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UNIFIED APPROACH TO DOMAIN DECOMPOSITION

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Abstract. *Domain decomposition methods have been extensively studied, specially during the last decade, as a very effective tool for parallelizing models of continuous (macroscopic) systems. In general, there are two approaches to develop domain decomposition methods: One starts with the discretized version of the model and the other one with the partial differential equations, before they are discretized. In this paper a very general formulation of this latter approach, which is applicable to any partial differential equation or system of such equations -symmetric or non-symmetric with coefficients that may be discontinuous-, is presented. The basis for the analysis are theories previously developed by the author - specially Trefftz-Herrera method.*

1 INTRODUCTION

In recent years domain decomposition methods have received much attention, as a tool for solving partial differential equations. This is mainly due to the development of parallel machines, since such methods are efficient for parallelizing numerical algorithms. In addition, they can be used to design adaptive algorithms which capture steep fronts that appear in many problems, such as modeling of transport. Domain decomposition methods are also used to simplify problems with complicated geometries or match regions with different physical parameters or different types of differential equations. A wealth of literature on the subject has appeared in recent years (see for example[1-9]).

The author's generalized version of Trefftz method in which discontinuous trial and test functions are admitted, leads in a direct manner to domain decomposition procedures. Such methodology was advanced in previous publications[10-16], although only recently research on the procedure, as an approach to domain decomposition methods, was initiated[8,9,17].

In this paper a very brief exposition of Trefftz-Herrera general formulation of domain decomposition is presented. More details may be found in some of the material already published[8,9,17].

The theory is presented in a relatively abstract manner in Sections 2-4 and its application is illustrated through specific examples in Section 5.

2. GENERAL FORMULATION OF BOUNDARY VALUE PROBLEMS

Let Ω be a region (Fig. 1) and $\{\Omega_1, ..., \Omega_N\}$ a 'domain decomposition' (i.e., a partition of Ω). The union of the internal boundaries separating the elements of the domain decomposition will be denoted by Σ (see Fig. 1).

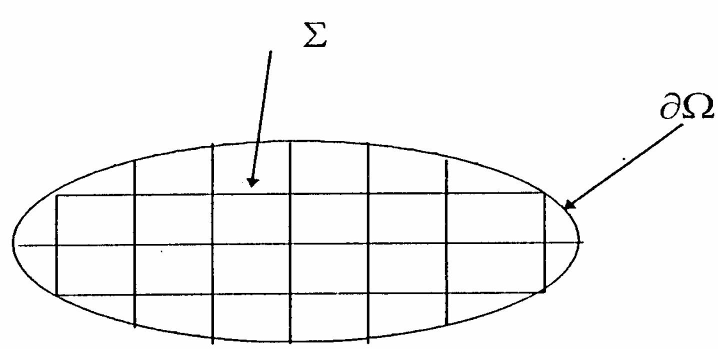

Figure 1: The region Ω

In what follows two linear spaces of functions defined in Ω will be considered. D_1 and D_2, the spaces of trial and test (or weighting) functions, respectively. Let $\mathcal{L}$ be a differential operator, then the definition of formal adjoint $\mathcal{L}^*$, of $\mathcal{L}$, requires that

$$w\mathcal{L}u - u\mathcal{L}^*w \equiv \nabla\cdot\{\underline{\mathcal{D}}(u, w)\}\ ;\ \forall\, u \in D_1\ \&\ w \in D_2 \tag{2.1}$$

for a suitable vector-valued bilinear function $\underline{\mathcal{D}}(u, w)$.

The general boundary value problem to be considered is one with prescribed jumps. Thus, it will be assumed that functions belonging to the spaces D_1 and D_2, may have jump discontinuities across Σ and the solution of the boundary value problem will be required to satisfy some jump conditions on Σ. Then the general expression of such problem is:

$$\mathcal{L}u = f_\Omega, \qquad \text{in } \Omega \tag{2.2}$$

$$\mathcal{B}(u,\cdot) = g_\partial(\cdot), \quad \text{on } \partial\Omega \tag{2.3}$$

and

$$\mathcal{J}(u,\cdot) = j_\Sigma(\cdot), \quad \text{on } \Sigma \tag{2.4}$$

Here $\mathcal{B}(u,\cdot)$ and $\mathcal{J}(u,\cdot)$ are suitable bilinear functionals defined point-wise, on $\partial\Omega$ and Σ, respectively. Similarly, $g_\partial(\cdot)$ and $j_\Sigma(\cdot)$ are linear functionals defined on $\partial\Omega$ and Σ, respectively. If, u_∂ and u_Σ are functions satisfying the boundary conditions and jump conditions, respectively, they can be defined by

$$g_\partial(w) \equiv \mathcal{B}(u_\partial, w) \text{ and } j_\Sigma(w) \equiv \mathcal{J}(u_\Sigma, w) \tag{2.5}$$

The linear functionals $f \in D_2^*$, $g \in D_2^*$ and $j \in D_2^*$, are defined by

$$\langle f, w\rangle = \int_\Omega w\, f_\Omega\, dx\ ;\ \langle g, w\rangle = \int_{\partial\Omega} g_\partial(w)\, dx\ ;\ \langle j, w\rangle = \int_\Sigma j_\Sigma(w)\, dx; \tag{2.6}$$

In previous articles[10,17], Green-Herrera formulas have been introduced. The general form of such formulas is:

$$\int_\Omega w\mathcal{L}u\, dx - \int_{\partial\Omega}\mathcal{B}(u, w)\, dx - \int_\Sigma \mathcal{J}(u, w)\, dx =$$

$$\int_\Omega u\mathcal{L}^*w\, dx - \int_{\partial\Omega}\mathcal{C}^*(u, w)\, dx - \int_\Sigma \mathcal{K}^*(u, w)\, dx \tag{2.7}$$

Here, $\mathcal{C}(w, u)$ and $\mathcal{K}(w, u)$ are suitable bilinear functionals, while $\mathcal{C}^*$ and $\mathcal{K}^*$ denote their transposes. Fundamental properties of these bilinear functionals are that $\mathcal{C}^*(u, \cdot)$ characterizes the "complementary boundary values, while $\mathcal{K}^*(u, \cdot)$ characterizes the "generalized averages". In the case when the differential operator $\mathcal{L}$ possesses continuous coefficients, explicit expressions for $\mathcal{J}$ and $\mathcal{K}$ are:

$$\mathcal{J}(u, w) \equiv -\underline{\mathcal{D}}([u], \dot{w})\cdot\underline{n} \tag{2.8a}$$

and

$$\mathcal{K}^*(u, w) \equiv \underline{\mathcal{D}}(\dot{u}, [w])\cdot\underline{n} \tag{2.8b}$$

The square brackets stand for the "jump" of the function contained inside and the dot on top refers to the 'average' across Σ; i.e.:

$$[u] = u_+ - u_- ; \qquad \dot{u} = 1/2\,(u_+ + u_-) \tag{2.9}$$

The unit normal vector $\underline{n}$, on Σ, is chosen arbitrarily and, by definition, it points toward the positive side of Σ.

Introducing the bilinear functionals

$$\langle Pu, w\rangle = \int_\Omega w \mathcal{L} u \, dx; \langle Q^* u, w\rangle = \int_\Omega u \mathcal{L}^* w \, dx \tag{2.10a}$$

$$\langle Bu, w\rangle = \int_{\partial\Omega} \mathcal{B}(u, w) \, dx \, ; \langle C^* u, w\rangle = \int_{\partial\Omega} \mathcal{C}^*(u, w) \, dx \tag{2.10b}$$

$$\langle Ju, w\rangle = \int_\Sigma \mathcal{J}(u, w) \, dx \, ; \langle K^* u, w\rangle = \int_\Sigma \mathcal{K}^*(u, w) \, dx \tag{2.10c}$$

In terms of such functionals the Green-Herrera formula of Eq. (2.6) becomes

$$P - B - J \equiv Q^* - C^* - K^* \tag{2.11}$$

3 VARIATIONAL FORMULATIONS

Boundary value problems of the kind considered in Section 2, admit two variational formulations, one in <u>terms of the data of the problem</u> and the other one in <u>terms of the sought information</u>. The first one is:

$$\langle (P - B - J)\, u, w\rangle = \langle f - g - j, w\rangle; \quad \forall\, w \in D_2 \tag{3.1}$$

and the second one is

$$\langle (Q^* - C^* - K^*)\, u, w\rangle = \langle f - g - j, w\rangle; \quad \forall\, w \in D_2 \tag{3.2}$$

More briefly, they are

$$(P - B - J)\, u = f - g - j \tag{3.3}$$

and

$$(Q^* - C^* - K^*)\, u = f - g - j \tag{3.4}$$

respectively.

4 TREFFTZ-HERRERA FORMULATION OF DOMAIN DECOMPOSITION

In the formulation that will be presented in this Section, the concept of "internal boundary solution" (IBS) will be required.

<u>Definition 4.1</u>. A function $\hat{u} \in D_1$ is said to be an "internal boundary solution" when there exists a solution of the boundary value problem with prescribed jumps, such that

$$K^* \hat{u} = K^* u \tag{4.1}$$

Let $\mathcal{E} \subset D_2$ be a system of functions such that

$$\mathcal{L}^* w \equiv 0, \text{ in } \Omega, \quad \text{and} \quad \mathcal{C}(w, \cdot) \equiv 0, \text{ on } \Sigma \tag{4.2}$$

whenever $w \in \mathcal{E}$. Eq. (4.1) implies

$$Qw = 0 \ \& \ Cw = 0 \tag{4.3}$$

so that $\mathcal{E} \subset N_Q \cap N_C \subset D_2$. Here N_Q and N_C denote the null subspaces of Q and Ċ, respectively.

When $\mathcal{E} \subset N_Q \cap N_C \subset D_2$, Eq. (3.2), implies

$$-\langle K^* u, w\rangle = \langle f - g - j, w\rangle, \ \forall \ w \in \mathcal{E} \tag{4.4}$$

Thus, Eq. (4.4) can be taken as a necessary condition for a function being an internal boundary solution. However, it is not sufficient, unless $\mathcal{E} \subset N_Q \cap N_C$ satisfies some additional conditions and the following concept of completeness, which was first introduced by the author[18] in a more limited context, will be useful.

<u>Definition 4.2.</u> A system of functions $\mathcal{E} \subset N_Q \cap N_C$ is said to be TH-complete when for every $\hat{u} \in D_1$, one has

$$-\langle K^* \hat{u}, w\rangle = \langle f - g - j, w\rangle, \ \forall \ w \in \mathcal{E} \Rightarrow \hat{u} \text{ is an IBS} \tag{4.5}$$

Using this concept, the following characterization of internal boundary solutions is clear.

<u>Theorem 4.1.</u> Let $\mathcal{E} \subset N_Q \cap N_C$, be TH-complete. Then, any $\hat{u} \in D_1$, is an IBS, if and only if

$$-\langle K^* \hat{u}, w\rangle = \langle f - g - j, w\rangle, \ \forall \ w \in \mathcal{E} \tag{4.6}$$

A useful alternative form of Eq. (4.6) can be given. Let $u_\Sigma \in D_1$ be such that $Ju_\Sigma = j$. Assume further that $u_P \in D_1$ is such that

$$P\, u_P = f\, ; \ B\, u_P = g \text{ and } K^*\, u_P = 0 \tag{4.7}$$

In many applications the boundary value problem defined by Eq. (4.7) is well posed. Then, the characterization of internal boundary solutions of Eq. (4.6), becomes:

$$-\langle K^* \hat{u}, w\rangle = \langle J(u_p - u_\Sigma), w\rangle, \ \forall \ w \in \mathcal{E} \tag{4.8}$$

An important feature of Eq. (4.8), is that it only involves functions defined on Σ.

In the examples that follow, it will be seen that Eq. (4.8) implies a domain decomposition formulation.

5. EXAMPLES

A.- <u>Elliptic Equations</u>

Consider the most general elliptic equation of second order:

$$\mathcal{L} u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b}\, u) + c\, u = f_\Omega \tag{5.1}$$

Let $D_1 \equiv D_2$ be continuous across Σ, with first derivatives possibly discontinuous. For definiteness take Dirichlet boundary conditions:

$$u = u_\partial \; ; \quad \text{on } \partial\Omega \tag{5.2}$$

and jump conditions

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right]; \quad \text{on } \Sigma \tag{5.3}$$

where u_Σ is a given function.

Then functions $w \in N_Q \cap N_C$ satisfy

$$\mathcal{L}^* w \equiv - \nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + c\, w = 0; \quad \text{in } \Omega \tag{5.4a}$$

and

$$w = 0 \; ; \quad \text{on } \partial\Omega \tag{5.4b}$$

In addition, such functions are continuous across Σ.

In this case, Eq. (4.8) is:

$$- \int_\Sigma a_n \hat{u} \left[\frac{\partial w}{\partial n}\right] dx = \int_\Sigma a_n\, w \left[\frac{\partial (u_\Sigma - u_p)}{\partial n}\right] dx \; ; \; \forall\, w \in \mathcal{E} \tag{5.5}$$

where $a_n = \underline{n} \cdot \underline{\underline{a}} \cdot \underline{n}$. The procedure is a domain decomposition method because the weighting functions $w \in N_Q \cap N_C$, may be constructed locally. For example, consider the rectangle of Fig. 2, divided into

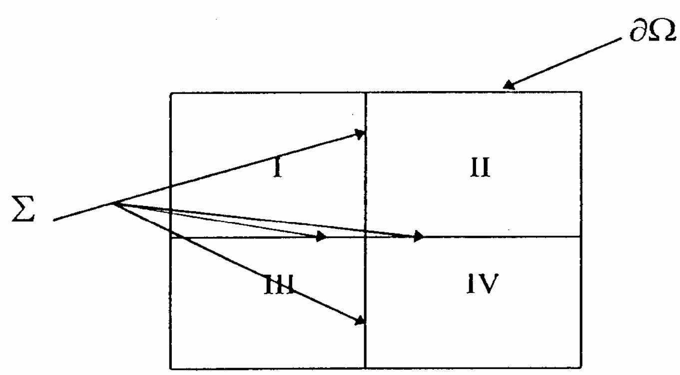

Figure 2: Domain decomposition of the example

four subregions. If the value of w is prescribed on Σ, we have well posed problems defined on each one of the subregions I to IV, because Eqs. (5.4) must be satisfied. Thus, w can be constructed in each one of the subregions separately.

Another example, is the system of equations of mixed methods. The single equation:

$$-\Delta u = f_\Omega \tag{5.6}$$

is equivalent to the system

$$-\underline{p} + \nabla u = 0 \tag{5.7a}$$

and

$$\nabla \cdot \underline{p} = f_\Omega \tag{5.7b}$$

Introduce the four dimensional vectors:

$$\underline{u} = (\underline{p}, u) \tag{5.8a}$$

and

$$\underline{f}_\Omega = (\underline{0}, f_\Omega) \tag{5.8b}$$

Define a vector valued differential operator by:

$$\underline{\underline{\mathcal{L}}} \cdot \underline{u} = \{-\underline{p} + \nabla u, -\nabla \cdot \underline{p}\} \tag{5.9}$$

Then, the system of Eqs. (5.7) can be written as

$$\underline{\underline{\mathcal{L}}} \cdot \underline{u} = \underline{f}_\Omega \tag{5.10}$$

In this case $\underline{\underline{\mathcal{L}}}$ is self-adjoint (i.e., $\underline{\underline{\mathcal{L}}}^* = \underline{\underline{\mathcal{L}}}$). Thus

$$\mathcal{D}(\underline{u}, \underline{w}) = u\,\underline{q} - w\,\underline{p} \tag{5.11}$$

is a three dimensional vector. Here, the notation $\underline{w} = (\underline{q}, w)$ is assumed.

Then

$$\mathcal{K}^*(\underline{u}, \underline{w}) = \dot{u}\,[\underline{q}] \cdot \underline{n} - [w]\,\dot{\underline{p}} \cdot \underline{n}; \text{ on } \Sigma \tag{5.12}$$

The information on Σ refers to the average $\dot{u}$ and to the flux $\dot{\underline{p}} \cdot \underline{n}$. If it is desired to concentrate the information on the flux, the additional condition

$$[\underline{q}] \cdot \underline{n} = 0; \quad \text{on } \Sigma \tag{5.13}$$

on the weighting functions $\underline{w} \equiv (\underline{q}, w)$, must be imposed. Observe that this condition is continuity of flux. Then Eq. (4.8) is:

$$\int_{\Sigma} [w]\ \underline{\hat{p}} \cdot \underline{n}\, dx = \int_{\Sigma} \left[u_{\Sigma} - u_{P} \right]\ \underline{q} \cdot \underline{n}\, dx \tag{5.14}$$

This is a mixed-method formulation because the information has been concentrated on the flux $\underline{\hat{p}} \cdot \underline{n}$, exclusively. Observe that Eq. (5.14) is quite similar to Eq. (5.5), except that now the spaces of functions are discontinuous while the fluxes are required to be continuous.

Here, Eq. (4.7) defining $\underline{u}_P = (\underline{p}_P, u_P)$, are

$$-\underline{p}_P + \nabla u_P = \underline{0} \tag{5.15a}$$

and

$$\nabla \cdot \underline{p}_P = f_{\Omega} \tag{5.15b}$$

subject to the boundary condition:

$$u_{\underline{P}} = u_{\partial}\ ; \quad \text{on} \quad \partial\Omega \tag{5.16}$$

and an internal condition:

$$\underline{p} \cdot \underline{n} = 0\ ; \quad \text{on} \quad \Sigma \tag{5.17}$$

Referring to Fig. 2, this is a well posed problem for each one of the subregions I to IV.

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