\,\mathrm{d}\underline{x}\right\} \tag{40b}$$

The minimum principle states that either one of these expressions attains its minimum on D_H, if and only if $\hat{u}_H \in D_H$ satisfies eqn (26) for a solution $u \in D_1$ of the boundary value problem. Again, functional $\mathscr{F}_1(\hat{u}_H)$ of eqn (40a), can be expressed in terms of integrals over Σ, when use is made of eqn (38).

5 DISCRETIZATION

In actual applications, the variational principles in Section 4 must be discretized. Discretization is specially simple when eqn (39a) is applied.

In what follows the $L^2(\Sigma)$ inner product, for functions defined on Σ, will be considered, and projections of such functions will be taken with respect to that inner product. For simplicity, it will be assumed that the function u_P, whose definition was given in Section 2, is already known. This is reasonable when discussing domain-decomposition methods, since its construction involves exclusively solving boundary value problems which are local, i.e. defined in each one of the subregions of the domain decomposition, separately.

Let $\bar{S}^0(\Sigma)$ be a finite-dimensional linear space of functions defined on Σ. This may be made, for example, by functions which are piece-wise polynomials, and an example in which they were taken as piece-wise bicubic polynomials, was given in Refs. 6 & 19. Such a finite-dimensional space induces a unique sub-space $\hat{D}_H(\Omega) \subset D_H$, also finite dimensional, defined by the properties that a function $v \in \hat{D}_H(\Omega)$, if and only if its trace on Σ belongs to $\bar{S}^0(\Sigma)$.

In the space $\bar{S}^0(\Sigma)$, the variational formulation (39a) may be written as

$$\underline{\underline{A}}\mathrm{U} = \underline{b} \tag{41}$$

where $\underline{b} \in \bar{S}^0(\Sigma)$ is the projection on $\bar{S}^0(\Sigma)$, of the function $[\mathrm{a}_n \partial u_P/\partial n]$ defined on Σ, while the finite dimensional linear transformation $\underline{\underline{A}}$ of $\bar{S}^0(\Sigma)$ into itself, is defined as follows:

Given any $\mathrm{U} \in \bar{S}^0(\Sigma)$, *let* $\hat{u}_H$ *be the unique element of* $\hat{D}_H(\Omega)$, *such that its trace on* Σ, *equals U. Then,* $\underline{\underline{A}}\mathrm{U} \in \bar{S}^0(\Sigma)$ *is defined as the projection, on* $\bar{S}^0(\Sigma)$, *of the function* $-[\mathrm{a}_n \partial \hat{u}_H/\partial n]$ *defined on* Σ.

It can be seen that the finite-dimensional mapping $\underline{\underline{A}}$, so defined, is symmetric and positive definite, since

$$(\mathrm{U}, \underline{\underline{A}}\mathrm{U}) = -\int_\Sigma \hat{u}_H[\mathrm{a}_n \partial \hat{u}_H/\partial n]\,\mathrm{d}\underline{x} \tag{42}$$

Thus, conjugate gradient methods, or preconditioned conjugate gradient methods, are applicable.

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