A Consistent Approach to Variable Bubble-Point Systems

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Here, it is shown that the ‘‘traditional approach’’ to variable bubble-point problems, using black-oil models, is not consistent, because it violates the ‘‘bubble-point conservation law.’’ In order to have a consistent approach, it is necessary to incorporate shocks—discussed in previous papers—in which the bubble-point is discontinuous. A ‘‘consistent approach’’ is applied to specific examples, and results compared with those of the ‘‘traditional’’ one. The conclusion that the ‘‘traditional approach’’ generally yields large errors for the production rates and other parameters of interest in the oil industry, is reached. © 1997 John Wiley & Sons, Inc.

I. INTRODUCTION

In this article, using results of previous research [1–5], it is shown that the ‘‘traditional’’ black-oil model approach to variable bubble-point problems [6–9] is *inconsistent,* and computations are carried out to demonstrate that such inconsistency generally yields large errors in the evaluation of production rates and other parameters of interest for the oil industry. In addition, a *consistent* formulation of black-oil models, suitable for application to variable bubble-point problems, is supplied.

When applying black-oil models to variable bubble-point problems, it is frequently assumed that the bubble-point pressure may vary inside the undersaturated region [6–9]. However, such an assumption is incorrect, because it contradicts the basic postulates on which black-oil models are built. Indeed, such postulates do not include molecular diffusion, nor mechanical dispersion, and it has been shown that a consequence of such omission is the ‘‘bubble-point conservation law,’’ according to which: when a gas-phase is not present, oil-particles conserve their gas content (dissolved gas:oil ratio). This result imposes very severe restrictions to the manners in which the dissolved gas:oil ratio of an oil-particle can vary, when a gas-phase is absent; physically, it means that when a gas-phase is not present, two oil particles cannot exchange dissolved gas, even if they

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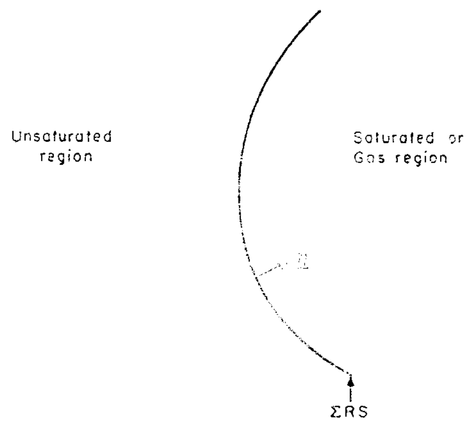

FIG. 1. Subregions of a reservoir with variable bubble-point.

are very close. In particular, such restrictions are not respected in the ''traditional'' approach, as explained in Section III of this article.

When the bubble-point is variable, it is necessary to introduce shocks in which the bubble-point, i.e., the solution gas-oil ratio: R_s^-, is discontinuous. However, this is not done in the ''traditional'' approach [6–9]. Since the classical Buckley–Leverett theory was developed [10, 11], modelers are prepared to deal with discontinuities of the saturation, but jumps of R_s are not included.

In this respect, it has been shown (see, for example, [4]), and it will be further discussed in this article, that when dealing with variable bubble-point problems, in general, in which free-gas may, or may not, coexist with liquid oil, the region of definition of the problem may be decomposed into three parts (Fig. 1):

a. A region where free-gas is absent, the oil-region, usually undersaturated;
b. A region where free-gas is present, the gas-region, necessarily saturated; and
c. A gas-front; i.e., a boundary between the oil and the gas regions.

Then:

—**In the oil-region,** only the bubble-point may have jump discontinuities and such discontinuities propagate with the velocity of oil-particles.
—**In the gas-region,** only the saturation may have jump discontinuities, and the shocks that can occur are essentially of the type described by the Buckley–Leverett theory. They occur only if capillary pressure is neglected.
—**At a gas-front,** two possible situations must be distinguished:

a. A front that **advances** into an undersaturated region. At such a front, in general, both the bubble-point and the saturation are discontinuous and the motion of the front is retarded with respect to the gas particles.
b. A front that **recedes** from an undersaturated region. At such a front, only the saturation is discontinuous and the front moves with the velocity of the gas particles.

—**A bifurcation** occurs when an advancing gas-front, which carries a double discontinuity: discontinuous saturation and discontinuous bubble-point, changes its sense of motion and starts to recede; then, the two discontinuities start to move with different velocities and, in this manner, they separate. In such a case, the bubble-point discontinuity moves with the oil-particle velocity, while the saturation discontinuity moves with the gas-particle velocity.

It is important to stress that only in one case does the presence of capillary pressure preclude the occurrence of shocks, i.e., in the saturated region, where free gas is present, and shocks occurring in that region are essentially of the Buckley–Leverett type. Other kinds of shocks may occur even if capillary forces are incorporated in the model.

II. BLACK-OIL MODEL

For the present analysis, it is enough to consider a ''black-oil'' or ''beta'' model, consisting of two phases, liquid oil and gas, whose Darcy velocities are denoted by $\underline{u}_o$ and $\underline{u}_g$, respectively. Black-oil models are based on the following assumptions [6]:

a. Gas is soluble in liquid oil, i.e., the gas-phase consists of only one component, while the liquid oil is made up of two components (dissolved gas and nonvolatile oil);
b. No physical diffusion is present. This includes both molecular diffusion and that induced by the randomness of the porous medium (mechanical dispersion).

In addition, the multiphase form of Darcy's Law is adopted; in particular, pressures are assumed to be continuous everywhere. Black-oil models, in general, include the possibility of nonvanishing capillary pressure, as is done here.

We use the notations $\bar{\rho}_o$ and $\bar{\rho}_{dg}$ for the effective densities of nonvolatile oil and dissolved gas, respectively, together with the relation:

$$\bar{\rho}_{dg} = \bar{R}_s \bar{\rho}_o, \qquad \text{where } \bar{R}_s \equiv \frac{\rho_{gSTC}}{\rho_{oSTC}} R_s. \tag{1}$$

The factor R_s is the ''solution gas:oil ratio'' [6], and $\bar{R}_s$ is introduced, because it simplifies the mathematical analysis, and some formulas adopt a simpler form, when it is used. Then the governing equations of the black-oil model may be written as:

$$(\phi S_o \bar{\rho}_o)_t + div(\bar{\rho}_o \underline{u}_o) = 0 \tag{2a}$$

$$(\phi S_o \bar{R}_s \bar{\rho}_o)_t + div(\bar{\rho}_o \bar{R}_s \underline{u}_o) + (\phi S_g \rho_g)_t + \text{div}(\rho_g \underline{u}_g) = 0, \tag{2b}$$

where ϕ is porosity and S_α represents saturation for phase α (oil or gas). In the following discussions, in addition to Darcy velocities, $\underline{u}_o$ and $\underline{u}_g$, ''particle velocities'' for each phase, will be considered in some instances; they are defined by

$$\underline{v}_\alpha = \underline{u}_\alpha / \phi S_\alpha; \qquad \alpha = o \text{ and } g. \tag{3}$$

As mentioned in the Introduction, when dealing with variable bubble-point problems, in general, in which free gas may or may not coexist with liquid oil, the region of definition of the problem may be decomposed into three parts (Fig. 1): one in which free-gas is present (the ''gas-region''), another in which free-gas is absent (the ''oil-region''), and a gas-front, which limits these two regions. To be specific, at the gas-front, the unit normal vector ($\underline{n}$) will be taken as pointing towards the gas region. When considering discontinuities across a surface Σ, the velocity of Σ will be $\underline{v}_\Sigma$, and the unit normal vector will be taken pointing toward the positive side. In particular, at the gas-front the gas-side will be the positive side (Fig. 1). Also, square brackets stand for the ''jump'' of a function; thus, for example: $[R_s] = R_s^+ - R_s^-$.

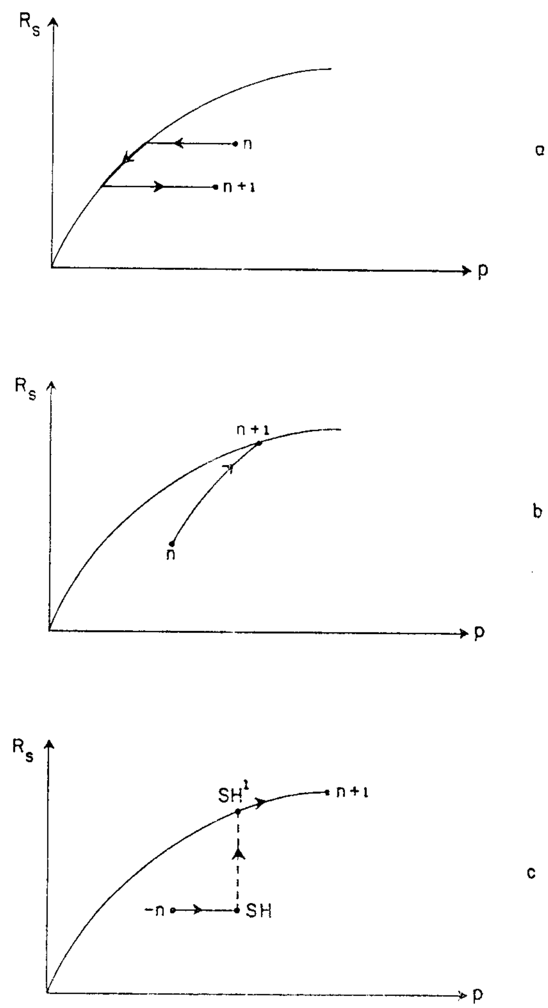

FIG. 2. Paths in the $R_s - p$ plane.

III. INCONSISTENCY OF THE TRADITIONAL APPROACH

When applying black-oil models to variable bubble-point problems, it is frequently assumed that the bubble-point pressure may vary inside the—undersaturated—''oil-region'' [6–9]. However, such an assumption is incorrect, because it violates the ''bubble-point conservation law.''

Bubble Point Conservation Law:

In the absence of a gas phase, oil-particles conserve their bubble-point.

Proof. Combining Eqs. (2), it is seen that

$$\phi S_o \bar{\rho}_o \{(\bar{R}_s)_t + \underline{v}_o \cdot \mathit{grad}(\bar{R}_s)\} + (\phi S_g \rho_g)_t + \text{div}(\rho_g \underline{u}_g) = 0. \tag{4}$$

When the gas-phase is absent, this equation reduces to

$$(\bar{R}_s)_t + v_o \cdot grad(\bar{R}_s) = 0. \tag{5}$$

In words: ‘‘the material oil-particle derivative of $\bar{R}_s$ (hence of R_s) vanishes.’’ Clearly, this implies that R_s (i.e., the bubble-point), remains constant on liquid-oil particles. ∎

The bubble-point conservation law, just established, is a very restrictive condition; physically it means that, when a gas-phase is not present, two oil-particles cannot exchange dissolved gas, even if they are very close. In particular, this property leads to preservation of discontinuities of the bubble-point.

Due to ‘‘bubble-point conservation law,’’ the paths that an oil-particle can describe on the $R_s - p_o$ plane consist of fragments of the saturation curve or of horizontal segments only [Fig. 2(a)]. The first ones take place in periods spent by the particle in regions where the gas-phase is present, while the latter ones correspond to periods spent by the particle in undersaturated regions, where the gas-phase is absent. Thus, if a particle starts at state ‘‘n’’ [Fig. 2(a)], so that it is undersaturated initially, and it is then depressurized, it moves along a horizontal line toward the left, until it reaches the saturation curve. If depressurization of the particle continues, it bubbles and liberates gas. At such point, the state of the oil-particle lies on the saturation curve. If it is repressurized, at first it will move along the saturation curve, until all the free-gas is removed, and at such point it will start to move along a horizontal line, toward the right, leaving the saturation curve, until it finally reaches a state such as ‘‘$n+1$’’ [Fig. 2(a)]. This path is reversible: we could start at state ‘‘$n+1$’’ and by successive depressurization and repressurization, reach state ‘‘n’’ . The point at which the saturation curve is left by the oil-particle, under repressurization, depends on the amount of free-gas available. In actual reservoirs, this amount of gas is supplied by the gas-phase, and it is determined by the relative motion of the oil-phase with respect to the gas-phase.

On the other hand, on the $R_s - p_o$ plane, the states of an oil-particle cannot follow a path such as the one joining states ‘‘n’’ and ‘‘$n+1$’’ [Fig. 2(b)], since this would imply that R_s changes without reaching the bubble-point. That is, R_s would change when the gas-phase is absent and the bubble-point conservation law would be violated. However, in the ‘‘traditional’’ approach to variable bubble-point reservoirs, such paths on the $R_s - p_o$ plane are admitted [6]. This is the *inconsistency* of the ‘‘traditional’’ approach that we have been referring to.

IV. SHOCKS REQUIRED IN A CONSISTENT APPROACH

At first a glance, the previous discussion suggests that in a black-oil model the only way in which an undersaturated oil-particle may become saturated is by depressurization to the bubble-point. This, however, is not correct, because in a black-oil model an oil-particle may become saturated in another manner: it may follow a discontinuous path on the $R_s - p_o$ plane, such as SH-SH$'$, in Fig. 2(c). This corresponds to an oil-particle that is initially undersaturated (point ‘‘n’’), so that the gas-phase is absent. At some point the oil-particle is reached by a gas-front (point SH) and becomes suddenly saturated (point SH$'$); under further pressurization R_s moves along the saturation curve. Such a path has the discontinuity SH-SH$'$, and, therefore, $[R_s] \neq 0$ there. In actual reservoir models, this corresponds to a discontinuity of R_s at the gas-front. The physical implication is that the gas transferance, at the gas-front from the gas-phase into the liquid-oil-phase, is so intense that it gives rise to a discontinuity in R_s.

The previous discussion indicates that a consistent formulation of black-oil models requires the incorporation of shocks, in which not only the saturation, but also the dissolved gas-oil ratio R_s may be discontinuous. Thus, the remainder of this section is devoted to presenting an exhaustive description of the shocks required, in a consistent formulation of black-oil models. In particular, the kind of shocks that can occur and the conditions under which they are generated, in each one of the three parts in which the region of study has been divided, will be discussed.

A. At a Gas-Front

Two situations must be distinguished.

1. At an Advancing Gas Front

According to the previous discussion, at an advancing gas-front, when oil particles carrying values of R_s below the saturation value reach the gas-front, where R_s necessarily equals the saturation value, a discontinuity will be produced (segment SH-SH′, of Fig. 2). This is the mechanism of shock generation at an advancing gas-front. Observe that, in general, at such a shock there are two discontinuous variables: $[R_s] \neq 0$ and $[S_o] \neq 0$.

2. At a Receding Gas Front

At a front that is receding, on the contrary, R_s is necessarily continuous, because the gas-phase leaves saturated oil behind it, as it goes away. In this case, the only discontinuous variable at the front is the saturation, since R_s is continuous there.

B. In the Oil-Region

In the interior of that region, the oil saturation equals to one, necessarily. Thus, the only possible discontinuous variable is R_s. In contrast to the ‘‘gas-region,’’ where oil is necessarily saturated and R_s is uniquely determined by pressure, when the gas-phase is absent the liquid oil will usually be undersaturated, and R_s can take any value below the saturation curve [Fig. 2(a)].

This is related to the initial conditions associated with well-posed problems. When free gas is present, the oil is saturated and the pressure determines R_s, and it is not necessary to include this parameter in the initial conditions. On the contrary, if the oil is undersaturated, R_s is not determined by pressure and must be prescribed as an initial condition. In particular, if the prescribed initial or boundary values of R_s are discontinuous, a shock in the undersaturated region would be introduced, because, by virtue of the ‘‘bubble-point conservation law,’’ the liquid-oil particles will retain their R_s values; observe that such discontinuity will propagate with the velocity, v_o, of oil particles. Therefore, shocks of this kind can be produced by the initial conditions—or possibly, by the boundary conditions.

Later, it will be seen that another generation mechanism for this kind of shocks occurs when an advancing gas-front changes its sense of motion—that is, when it stops and starts to recede. When this happens, the gas-front shock, which carries two discontinuities, one of R_s and another one of saturation, bifurcates. At the point of reversal, the discontinuity in gas:oil ratio starts to move with the velocity of the oil, while the discontinuity in saturation starts to move with the velocity of the gas.

C. At the Gas-Region

In this region, the oil is saturated and R_s is determined by oil pressure. Even more, R_s is usually taken to be continuous as a function of oil pressure. In turn, the oil pressure is necessarily continuous as a function of position. Thus, R_s is continuous in this region and the only possible discontinuous variable is saturation.

Shocks in this region, in which the only discontinuous variable is the saturation, were originally described by Buckley and Leverett [10, 11], and further discussed by many authors. They are generated when characteristic curves carrying different values of saturation intersect, giving rise to multivalued solutions that are not physically admissible. A very clear discussion of this process was presented by Sheldon and Cardwell [12, 13]. A recent account, from a present-day perspective, is given in [2].

Note that, as generally recognized, such shocks are developed only when capillary pressure is neglected. However, the other kinds of discontinuities, which have been discussed above, may occur even if capillary pressure is incorporated in the model.

V. CONSISTENT APPROACH

The point of view adopted in the black-oil model formulation is to consider all reservoirs as continuous systems. As is well known, the general theory of such systems is ‘‘Continuum Mechanics.’’ Thus, to overcome the inconsistency of the ‘‘traditional’’ approach discussed in Section III, it will be necessary to resort to the basic principles of Continuum Mechanics.

The systematic procedure for deriving the basic equations satisfied by any continuous system has two parts (see for example Chapter 1 of [14], for a description suitable for our purposes): the ‘‘general local balance laws’’ and, when discontinuous solutions (shocks) occur, such local laws must be complemented with the ‘‘general jump conditions.’’ The general local balance laws reduce to Eqs. (2), when the assumptions of the black-oil model, specified in Section II, hold. However, as explained in Section IV, for developing a consistent approach to variable bubble-point reservoirs it will be necessary to include shocks of quite general character. Thus, Eqs. (2) must be complemented with the equations derived from the ‘‘general jump conditions,’’ for each component of the system, The mathematical expression of the ‘‘general jump conditions’’ is (see [14], and the Appendix):

$$[\psi(\underline{v} - v_\Sigma) - \underline{\tau}] \cdot \underline{n}_\Sigma = q. \tag{6}$$

Here, ψ is the intensive property associated with the component under consideration, $\underline{v}$ is the particle velocity of the phase in which the component lies, $\underline{v}_\Sigma$ is the shock velocity, $\underline{\tau}$ is diffusive flux, and q is the rate, per unit area of the shock, at which the mass of the component is being supplied to the phase under consideration.

In the case of the two-phase (free-gas and liquid-oil) system, which constitutes the black-oil model, the gas-phase is made of only one component, gas; while the liquid-phase is made of two components, dissolved gas and nonvolatile oil. Thus, there are three components altogether. The extensive properties associated with them, relevant for our purposes, are: mass of free-gas (M_g), mass of dissolved gas (M_{dg}), and mass of nonvolatile oil (M_o). Each one of them can be expressed as a volume integral over the region occupied by the system ($\mathcal{B}$):

$$M_g = \int_{\mathcal{B}} \phi S_g \rho_g d\underline{x}, \qquad M_o = \int_{\mathcal{B}} \phi S_o \bar{\rho}_o d\underline{x}, \qquad M_{dg} = \int_{\mathcal{B}} \phi S_o R_S \bar{\rho}_o d\underline{x}. \tag{7}$$

The intensive property associated with each component is its mass per unit volume; i.e., the integrand in each one of these expressions.

On the other hand, $\underline{\tau} \equiv 0$, since the black-oil model does not incorporate diffusive flux. Regarding q, we will write q_o^g for the rate per unit area, at which mass is being supplied through the shock, to the gas-phase by the liquid-oil-phase. Conversely, q_g^o will be the rate per unit shock area, at which mass is being supplied to the liquid-phase by the gas-phase. Observe that this mass goes into the liquid-phase as dissolved-gas and that the mass received by the nonvolatile oil component is null; i.e., $q \equiv 0$ in the jump condition [Eq. (6)] corresponding to the nonvolatile oil component. Thus, the jump conditions satisfied by gas, dissolved gas, and nonvolatile oil components are:

$$[\phi S_g \rho_g (\underline{v}_g - v_\Sigma)] \cdot \underline{n}_\Sigma = q_o^g \tag{8a}$$

$$[\phi S_o R_S \bar{\rho}_o(\underline{v}_o - v_\Sigma)] \cdot \underline{n}_\Sigma = q_g^o \tag{8b}$$

$$[\phi S_o \bar{\rho}_o(\underline{v}_o - v_\Sigma)] \cdot \underline{n}_\Sigma = 0, \tag{8c}$$

respectively.

Observe, that the rates of mass-exchange between the two phases, q_o^g and q_g^o, that occur through the shock are not known. Actually, they can be evaluated only after the problem has been solved. However, for applying Eqs. (8), they do not need to be known, because adding Eqs. (8a) and (8b), and using the fact that $q_o^g + q_g^o = 0$, it is obtained:

$$[\phi S_g \rho_g(\underline{v}_g - v_\Sigma)] \cdot \underline{n}_\Sigma + [\phi S_o R_S \bar{\rho}_o(\underline{v}_o - v_\Sigma)] \cdot \underline{n}_\Sigma = 0. \tag{9}$$

Equation (8c) can be written as

$$\bar{\rho}_o^+(\underline{u}_o - \phi \underline{v}_\Sigma S_o)^+ \cdot \underline{n} = \bar{\rho}_o^-(\underline{u}_o - \phi \underline{v}_\Sigma S_o)^- \cdot \underline{n}. \tag{10a}$$

Using this latter equation, Eq. (9) becomes:

$$(\bar{\rho}_o^+/\rho_g)[\bar{R}_s](\underline{u}_o - \phi \underline{v}_\Sigma S_o)^+ \cdot \underline{n} + (\underline{u}_g - \phi \underline{v}_\Sigma S_g)^+ \cdot \underline{n} = (\underline{u}_g - \phi \underline{v}_\Sigma S_g)^- \cdot \underline{n}. \tag{10b}$$

Equations (2) must be complemented with Eqs. (10) to obtain a consistent black-oil model for variable bubble-point problems. Equations (2) apply everywhere in the interior of the definition of the problem, while the jump conditions (10) apply in the shock. In addition, Eqs. (8a) or (8b) can be used, after the problem has been solved, to obtain the rate of mass-exchange through the shock between the phases.

VI. SHOCK VELOCITIES

As a first illustration of the application of the consistent approach presented in Section V, in this section the velocities for each one of the shocks described in Section IV will be derived.

A. At a Gas-Front

A gas-front is characterized by $S_g^- = 0$. —recall that the normal vector on Σ has been taken pointing toward the region occupied by the gas-phase—and the right-hand side of (10b) vanishes. Using Eq. (2), it is obtained:

$$[R_s]\frac{\bar{\rho}_o^+ S_o^+}{\rho_g S_g}(\underline{v}_o^+ - \underline{v}_\Sigma) \cdot \underline{n} + (\underline{v}_g - \underline{v}_\Sigma) \cdot \underline{n} = 0. \tag{11}$$

Making use of the relation

$$\underline{v}_g - \underline{v}_\Sigma = \underline{v}_g - \underline{v}_o^+ + \underline{v}_o^+ - \underline{v}_\Sigma \tag{12}$$

yields

$$\left(1 + [R_s]\frac{\bar{\rho}_o^+ S_o^+}{\rho_g S_g}\right)(\underline{v}_o^+ - \underline{v}_\Sigma) \cdot \underline{n} = -(\underline{v}_g - \underline{v}_o^+) \cdot \underline{n}. \tag{13}$$

This equation can be written as:

$$(\underline{v}_\Sigma - \underline{v}_o^+) \cdot \underline{n} = \varepsilon(\underline{v}_g - \underline{v}_o^+) \cdot \underline{n}, \tag{14}$$

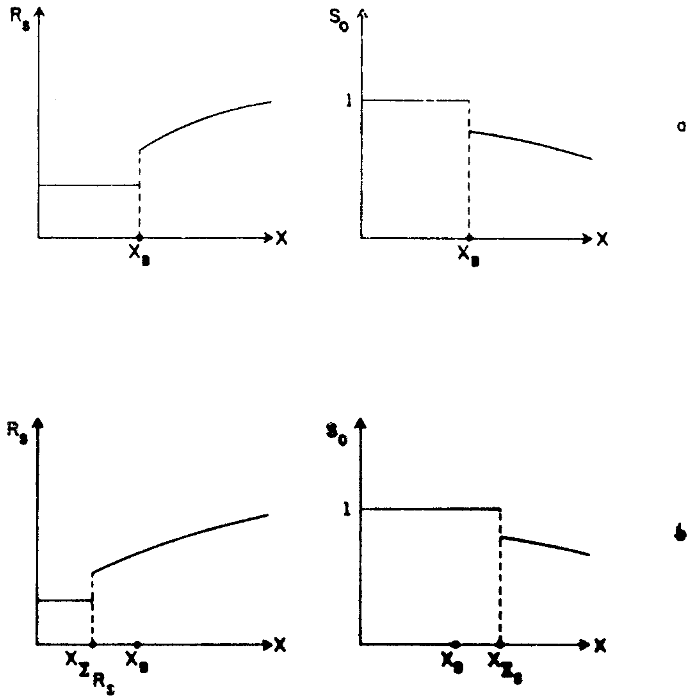

FIG. 3. The bifurcation mechanism: (a) the bifurcation point at x_B; (b) the two shocks after bifurcation.

if the ‘‘retardation factor,’’ ε, is defined by

$$\varepsilon = \frac{1}{1 + [\bar{R}_S] \frac{\bar{\rho}_o^+ S_o^+}{\rho_g S_g}}. \tag{15}$$

Observe that the $0 < \varepsilon \leq 1$, and, therefore, ε can be interpreted as a retardation factor of the shock motion with respect to the gas. In particular, when $\varepsilon = 1$, there is no retardation effect.

1. At a Gas-Front Advancing into a Region of Undersaturated Oil

In this case $[\bar{R}_s] \equiv \bar{R}_s^+ - \bar{R}_s^- > 0$, so that $\varepsilon < 1$, and the motion of the shock is retarded with respect to the gas.

2. At a Gas-Front Receding from a Region of Undersaturated Oil

This situation is characterized by $[R_s] = 0$, so that $\varepsilon = 1$. In this case, the shock is not retarded and $\underline{v}_\Sigma \cdot \underline{n} = \underline{v}_o \cdot \underline{n}$.

B. In the Oil-Region

In this case, the gas-phase is absent, so that the terms involving it, in Eq. (10b), vanish. Also, the only possible discontinuous variable is R_s, since $S_o = 1$, at both sides of the shock. Hence, Eq. (10b) implies

$$\underline{v}_\Sigma \cdot \underline{n} = \underline{v}_o \cdot \underline{n}, \qquad \text{on } \Sigma, \tag{16}$$

since $[R_s] \neq 0$.

C. At the Gas-Region

The oil is saturated in this region, so that $[R_s] = 0$, since the pressure is continuous and, in turn, R_s is continuous as a function of pressure. Thus, Eq. (10b) reduces to

$$\underline{v}_\Sigma \cdot \underline{n} = \frac{[\underline{u}_g] \cdot \underline{n}}{\phi [S_g]}. \tag{17}$$

A special case of this equation is the immiscible and incompressible case considered by the classical Buckley–Leverett theory, for which Eq. (17) becomes the well-known relation (see, for example, [2]):

$$\underline{v}_\Sigma \cdot \underline{n} = \phi^{-1} \frac{[f_g]}{[S_g]} \underline{u}_T \cdot \underline{n}, \tag{18a}$$

where

$$\underline{u}_T = \underline{u}_o + \underline{u}_g \text{ and } \underline{u}_g = f_g \underline{u}_T. \tag{18b}$$

VII. SHOCK BIFURCATION

The case when a gas-front advancing into a region of undersaturated gas changes its sense of motion, thus becoming a receding one, has special interest. Indeed, at the point where it stops and starts to recede, a discontinuity of R_s at the oil phase remains. Such discontinuity will move with velocity, $\underline{v}_o$, as shown in Section VI, thus, it will move towards the interior of the ''oil-region'' and, at later times, it will be located in its interior. On the other hand, also according to the results of that section, the saturation discontinuity, at the receding gas-front, will move with the velocity of the gas. Therefore, at a point where an ''advancing'' gas-front changes its sense of motion, it ''bifurcates,'' giving rise to two shocks: one in which the only discontinuous variable is the saturation, and the other one in which the only discontinuous variable is the bubble-point. This phenomenon is illustrated in Fig. 3. If the front, when it starts to recede, is located at x_B, then the discontinuities of R_s and S_o are together there [Fig. 3(a)]. However, as the receding motion of the front progresses, these discontinuities separate, because they move with different velocities; this is shown in Fig. 3(b), where the discontinuity of the bubble-point (R_S) is located at $x_{\Sigma R}$, to the left of the gas-front, where the saturation discontinuity $(x_{\Sigma S})$ lies. Figure 4—corresponding to the illustrative example to be explained next—shows the paths of the shocks and the bifurcation point, in space–time.

A. Illustrative Example

The bifurcation point and the discontinuity paths in the x–t plane, corresponding to a simple, one-dimensional numerical example, in which capillary pressure is neglected, $(p_o = p_g = p)$, are illustrated in Fig. 4. The parameter values used in the example are given in Table I; also, a simplified equation of state was assumed in which there is a critical pressure, p^*, such that R_S is constant whenever $p > p^*$; this leads to a constant jump in $R_s([R_s] = 1.78 \times 10^{-2})$. In addition, it is assumed that $p > p^*$ throughout. A positive pressure gradient, $\partial p / \partial x$, is initially applied at the gas-side, and smoothly decreased, until at time t_B it changes sign. The phases are assumed to be incompressible and, therefore, at t_B the gas-phase also changes its sense of motion. This, in turn, implies that t_B is also the bifurcation time. The initial distributions were taken as uniform, but different on each subregion. The retardation of the advancing front is quite

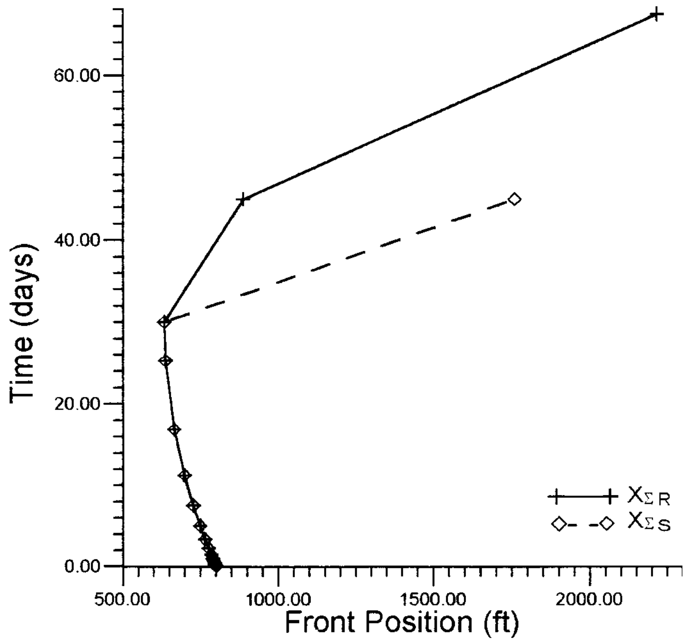

FIG. 4. Front position for Example 1 with bifurcation time.

TABLE I. Parameter values for the illustrative example.[a]

PROPERTIES
$\rho_o = 47$ pound/ft^3; $\rho_g = 0.02$ pound/ft^3; $\mu_o = 0.3$ cp; $\mu_g = 0.03$ cp $\phi = 0.05; k = 100$ m darcies; $k_{ro} = 0.5; k_{rg} = 0.4$
DATA
$S_g = 0.2; S_o^+ = 0.8; [\bar{R}_s] = 1.78 \times 10^{-2}$; $v_g = 10(t - t_B)$ $x_O = 800; x_B = 633.5; t_B = 30$
DERIVED PARAMETERS
$v_o^+ = 0.3125(t - t_B); \varepsilon = 5.94 \times 10^{-3}$;
ADVANCING $(t < t_B)$
$v_o^- = 3.239 \times 10^{-1}(t - t_B)$ $v_{\Sigma SR} = 0.3697(t - t_B); x_{\Sigma SR} = 633.5 + 0.1848(t - t_B)^2$
RECEDING $(t > t_B)$
$v_o^- = 2.25(t - t_B)$ $v_{\Sigma S} = 10(t - t_B); v_{\Sigma R} = 2.25(t - t_B)$ $x_{\Sigma S} = 633.5 + 5(t - t_B)^2$ $x_{\Sigma R} = 633.5 + 1.125(t - t_B)^2$

[a] Distances are in feet and times in days.

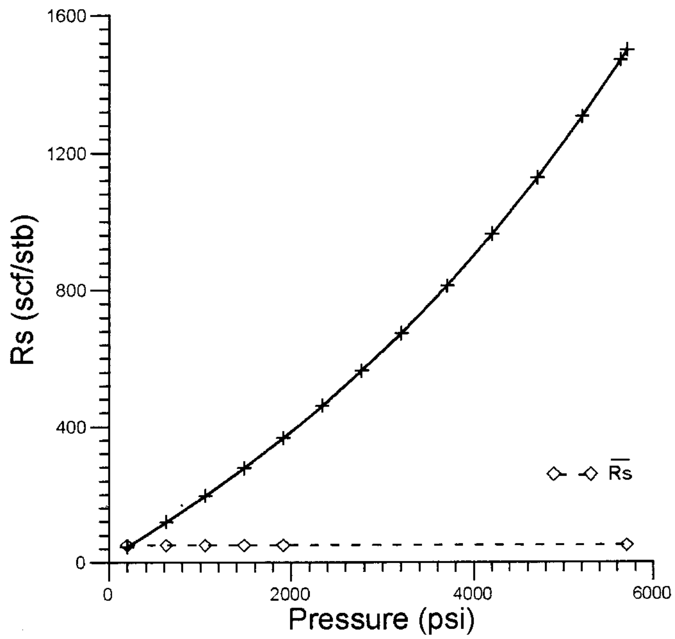

FIG. 5. Curve of gas in solution.

significant since $\varepsilon = 0.0059$. Recalling Eq. (15), it is seen that ε is controlled by the ratios ρ_o^+/ρ_g and S_o^+/S_g, as well as $[\bar{R}_S] = [R_S](\rho_{gSTC}/\rho_{oSTC})$, which for this case take the values 2350, 4, and 0.0178, respectively, giving a fairly large product: 167.32. Due to the drastic retardation effect, the velocity of the receding gas-front is much larger than that of the advancing one, and the scales involved when the fronts are advancing and receding are quite different, as can be clearly appreciated in Fig. 4. On the other hand, for this example, the contrast between $v_{\Sigma R}$ and $v_{\Sigma S}$ is not as sharp: $v_{\Sigma R}/v_{\Sigma S} \cong 0.2$. Of course, if the parameters are varied, the behavior of the fronts and the bifurcation may change very much.

VIII. NUMERICAL EXAMPLE

The illustrative example presented in Section IV (Fig. 4)—referred to as Example 1—, in spite of its simplifying assumptions shows that the effects described by the authors' theory, may be quite significant, and their numerical implications deserve to be studied more thoroughly. In particular, the magnitude of the numerical errors introduced by the application of a ‘‘traditional’’ black-oil model, which has been shown to be *inconsistent,* to variable bubble-point problems should be clarified. Thus, in this section an example, in which shocks and a bifurcation occur, will be numerically treated by two methods: one in which shocks are explicitly incorporated using the authors' theory, and the other will be a ‘‘traditional’’ formulation [6–9], in which shocks are not incorporated explicitly. Then, comparisons of the results of these methods will be made.

The example to be treated—referred as Example 2—is quite similar to that of Section VII, but to make it more realistic, the saturation $R_s - p_o$ curve of Fig. 5, for which R_s is pressure dependent everywhere, will be used. Except for such replacement, everything else will be as before. Thus, a

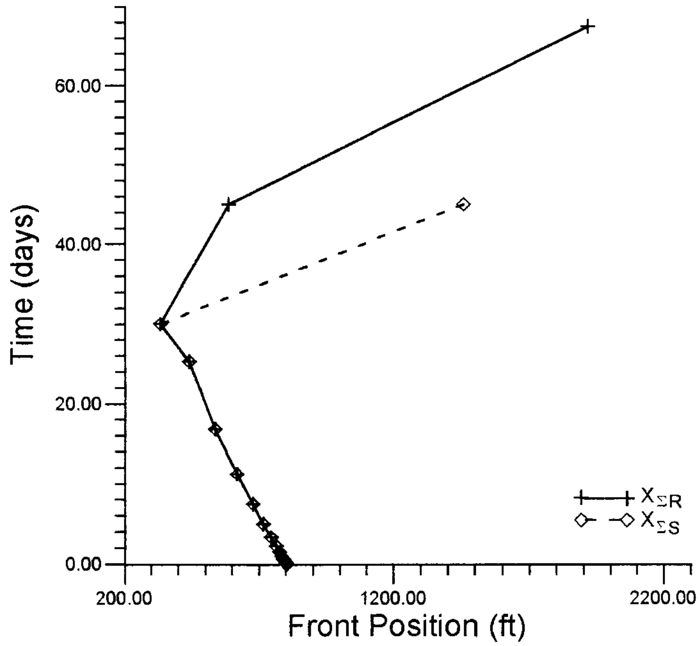

FIG. 6. Front position for Example 2 with bifurcation time.

composite system consisting of two regions—the gas and oil regions, respectively—is considered, and its basic properties are chosen to be those of Table I. In particular, initially there is a left-region of undersaturated oil and a right-region containing a gas-phase, where the oil is saturated. Again, production is obtained from the left-boundary at a constant pressure, while gas is injected on the right-boundary at a variable rate: $v_g = 10(t - 30)$. Therefore, at the gas-front $(x_{\Sigma SR})$, when it is advancing into the region of undersaturated oil, two discontinuities coexist—one of saturation (S_o) and the other one of the solution gas:oil ratio (R_S). Initially, the gas-front moves toward the left with variable velocity, which is not given by a simple closed expression (see Fig. 6), because the retardation factor, ε, as given by Eq. (3b), is fully dependent on the detailed shape of the $R_S - p$ curve, illustrated in Fig. 5. At $t = t_B = 30$ days, the sense of motion changes and the shock bifurcates, since the discontinuities start to recede with different velocities.

First, let us discuss briefly the effect of incorporating a more realistic $R_S - p_o$ curve. Qualitatively, there is little change: again, as in Example 1, two discontinuities coexist at the gas-front when it is advancing toward the region of undersaturated oil, and its motion is retarded (i.e., $\varepsilon < 1$); there is a time, t_B, when the sense of motion is reversed and, then, the shock bifurcates. However, in Example 2 (in contrast with Example 1), the velocity of the front is non-constant and it is not possible to give a simple, closed expression for it. Quantitatively, the main change is also in the velocity, $v_{\Sigma RS}$, of the advancing gas-front, which is quite significant (Fig. 7). This velocity is evaluated with Eq. (14). The difference in velocities is due to the fact that the retardation factor ε is no longer constant with time (Fig. 8), and it is strongly influenced by the detailed shape of the saturation $R_S - p_o$ curve. The trajectories of the shocks, for $t < t_B$, and the position of the bifurcation point x_B differ considerably in Examples 1 and 2, because the velocities of the advancing fronts differ significantly in both examples. However, the time of bifurcation is the same: $t = t_B = 30$ days. On the other hand, the corresponding velocities of each one of the receding shocks are essentially the same, in both examples; the shock in R_S, moving with the oil velocity, while the speed of the saturation discontinuity is that of the gas—there is no retardation effect.

Secondly, the main objective of the numerical comparisons is addressed: to establish the significance of using a ‘‘consistent’’ black-oil model when treating variable bubble-point problems. For this purpose, the example that has just been described was also treated using the ‘‘traditional numerical formulation’’ [6–9]. When a ‘‘traditional formulation,’’ in which no discontinuity in R_S is incorporated, the saturation front advances toward the undersaturated region much faster than it should, as illustrated in Fig. 9. Thus, in about three days the saturation front reaches the left-boundary; this is about one tenth of the bifurcation time t_B of our ‘‘consistent’’ black-oil model, in which the gas-front begins to recede without reaching the left-boundary, and, therefore, it never reaches this boundary. Obvious consequences of this drastic difference in the front velocities, when the *inconsistent* ‘‘traditional’’ black-oil model is applied, are big inaccuracies in the evaluation of gas production at the boundary and many other quantities of interest for the oil industry.

Also shown in Fig. 9 is the front position corresponding to Example 1, which only considers two values of R_S, one at each side of the front. We readily observe the influence of the R_S curve.

IX. DISCUSSION AND CONCLUSIONS

When applying black-oil models to variable bubble-point problems, it is frequently assumed that the bubble-point pressure may vary inside the undersaturated region [6–9]. However, it has been shown that such an assumption is *inconsistent,* for black-oil models, and that this

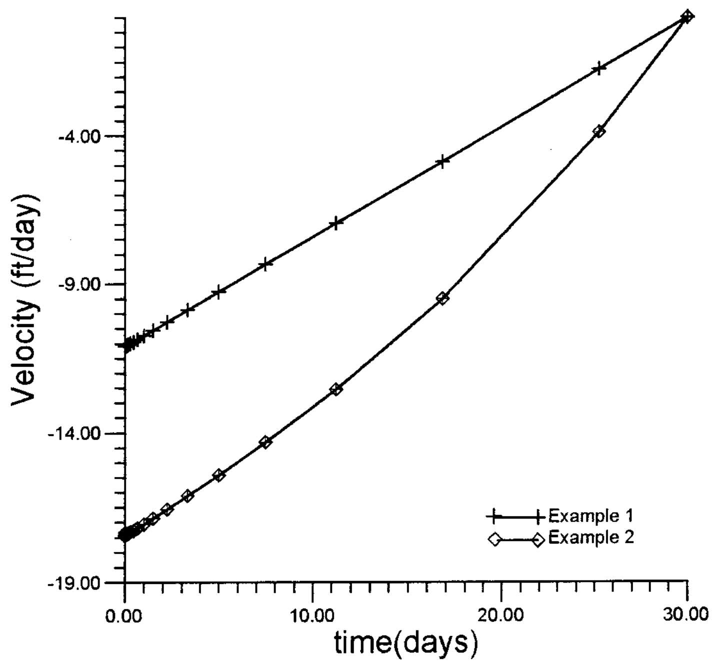

FIG. 7. Comparison of front velocity $(t < t_B)$.

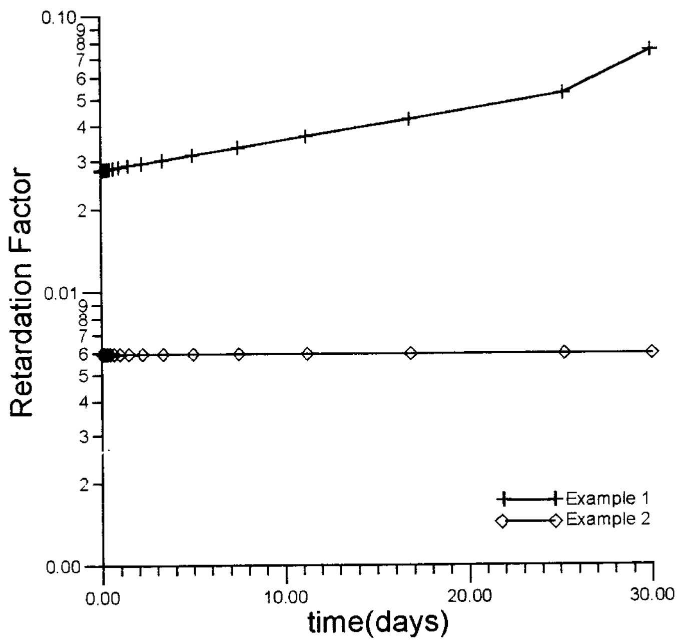

FIG. 8. Comparison of retardation factor $(t < t_B)$.

is due to the fact that no diffusion mechanism—neither molecular diffusion nor mechanical dispersion—is included in the formulation of black-oil models. A *consistent* numerical treatment of variable bubble-point problems, in black-oil models, generally requires the incorporation of discontinuities in the solution gas:oil ratio R_S. It has also been shown that numerical predictions

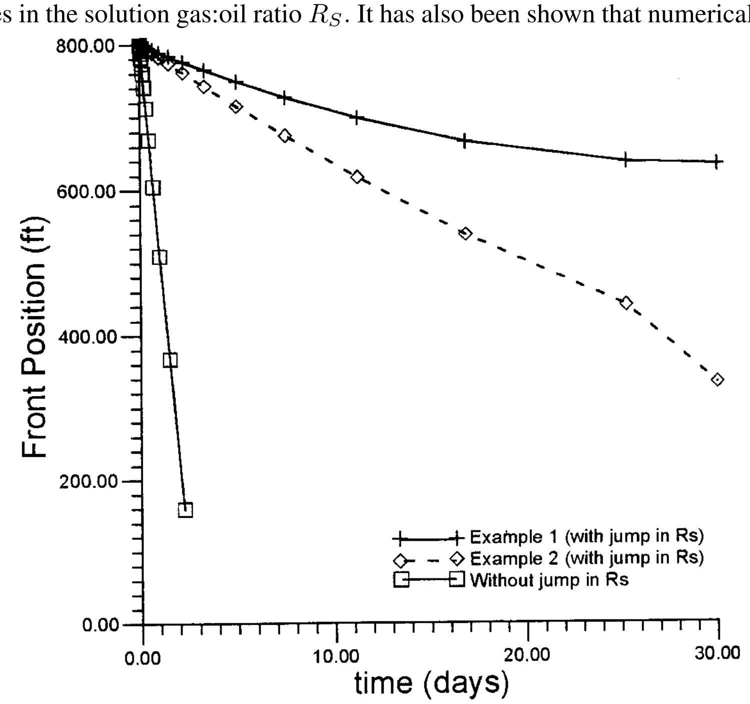

FIG. 9. Comparison of front position with and without jump.

made using the ‘‘traditional’’—inconsistent—approach [6–9], in which no discontinuities in R_S are incorporated, may differ drastically from those made using a ‘‘consistent approach,’’ in which discontinuities of R_S are included. Thus, to obtain a satisfactory accuracy, it is necessary to include discontinuities in R_S. To achieve a fuller and deeper understanding of these facts—and to suggest to modelers proper ways of handling them—the following discussion is presented.

Generally, in the physical reality, diffusion mechanisms are present, even if in some cases they may be rather weak. Mathematical models are only an approximation, in which all the complexity of the physical reality is never incorporated. For a ‘‘modeler,’’ a wise strategy to be followed is to incorporate only those processes that are relevant, at the level of accuracy required of the results of the model. If, in some application, diffusion is not relevant, it is correct to neglect it and apply the black-oil model, in a consistent manner—incorporating shocks—as it has been explained by the authors here and in previous articles [1–5]. However, if, on the contrary, diffusion processes are relevant, the model must be modified in a consistent manner, and diffusion processes—either molecular diffusion, mechanical dispersion, or both—must be incorporated explicitly.

What must *not* be done, however, is to apply an inconsistent model, because numerical results obtained using inconsistent models are unwarranted. Indeed, for example, the velocity of the gas-front predicted by the *inconsistent* traditional approach is the same as the velocity of the gas-phase. However, velocity of the gas-front predicted by a consistent approach differs considerably from that of the gas-phase. Preliminary analysis, already made [15], indicates that this is the case even if diffusion mechanisms are incorporated, and that the actual velocity of the front cannot be determined unless the diffusion mechanism is clearly defined.

Using results obtained in previous articles [1–5], in the present article a *consistent* black-oil model has been formulated, which is adequate for applications to variable bubble-point problems when molecular diffusion and mechanical dispersion can be neglected. Such a model incorporates three kinds of possible shocks and a *bifurcation* mechanism, which also may occur. The shocks can be classified according to whether they occur at: (1) a gas-front, (2) a region occupied by undersaturated oil, and (3) a gas-region. Simple expressions have been derived for the velocities of each one of them. For shocks of type (3), only the saturation can jump and they are essentially described by Buckley–Leverett theory. In case (2), only the bubble-point [i.e., the solution oil:gas ratio (R_s)] can jump. In case (1), at an advancing gas-front, both S_o and R_s are discontinuous, but if the gas-front is receding, then only the saturation can jump. In addition, when an advancing gas-front changes its sense of motion and starts to recede, the shock *bifurcates,* giving rise to two shocks, one moving with the oil velocity, where only R_s is discontinuous, and the other one with the velocity of the gas, where only S_o jumps.

From a different point of view, the results and discussions that have been presented in this article contribute to clarify some aspects of black-oil models. In particular, a consequence of omitting molecular diffusion and mechanical dispersion, as is done in black-oil models, is the ‘‘bubble-point conservation law,’’ according to which: when a gas-phase is not present, oil-particles conserve their gas content (bubble-point). This is, indeed, a very severe restriction to the manners in which the solution gas:oil ratio of an oil-particle can vary, when a gas-phase is absent. Physically, this means that, when a gas-phase is not present, two oil particles cannot exchange dissolved gas, even if they are very close. This property leads to preservation of discontinuities (shocks) of the bubble-point.

Frequently, capillary forces are perceived as diffusive, and in some instances such perception is justified. However, it must be stressed that the ‘‘bubble-point conservation law’’ holds even in the presence of capillary forces. Thus, such forces do not preclude the validity of the ‘‘bubble-point conservation law.’’ On the other hand, when either molecular diffusion, mechanical dispersion, or both are included in the models, the ‘‘bubble-point conservation law’’ ceases to be valid.

APPENDIX

There are some differences between our Eq. (6) and Eq. (1.3.6) of Ref. [15]. These differences are explained in this appendix.

First, there is a change in notation: our Greek letter, ψ, stands for the product $\rho\psi$ of [15]. Secondly, the right-hand side of Eq. (6), is 'q' while that of Eq. (1.3.6) is zero. This is because this latter equation is not as general as our Eq. (6), which includes the possibility of a concentrated mass exchange, with the exterior—or other components of the system—through the shock surface Σ. The possibility of having such kind of exchange, in some models, is similar to the limiting cases of forces concentrated on a surface, acting on a body, studied in Mechanics. When a gas-front advances into a region of undersaturated oil, mass exchange between the gas and the liquid-oil-phase is very intense at the front and can be mimicked only if a nonvanishing 'q' in Eq. (6), is incorporated. This is the reason for taking q_o^g and q_g^o, different to zero in Eqs. (8a) and (8b). As mentioned before, they represent the rate of mass-exchange, through the shock, between the gas and liquid phases.

References

1. I. Herrera, A. Galindo, and R. Camacho, ''Shock modelling in variable bubble point problems of petroleum engineering,'' in *Computational Modelling of Free and Moving Boundary Problems,* Vol. 1: Fluid Flow, L. C. Wrobel and C. A. Brebbia, Eds., Computational Mechanics Publication/de Gruyter, 1991, pp. 399–415.
2. I. Herrera, A. Galindo, and R. Camacho, ''Shock modelling in petroleum engineering,'' in *Computational Methods for Free and Moving Boundary Problems in Heat and Fluid Flow,* L. C. Wrobel and C. A. Brebbia, Eds., Computational Mechanics Publications, 1993, pp. 143–170 (Invited chapter).
3. I. Herrera, R. Camacho, and A. Galindo, ''Mechanisms of shock generation in variable bubble-point systems,'' in *Computational Modelling of Free and Moving Boundary Problems,* Vol. 2, L. C. Wrobel and C. A. Brebbia, Eds., Computational Mechanics Publications, Southampton & Boston, 1993, pp. 435–453 (Invited paper).
4. I. Herrera and L. Chargoy, ''Shocks in solution gas-drive reservoirs,'' SPE Paper No. 029136, *SPE Thirteen Symposium in Reservoir Simulation,* San Antonio, 1995, pp. 429–440.
5. I. Herrera, ''Shocks and bifurcations in black-oil models,'' *SPE Journal in Reservoir Simulation,* March, 1996.
6. K. Aziz and A. Settari, Eds., *Petroleum Reservoir Simulation,* Applied Science Publishers, London, 1979.
7. L. K. Thomas, W. B. Lumpkin, and G. M. Reheis, ''Reservoir simulation of variable bubble-point problems,'' *Trans., AIME,* **253,** 10–16 (1976).
8. D. S. Stright, Jr., K. Aziz, A. Settari, and F. Sarrat, ''Carbon dioxide injection into bottom-water, undersaturated viscous oil reservoirs,'' *JPT,* **29,** 1248–58 (1977).
9. G. R. Huan, ''A flash black oil model,'' *Paper SPE 13521,* presented at the SPE Middle East Oil Technical Conference and Exhibition, Bahrain, March 11–14, 1985.
10. M. C. Leverett, ''Capillary behaviour in porous solids,'' *Trns. AIME,* **142,** 152 (1941).
11. S. E. Buckley and M. C. Leverett, ''Mechanics of fluid displacement in sands,'' *Trans., AIME* **146,** 107 (1942).
12. W. T. Cardwell, ''The meaning of the triple value in noncapillary Buckley–Leverett theory,'' T.P. 808, *Trans. AIME,* **216,** 271–276 (1959).
13. J. W. Sheldon and W. T. Cardwell, ''One-dimensional, incompressible, noncapillary, two-phase fluid in a porous medium,'' *Petroleum Trans. AIME,* **216,** 290–296 (1959).

14. M. B. Allen, I. Herrera, and G. F. Pinder, *Numerical modeling in Science and Engineering,* John Wiley, New York, 1988.

15. I. Herrera, ''The effect of diffusion in variable bubble-point problems,'' to appear.