Indirect Methods of Collocation: Trefftz–Herrera Collocation

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A nonstandard collocation method (TH-collocation) is presented, where collocation is used to construct specialized weighting functions instead of the solution itself, as it is usual, so that in this sense it is an indirect method. TH-collocation is shown to be as accurate as standard collocation, but computationally far more efficient. The present article is the first of a series devoted to explore thoroughly collocation methods. The following classification of collocation methods is introduced: direct-nonoverlapping; indirect-nonoverlapping; direct-overlapping; and indirect-overlapping. Most of the effort reported in the literature has gone to direct-nonoverlapping methods. The procedure presented in this article falls into the indirect-nonoverlapping category and it is based on Trefftz–Herrera formulation. © 1999 John Wiley & Sons, Inc.
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I. INTRODUCTION

Collocation is known as an efficient and highly accurate numerical solution procedure for partial differential equations. Another attractive feature is that its formulation is very simple.

Usually this kind of method is applied using splines. However, a more general point of view is obtained when it is formulated in spaces of fully discontinuous functions, i.e., spaces in which the functions and their derivatives may have jump discontinuities. This latter approach has been proposed by Herrera [1–4], and research on it has been carried out for some years [5]. From this more general perspective, the standard formulation using splines is seen as a particular case, which can be obtained when a suitable strategy for solving the final system of equations is followed.

The great generality of this framework permits classifying collocation methods into two large groups: direct and indirect methods. Direct methods are those in which collocation is used to construct the solution directly, while indirect methods are those in which collocation is applied to construct specialized test functions. In particular, the conventional collocation method [6–11] is

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a direct method, while Trefftz–Herrera collocation [12–15] is an indirect one. In turn, each one of these methods can be divided into two large subgroups, depending on whether the subregions used in the construction of the solution are disjoint or overlapping.

When collocation methods are seen from this perspective, it becomes apparent that its study thus far has been quite incomplete, in spite of its obvious interest. Attention has been given to direct methods mainly and, to our knowledge, only the disjoint class has been reported and discussed in the literature, while the best known formulations are based on the use of splines [6–11]. Due to this fact, Herrera and his collaborators have started a line of research to explore this wide class of methods and compare their relative merits. The results thus far obtained will be reported in a series of articles that are now being prepared. In the case of elliptic equations of second-order, it is standard to require continuity of both the function and its derivative [6–11] when formulating direct methods. However, these conditions can be relaxed when the Trefftz–Herrera method, an indirect method, is applied [12–15]. But even in the realm of direct collocation methods, it is possible to relax such conditions when a wider class of direct methods, to be introduced in later articles of this series, is considered. Similar results can be derived for higher-order equations and systems of equations.

The present article, first of the series, is devoted to start a systematic formulation of the Trefftz–Herrera collocation (TH). The basis of this procedure stems from two sources: the Trefftz method and Herrera's algebraic theory of boundary value problems. The first method originates in the work of Trefftz [16]. According to Jirousek and Zielinski [17], the end of the 1970s can be considered as the beginning of the modern Trefftz-type methods, since during this period the theory of the TH-complete sets of test functions was established by Herrera [18]. Jirousek himself, with the collaboration of Zielinski, lead the application of these methods to plate bending and elasticity problems [19–20]. From the beginning of the 1980s, this approach attracted a growing number of researchers (for an extensive list of contributors, see [17]).

In Herrera's theoretical foundations for Trefftz method a fundamental concept is ‘‘TH-completeness,’’ which was introduced under the name of C-completeness [21]. Very important contributions to the development of TH-complete systems of analytical solutions were made by several authors and a systematic presentation of the subject, summarizing such work up to 1992, may be found in Begher and Gilbert [22]. According to these authors:

> ‘‘The function theoretic approach which was pioneered by Bergman and Vekua and further developed by Colton, Gilbert, Kracht–Kreyszig, Lanckau and others, may now be effectively applied because of this result (the TH-completeness concept) of the formulation by Herrera, as an effective means to solving boundary value problems.’’

Several presentations of Herrera's theoretical framework of Trefftz method are available [23–24] and a systematic description of this theory appeared in book form [18]. A basic ingredient of this framework is the ‘‘Algebraic Theory of Boundary Value Problems’’ [1–4, 18]. Later, some aspects of this theory were generalized and, in this more general form, it supplies the basis for the systematic use of fully discontinuous functions in the treatment of partial differential equations. This scheme, which possesses great generality because it is applicable to equations of any order and systems of such equations, has lead to a great variety of numerical methods, among those best known are Localized Adjoint Methods (LAM) [25–33] and Eulerian–Lagrangian Localized Adjoint Methods (ELLAM) [27, 31, 34–46].

Although the possibility of using this framework as a basis for a new type of collocation methods was first suggested in [26], and more recently in [12–15], it has not been fully developed up to know. Thus, in this article this procedure is applied to ordinary differential equations and its merits are compared with those of ‘‘standard collocation.’’ In another article of this series, similar developments will be presented for multidimensional problems.

In Section II of this article, the more conventional methods of collocation, standard collocation, are briefly explained. The Trefftz–Herrera method is presented in Section III, as an exact method of solution. However, generally the specialized test functions in which the method is based are not known exactly, and it is necessary to resort to approximate methods to construct them. These may be any, but, in particular, when the approximate method used is collocation, the resulting numerical procedure will be referred to as the Trefftz–Herrera Collocation (TH-collocation). The construction of the specialized test functions by means of collocation will be explained in Section IV. The price paid for using approximate weighting functions is the introduction of an error, and bounds for the error are derived in Section V. Numerical experiments in which these theoretical bounds are tested are carried out in Section VI, and the performances of TH-collocation and ‘‘standard collocation’’ are also compared there. The conclusions are presented in Section VII.

II. STANDARD COLLOCATION METHOD

The method of collocation most widely known and used has been described by several authors. A particular version of this method, apparently introduced by Carey and Finlayson [8], performs collocation on finite elements using splines. For a description of this procedure see [6–11], and a brief historical description is presented in [47]. For comparison purposes, in this section, this method of collocation is applied to the most general second-order differential equation, in one independent variable, which is linear. In the following section, TH-collocation (Trefftz–Herrera Collocation [12–15]) will be introduced and applied to a more general version of the same problem.

Consider the differential equation

$$\mathcal{L}u \equiv -\frac{d}{dx}\left(a\frac{du}{dx}\right) + \frac{d}{dx}(bu) + cu = f_\Omega \tag{2.1}$$

in an interval $[0, l]$ of the real line and subjected to Dirichlet boundary conditions:

$$u = u_0, \quad \text{at } x = 0 \quad \text{and} \quad u = u_l \quad \text{at } x = l. \tag{2.2}$$

To formulate this problem using piecewise cubic polynomials and orthogonal collocation, a partition $\Pi \equiv \{x_0, x_1, \ldots, x_E\}$ is introduced. In addition, define $x_0 = 0$ and $x_E = l$. Then the collocation approximate solution $\hat{u}(x)$ is represented by [47]:

$$\hat{u}(x) = \sum_{i=0}^{E} \hat{\mathbf{u}}_i \cdot \boldsymbol{\phi}_i(\mathbf{x}) = \sum_{i=0}^{E} \{\hat{u}_i h_i^0 + \hat{u}_i' h_i^1\}. \tag{2.3}$$

Here, as in what follows, the notation is $\mathbf{u}_i = (u_i, u_i')$ and $\boldsymbol{\phi}_i = (h_i^0, h_i^1)$, where $h_i^0(\mathbf{x})$ and $h_i^1(\mathbf{x})$ are piecewise Hermite cubic polynomials with support in the interval (x_{i-1}, x_{i+1}), except at ‘‘boundary nodes’’ (i.e., when $i = 0$ or E), in which case the support has to be modified in an obvious manner. Clearly, when the approximate solution $\hat{u}(x)$ is given by Eq. (2.3), it belongs to $C^1([0, l])$.

In addition, the approximate solution $\hat{u}(x)$ must fulfill the *collocation equations*:

$$[\mathcal{L}\hat{u} - f_\Omega]_{x_j^e} = 0 \quad e = 1, \ldots, E, j = 1, 2, \tag{2.4}$$

where, for each $e = 1, 2, \ldots, E, x_j^e$ $(j = 1, 2)$ are the Gaussian points of the interval (x_{e-1}, x_e). If we substitute the expression of Eq. (2.3) in this one, then the resulting system of equations is

$$\left[\sum_{i=0}^{N}\{\hat{u}_i\mathcal{L}h_i^0 + \hat{u}_i'\mathcal{L}h_i^1\} - f_\Omega\right]_{x_j^e} = 0 \quad e = 1, \ldots, E, j = 1, 2. \tag{2.5}$$

In matrix form this is

$$\left[\begin{array}{ccccccc} [\mathcal{L}h_1^0]_{x_1^1} & [\mathcal{L}h_1^1]_{x_1^1} & [\mathcal{L}h_2^0]_{x_1^1} & [\mathcal{L}h_2^1]_{x_1^1} & 0 & 0 & \cdots \\ [\mathcal{L}h_1^0]_{x_2^1} & [\mathcal{L}h_1^1]_{x_2^1} & [\mathcal{L}h_2^0]_{x_2^1} & [\mathcal{L}h_2^1]_{x_2^1} & 0 & 0 & \cdots . \\ 0 & 0 & [\mathcal{L}h_2^0]_{x_1^2} & [\mathcal{L}h_2^1]_{x_1^2} & [\mathcal{L}h_3^0]_{x_1^2} & [\mathcal{L}h_3^1]_{x_1^2} & \cdots \\ 0 & 0 & [\mathcal{L}h_2^0]_{x_2^2} & [\mathcal{L}h_2^1]_{x_2^2} & [\mathcal{L}h_3^0]_{x_2^2} & [\mathcal{L}h_3^1]_{x_2^2} & \cdots . \\ . & . & . & . & . & . & \cdots \\ . & . & . & . & . & . & \cdots . \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots . \end{array}\right.$$

$$\left.\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ . & . & . & . \\ . & . & . & . \\ [\mathcal{L}h_N^0]_{x_1^N} & [\mathcal{L}h_N^1]_{x_1^N} & [\mathcal{L}h_{N+1}^0]_{x_1^N} & [\mathcal{L}h_{N+1}^1]_{x_1^N} \\ [\mathcal{L}h_N^0]_{x_2^N} & [\mathcal{L}h_N^1]_{x_2^N} & [\mathcal{L}h_{N+1}^0]_{x_2^N} & [\mathcal{L}h_{N+1}^1]_{x_2^N} \end{array}\right] \left[\begin{array}{c} u_0 \\ u_0' \\ u_1 \\ u_1' \\ . \\ . \\ u_N \\ u_N' \end{array}\right] = \left[\begin{array}{c} f(x_1^1) \\ f(x_2^1) \\ f(x_1^2) \\ f(x_2^2) \\ . \\ . \\ f(x_1^N) \\ f(x_2^N) \end{array}\right]. \tag{2.6}$$

When this system of equations is complemented with the boundary conditions of Eq. (2.2), a $2(E+1)$ by $2(E+1)$ system is obtained, which can be solved for $\hat{u}_i$ and $\hat{u}_i'$.

For the standard method of collocation, applied using cubic polynomials and orthogonal collocation, the estimated error yielded by the approximate solution is of order $O(h^4)$, as it is shown in [48] (p. 304).

III. TREFFTZ–HERRERA METHOD

In this section, a slightly more general version of the problem presented in Section II, will be formulated using Trefftz–Herrera approach. For this purpose, write $\Omega_i = (x_{i-1}, x_i), i = 1, \ldots, E$, and let the spaces of trial and test functions be identical with D, where D is the space of functions whose elements belong to $H^2(\Omega_i)$, in each one of the subintervals, $i = 1, \ldots, E$. Observe that functions belonging to D may have jump discontinuities at internal nodes, since no connecting condition is imposed between different elements.

The general boundary value problem to be considered is one with prescribed jumps. The differential equation and boundary conditions will be the same as before, i.e., Eqs. (2.1) and (2.2). The systematic formulation of other kind of boundary conditions has been treated in other publications such as [3]. The jump conditions may be stated in many different manners. We do not intend here to present a form as general as it is possible. For this kind of development, the reader is referred to [1]. To be specific, it is assumed that they are given in the form:

$$[u]_i = j_i^0 \quad \text{and} \quad \left[a\frac{du}{dx}\right]_i = j_i^1, \quad \text{at } x_i, \quad i = 1, \ldots, E-1, \tag{3.1}$$

where the square brackets stand for the "jump" of the function contained inside, i.e., limit on the right minus limit on the left, and the "prescribed jumps," j_i^0 and j_i^1, for each $i = 1, \ldots, E-1$, are two given numbers. When the coefficient 'a' is continuous, prescribing the jump of $a\frac{du}{dx}$ is equivalent to prescribing the jump of $\frac{du}{dx}$. However, the developments that follow are applicable even if the coefficients of the differential equation are discontinuous. Observe also that the problem considered in Section II is the particular case of this more general problem, for which j_i^0 and j_i^1 vanish. Finally, a dot on top of an expression means the average of its limits, from the right and from the left; for example, $\dot{u} = \frac{1}{2}(u_+ + u_-)$.

To apply the Trefftz–Herrera formulation to this problem, for every $u \in D$ and $w \in D$ define (see Appendix):

$$\langle Kw, u\rangle = \langle K^0 w, u\rangle + \langle K^1 w, u\rangle \tag{3.2a}$$

$$\langle Jw, u\rangle = \langle J^0 w, u\rangle + \langle J^1 w, u\rangle \tag{3.2b}$$

with

$$\langle J^0 u, w\rangle = -\sum_{i=1}^{E-1}[u]_i\left\{a\frac{dw}{dx} + bw\right\}_i \quad \text{and} \quad \langle J^1 u, w\rangle = \sum_{i=1}^{E-1}\left[a\frac{du}{dx}\right]_i \dot{w}_i \tag{3.3a}$$

$$\langle K^0 w, u\rangle = \sum_{i=1}^{E-1}\dot{u}_i\left[a\frac{dw}{dx} + bw\right]_i \quad \text{and} \quad \langle K^1 w, u\rangle = -\sum_{i=1}^{E-1}\left(a\frac{du}{dx}\right)_i [w]_i. \tag{3.3b}$$

Also,

$$\langle f, w\rangle = \int_0^l w f_\Omega \, dx, \tag{3.4a}$$

$$\langle g, w\rangle = u_l\left(a\frac{dw}{dx} + bw\right)_{x=l} - u_0\left(a\frac{dw}{dx} + bw\right)_{x=0}, \tag{3.4b}$$

$$\langle j, w\rangle = \langle j^0, w\rangle + \langle j^1, w\rangle \tag{3.4c}$$

with

$$\langle j^0, w\rangle = -\sum_{i=1}^{E-1} j_i^0\left(\dot{\overbrace{a\frac{dw}{dx} + bw}}\right)_i \; ; \quad \langle j^1, w\rangle = \sum_{i=1}^{E-1} j_i^1 \dot{w}_i. \tag{3.4d}$$

The basic strategy of the TH-formulation is to concentrate all the information about the sought solution at the internal nodes. To this end, specialized test functions will be developed. They satisfy:

$$\mathcal{L}^* w \equiv -\frac{d}{dx}\left(a\frac{dw}{dx}\right) - b\frac{d}{dx}w + cw = 0 \tag{3.5a}$$

and

$$w(0) = w(l) = 0. \tag{3.5b}$$

Functions $w \in D$, which fulfill Eqs. (3.5), constitute the linear subspace $N_Q \cap N_C \subset D$ (see Appendix). Then, the TH-formulation of this problem is given by the variational principle [12–15]:

$$-\langle K^* u, w\rangle = \langle f - g - j, w\rangle; \quad \forall w \in N_Q \cap N_C \subset D. \tag{3.6}$$

Let $u \in D$ be the solution of the boundary value problem with prescribed jumps, then the general result is as follows (see Appendix): When $\mathcal{E} \subset N_Q \cap N_C$ is TH-complete and $\hat{u} \in D$, then

$$-\langle K^* \hat{u}, w\rangle = \langle f - g - j, w\rangle; \quad \forall w \in \mathcal{E} \tag{3.7}$$

if and only if

$$K^* \hat{u} = K^* u. \tag{3.8}$$

Observe that Eq. (3.8) must be understood as an equality between functionals, and it is the same as $\langle K^* \hat{u}, w\rangle = \langle K^* u, w\rangle, \forall w \in D$. In view of Eqs. (3.3) and (3.4), this is tantamount to

$$\dot{\hat{u}}_i = \dot{u}_i; \quad i = 1, \ldots, E-1 \tag{3.9a}$$

and

$$\frac{1}{2}\left\{\left(a\frac{d\hat{u}}{dx}\right)_+ + \left(a\frac{d\hat{u}}{dx}\right)_-\right\}_i = \frac{1}{2}\left\{\left(a\frac{du}{dx}\right)_+ + \left(a\frac{du}{dx}\right)_-\right\}_i, \quad i = 1, \ldots, E-1. \tag{3.9b}$$

It can be seen that

$$u_+ = \dot{u} + \frac{1}{2}[u] \quad \text{and} \quad u_- = \dot{u} - \frac{1}{2}[u]. \tag{3.10}$$

Therefore, both u_+ and u_- are determined by $\dot{u}$, when $[u]$ is data of the problem. In particular, when the sought solution is continuous, the prescribed jump of the function vanishes everywhere and $u = \frac{1}{2}(u_+ + u_-) = u_+ = u_-$. Similarly, the ‘‘fluxes’’ on both sides of each node can be derived from

$$\overset{\cdot}{\overbrace{a\frac{du}{dx}}}$$

and the jump conditions.

In more general situations, as when dealing with partial differential equations, the dimension of the linear subspace $N_Q \cap N_C$ is infinite. However, for ordinary differential equations its dimension is finite. In this special situation, a subset $\mathcal{E} \subset N_Q \cap N_C$ is TH-complete whenever $\mathcal{E}$ is a basis of $N_Q \cap N_C$. For the special case under consideration, its dimension is $2(E-1)$, and to build a basis it is only necessary to have a set of $2(E-1)$ linearly independent functions of $N_Q \cap N_C$. To this end, with each subinterval $(x_{i-1}, x_i) - i = 1, \ldots, E-$, associate two functions $w^{i\alpha}$ $(\alpha = 1, 2)$ which belong to $N_Q \cap N_C$ [i.e., that satisfy Eqs. (3.5)], vanish outside that subinterval and are uniquely defined by the conditions:

$$w^{i1}(x_{i-1}+) = 1; \quad w^{i1}(x_i-) = 0 \tag{3.11a}$$

$$w^{i2}(x_{i-1}+) = 0; \quad w^{i2}(x_i-) = 1. \tag{3.11b}$$

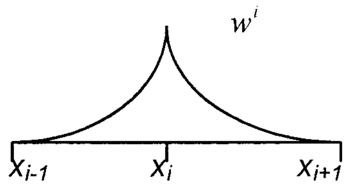


FIG. 1. A weighting function.

Observe that, except for w^{11} and w^{E2}, all those functions belong to $N_Q \cap N_C$. Thus, if $\mathcal{E} \subset N_Q \cap N_C$ is defined by

$$\mathcal{E} = \{w^{12}, w^{E1}\} \cup \{w^{i\alpha}; i = 2, \ldots, E-1; \alpha = 1, 2\}, \tag{3.12}$$

then $\mathcal{E}$ is TH-complete, since the number of elements of $\mathcal{E}$ is $2(E-1)$. In particular, the system of Eqs. (3.7) is a $2(E-1)$ by $2(E-1)$ system, because the unknowns are $\dot{\hat{u}}$ and

$$\overset{\cdot}{\overbrace{\left(a\frac{d\hat{u}}{dx}\right)}}_i,$$

with $i = 1, \ldots, E-1$, according to Eqs. (3.9). This is the system of equations supplied by the Trefftz–Herrera formulation of this problem.

A comment would helpful to better understanding these results. The relations given by Eqs. (3.9) are exact, since at no point have any approximations been introduced. This means that the solution of the system of Eqs. (3.7), which in the case being treated is a $2(E-1)$ by $2(E-1)$ system, yields exactly the averages of the solution and the flux (its derivative, in many instances), at the internal nodes. However, later it will be seen that the point at which approximations are required in applications is when developing the specialized test functions of $N_Q \cap N_C$; unless they are known exactly beforehand. Another point to be noticed is that no base functions are involved in the system (3.7).

Inspecting Eq. (3.4), it is seen that the information about the sought solution can be concentrated further. Indeed, if the specialized weighting functions, in addition to satisfying Eqs. (3.5), are chosen fulfilling the condition $[w]_i = 0$ at every internal node $(i = 1, \ldots, E-1)$, this is the condition for $w \in N_{K^1}$, where N_{K^1} is the null subspace of K^1, then only information about $\dot{\hat{u}}_i$ is retained. In this case, and this is the only one to be discussed in what follows, the specialized test functions can be characterized as being $H^0(0, l)$ and fulfilling Eqs. (3.5). A TH-complete system can be defined by

$$\mathcal{E} = \{w^1, \ldots, w^{E-1}\}, \tag{3.15}$$

where, for $i = 1, \ldots, E-1$,

$$w^i(x) = w^{i2}(x), \quad \text{when } x_{i-1} < x < x_i \tag{3.16a}$$

$$w^i(x) = w^{(i+1)1}(x), \quad \text{when } x_i < x < x_{i+1}, \tag{3.16b}$$

and each of these functions vanishes outside the interval (x_{i-1}, x_{i+1}), as illustrated in Fig. 1.

The resulting system of equations, derived from Eq. (3.7), is an $(E-1)$ by $(E-1)$ system, since the only unknowns are $\dot{u}_i$, for $i = 1, \ldots, E-1$. It is

$$-\langle K^* \hat{u}, w^k \rangle = \langle f - g - j, w^k \rangle; \quad k = 1, \ldots, E-1, \tag{3.17}$$

or, more explicitly,

$$-\sum_{i=1}^{E-1}\left[a\frac{dw^k}{dx}+bw^k\right]_i\dot{\hat{u}}_i=\langle f-g-j,w^k\rangle;\quad k=1,\ldots,E-1. \tag{3.18}$$

This is a system of equations whose matrix will be denoted by $\mathbf{M}$ and its elements are

$$M_{ki}=-\left[a\frac{dw^k}{dx}+bw^k\right]_i. \tag{3.19}$$

Observe that $\mathbf{M}$ is a three-diagonal matrix, because M_{ki} vanishes unless $i=k-1,k$ or $k+1$, since the support of w^k is the interval $[x_{k-1},x_{k+1}]$. It will be useful to notice that

$$\left[a\frac{dw^k}{dx}+bw^k\right]_i=\sum_{j=1}^{E-1}\left[a\frac{dw^k}{dx}+bw^k\right]_j w_i^j, \tag{3.20}$$

because $w_i^j=\delta_i^j$.

The right-hand side of Eq. (3.18) involves terms whose evaluation requires knowledge of w^k away from the internal nodes, as it is seen in Eq. (3.4a). There are instances in which it is better to be able to evaluate them using the values of w^k at the internal nodes, exclusively. This is possible. Indeed, let $u_P\in D$ be a function such that

$$\mathcal{L}u_P=f_\Omega;\quad \text{in } (0,l) \tag{3.21a}$$

$$u_P(0)=u_0\quad\text{and}\quad u_P(l)=u_l \tag{3.21b}$$

$$(\dot{u}_P)_i=0\quad\text{and}\quad [u_P]_i=j_i^0. \tag{3.21c}$$

In the Appendix it is shown that such function satisfies

$$\langle f-g-j^0,w^k\rangle=\langle J^1u_P,w^k\rangle;\quad k=1,\ldots,E-1, \tag{3.22}$$

and using it one can write

$$\sum_{i=1}^{E-1}M_{ki}\dot{\hat{u}}_i=\langle J^1u_P-j^1,w^k\rangle;\quad k=1,\ldots,E-1 \tag{3.23}$$

instead of Eq. (3.18). Recalling that $w_k^i=\delta_k^i$ and using Eqs. (3.3a) and (3.4d), it is seen that

$$\sum_{i=1}^{E-1}M_{ki}\dot{\hat{u}}_i=\left[a\frac{du_P}{dx}\right]_k-j_k^1,\quad k=1,\ldots,E-1. \tag{3.24}$$

When b is continuous (in particular, when the operator $\mathcal{L}$ is symmetric, since $b\equiv 0$, in this case), Eq. (3.19) reduces to

$$M_{ki}=-\left[a\frac{dw^k}{dx}\right]_i=-\sum_{j=1}^{E-1}\left[a\frac{dw^k}{dx}\right]_j w_i^j;\quad i,k=1,\ldots,E-1. \tag{3.25}$$

In the Appendix it is shown that when $\mathcal{L}$ is symmetric so is M_{ki}. Even more, when $\mathcal{L}$ is positive definite (this requires $c\geq 0$, in addition to $b\equiv 0$), then the matrix M_{ki} is also positive definite.

When this is the case, consider the space of $E-1$ dimensional vectors $\hat{\mathbf{u}} = (\hat{u}_1, \dots, \hat{u}_{E-1})$, and write

$$\mathbf{d} = \left(\left[a\frac{du_P}{dx} \right]_1 - j_1^1, \dots, \left[a\frac{du_P}{dx} \right]_{E-1} - j_{E-1}^1 \right).$$

Then the functional $\mathbf{X}(\hat{\mathbf{u}}) \equiv \mathbf{M}\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} - 2\mathbf{d} \cdot \hat{\mathbf{u}}$ attains a minimum in this space if and only if $\hat{\mathbf{u}} = \mathbf{u}$. Here $\mathbf{u} = (u_1, \dots, u_{E-1})$ are the values, at the internal nodes, of the exact solution of the boundary value problem with prescribed jumps.

IV. CONSTRUCTION OF SPECIALIZED TEST FUNCTIONS

The algorithms to be considered from now on yield information about the function itself exclusively; thus, the specialized weighting functions of Section III are taken from $N_Q \cap N_C \cap N_{K^1}$. In some special cases, such as when the coefficients are constant, it is possible to know the specialized weighting functions exactly. However, in general, it is necessary to resort to approximate methods to construct them. In this article, a collocation method is used; more specifically, orthogonal collocation. All that is required is to obtain the functions w^{i1} and w^{i2} $(i = 1, \dots, E)$, since the test functions w^i $(i = 1, \dots, E-1)$ are derived from them using Eqs. (3.16). They are approximated using polynomials of degree G, which, as it will be seen, allows allocating $N = G - 1$ collocation points at each subinterval (x_{i-1}, x_i).

Let us introduce the notation

$$l_{i,i-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \tag{4.1a}$$

$$l_{i-1,i}(x) = \frac{x - x_i}{x_{i-1} - x_i}, \tag{4.1b}$$

and

$$m_{i-1,i}(x) = l_{i-1,i}(x) l_{i,i-1}(x). \tag{4.1c}$$

Then define

$$\hat{w}^{i1}(x) = l_{i-1,i}(x) + m_{i-1,i}(x) P_i^{(1)}(x) \tag{4.2a}$$

and

$$\hat{w}^{i2}(x) = l_{i,i-1}(x) + m_{i-1,i}(x) P_i^{(2)}(x), \tag{4.2b}$$

where $P_i^{(1)}(x)$ and $P_i^{(2)}(x)$ are polynomials of degree $G-2$. The $G-1$ coefficients of each one of these polynomials can be determined by orthogonal collocation; that is, it is required that

$$\mathcal{L}^* \hat{w}^{i\alpha}(x) = 0; \quad (\alpha = 1, 2; x \in \mathcal{G}^i), \tag{4.3}$$

where $\mathcal{G}^i$ is the set of $G-1$ Gaussian points of the interval (x_{i-1}, x_i). Once the functions $\hat{w}^{i\alpha}$ $(i = 1, \dots, E; \alpha = 1, 2)$ have been constructed, the test functions $\hat{w}^i$, which approximate w^i, are defined using Eqs. (3.16). Observe that, when the coefficient 'a' is continuous, the nonvanishing elements of the matrix $\mathbf{M}$ [Eq. (3.19)] are

$$\left[a\frac{dw^k}{dx} \right]_{k-1} = a \left(\frac{1}{h_k} + \frac{1}{h_k} P_k^{(2)}(x_{k-1}) \right) \tag{4.4a}$$

$$\left[a\frac{dw^k}{dx}\right]_k = a\left(-\frac{1}{h_{k+1}} + \frac{1}{h_{k+1}}P_{k+1}^{(1)}(x_k) - \frac{1}{h_k} + \frac{1}{h_k}P_k^{(2)}(x_k)\right) \tag{4.4b}$$

$$\left[a\frac{dw^k}{dx}\right]_{k+1} = a\left(\frac{1}{h_{k+1}} + \frac{1}{h_{k+1}}P_{k+1}^{(1)}(x_{k+1})\right). \tag{4.4c}$$

Here, $h_k = x_k - x_{k-1}$ and, in what follows, $h = \underbrace{\max}_{k} h_k$.

The construction of the function u_P of Section III, is similar. Indeed, the conditions of Eqs. (3.21c) together, are equivalent to

$$u_P(x_i+) = \frac{1}{2}j_i^0 \quad u_P(x_i-) = -\frac{1}{2}j_i^0. \tag{4.5}$$

Therefore, the conditions of Eqs. (3.21), which include the boundary values at 0 and l, define well-posed problems locally, at each one of the $(x_{i-1}, x_i), i = 1, \ldots, E$, whose only solution is u_P. One can use the following approximation for u_P, in the subinterval (x_{i-1}, x_i), for $i = 2, \ldots, E-1$:

$$\hat{u}_P(x) = \frac{1}{2}j_{i-1}^0 l_{i-1,i}(x) - \frac{1}{2}j_i^0 l_{i,i-1}(x) + m_{i-1,i}(x)P_i(x), \tag{4.6}$$

where the coefficients of the polynomial $P_i(x)$, whose degree is again $G-2$, are determined by orthogonal collocation in the equation

$$\mathcal{L}\hat{u}_P(x) = f_\Omega(x). \tag{4.7}$$

When $i = 1$ or E, the subinterval (x_{i-1}, x_i) contains one of the end points of the interval $(0, l)$, and Eq. (4.6) must be modified in a suitable manner, incorporating the prescribed boundary values.

V. ERROR ESTIMATES

In this section, it is assumed that the coefficients of the operator $\mathcal{L}$ are $C^1(0, l)$. Let $\hat{u}$ and $\hat{w}^i$ be the approximations to u and w^i, respectively. In addition, define

$$e(x) = u(x) - \hat{u}(x) \quad \text{and} \quad v^i(x) = w^i(x) - \hat{w}^i(x), \tag{5.1}$$

and observe that the support of $v^i(x)$, is contained in (x_{i-1}, x_{i+1}). In addition, this function vanishes at x_{i-1}, x_i, and x_{i+1}. Equation (3.6) is

$$-\langle K^* u, w^i\rangle = \langle f - g - j, w^i\rangle \quad \forall i = 1, \ldots, E-1. \tag{5.2}$$

On the other hand, an approximate *internal boundary solution*, according to Eq. (3.7), fulfils

$$-\langle K^* \hat{u}, \hat{w}^i\rangle = \langle f - g - j, \hat{w}^i\rangle \quad \forall i = 1, \ldots, E-1, \tag{5.3}$$

which can be written as

$$-\langle K^* \hat{u}, w^i\rangle + \langle K^* \hat{u}, v^i\rangle = \langle f - g - j, \hat{w}^i\rangle \quad \forall i = 1, \ldots, E-1. \tag{5.4}$$

Subtracting Eq. (5.4) from Eq. (5.2) and rearranging, it is obtained:

$$-\langle K^* e, w^i\rangle = \langle K^* \hat{u}, v^i\rangle + \langle f - g - j, v^i\rangle \quad \forall i = 1, \ldots, E. \tag{5.5}$$

By Theorem A.1 of the Appendix, there exists a function $\epsilon_\Omega(x) \in H^0(0,l)$ and a generic constant M, such that $\|\epsilon_\Omega\|_\infty < Mh^{\lambda+2N}$ with the property that, for every $i = 1, \ldots, E-1$, one has

$$\int_0^l \epsilon_\Omega(\xi) w^i(\xi)\, d\xi = \langle K^* \hat{u}, v^i \rangle + \langle f - g - j, v^i \rangle. \tag{5.6}$$

Here, N is as was defined in Section IV, $\lambda = 0$ if $b + \frac{da}{dx} = 0$, or $G = 2$ (i.e., $N = 1$), and $\lambda = -1$ otherwise. Using such function, define the function $\hat{e}(x)$ by

$$\hat{e}(x) = \int_0^l G(x,\xi)\epsilon_\Omega(\xi)\, d\xi, \tag{5.7}$$

where $G(x,\xi)$ is the Green's function for the boundary value problem of Section III, when the boundary and jump conditions vanish. With this definition, $\hat{e}(x)$ fulfills the differential equation

$$\mathcal{L}\hat{e}(x) = \epsilon_\Omega(x) \tag{5.8}$$

together with the boundary conditions

$$\hat{e}(0) = \hat{e}(1) = 0 \tag{5.9}$$

and the continuity conditions:

$$[\hat{e}]_i = \left[a \frac{d\hat{e}}{dx} \right]_i = 0 \quad i = 1, \ldots, E-1. \tag{5.10}$$

Therefore, Eq. (3.7) can be applied to $\hat{e}$, with $g, j \in D^*$, equal to zero and

$$\langle f, w \rangle \equiv \int_0^l \epsilon_\Omega(\xi) w(\xi)\, d\xi. \tag{5.11}$$

It is

$$-\langle K^* \hat{e}, w^i \rangle = \int_0^l \epsilon_\Omega(\xi) w^i(\xi)\, d\xi; \quad i = 1, \ldots, E-1. \tag{5.12}$$

In view of this equation and using Eqs. (5.6) and (5.5), it is seen that

$$-\langle K^* \hat{e}, w^i \rangle = -\langle K^* e, w^i \rangle; \quad i = 1, \ldots, E-1. \tag{5.13}$$

This implies

$$K^{0*}\hat{e} = K^{0*} e, \tag{5.14}$$

because the system of test functions $\{w^1, \ldots, w^{E-1}\}$, is TH-complete. Hence,

$$\hat{e}(x_i) = \dot{\overline{e(x_i)}} \quad i = 1, \ldots, E-1. \tag{5.15}$$

Now

$$|\dot{\overline{e(x_i)}}| = |\hat{e}(x_i)| = \left| \int_0^l G(x_i,\xi)\epsilon_\Omega(\xi)\, d\xi \right|$$

$$\leq \|\epsilon_\Omega\|_\infty \int_0^l |G(x_i,\xi)| d\xi \leq M' \|\epsilon_\Omega\|_\infty \leq MM' h^{\lambda+2N}. \tag{5.16}$$

In conclusion, using the results of the Appendix, it has been shown that the error of TH-collocation, when the weighting functions are constructed applying orthogonal collocation in polynomials, is $O(h^{2N})$ if $\frac{da}{dx} + b = 0$ or $N = 1$, and it is $O(h^{2N-1})$ otherwise. Here, N is the number of collocation points at each subinterval of the partition. Recall that the degree G of the approximating polynomial is given by $G = N + 1$.

VI. NUMERICAL EXPERIMENTS

The numerical experiments that were performed consist in solving Eq. (2.1), subjected to Dirichlet boundary conditions, by TH-collocation and the ''standard collocation method'' of Section II. The examples considered correspond to several choices of the coefficients in Eq. (2.1), which are given in Table I and for each one of them the analytical solutions are known and are given in Table II. In all cases, the domain of definition was the interval [0, 1], the prescribed jumps are taken to be zero (i.e., the solution is required to be $C^1([0,1])$) and the prescribed boundary values are those implied by the analytical solutions of Table II. The number of elements (NE) was increased successively from 10 to 200.

The numerical results are summarized in Figs. 2–7. Each one of the Figs. 2–5, is composed of two parts. In part (a), the numerical efficiency of TH-collocation is compared with standard collocation, using cubic polynomials in each one of these methods. In part (b), the behavior of the error, measured in terms of the norm $\| \quad \|_\infty$, is illustrated.

The differential equation for Example 5 depends on the parameter α. This parameter was varied to take the values: 20, 40, 60, 80, and 100. The exact solutions for these choices of α are illustrated in Fig. 6. They exhibit a sharp front, which gets sharper as α increases. Figures 7(a)–(b) illustrate the variation of execution-time as the mesh is refined, while Figures 7(c) and (d) show the variation of the error as the number of elements is increased.

Because of the removal of continuity conditions in TH-collocation, required of the function

TABLE I. Definitions of the examples treated.[a]

Example	a	b	c	f_Ω
1	1	$2px/q$	$-\left\{\frac{4p(1+p)}{q^2} + \frac{2p^2}{q} + p^2\right\}$	0
2	1	0	$-40\pi^2$	0
3	$x^2 - 1$	0	$n(n+1)$	0
4	$4x^2 + 3$	$3x - 1$	$3x(x+1)$	$-(x+1)^2 e^x$
5	-1	$-\alpha$	0	0

[a] $p = \sqrt{40}\pi$; $q = 1 + p(1 + x^2)$.

TABLE II. Solution for each one of the examples.

Example	Exact solution
1	$\sin(px) + x\cos(px)$;
2	$\sin(\sqrt{40}\pi x)$
3	$(63x^5 - 70x^3 + 15x)/8$
4	e^x
5	$\frac{e^{\alpha x} - e^{\alpha}}{1 - e^{\alpha}}$; $\alpha = 20, 40, 60, 80, 100$

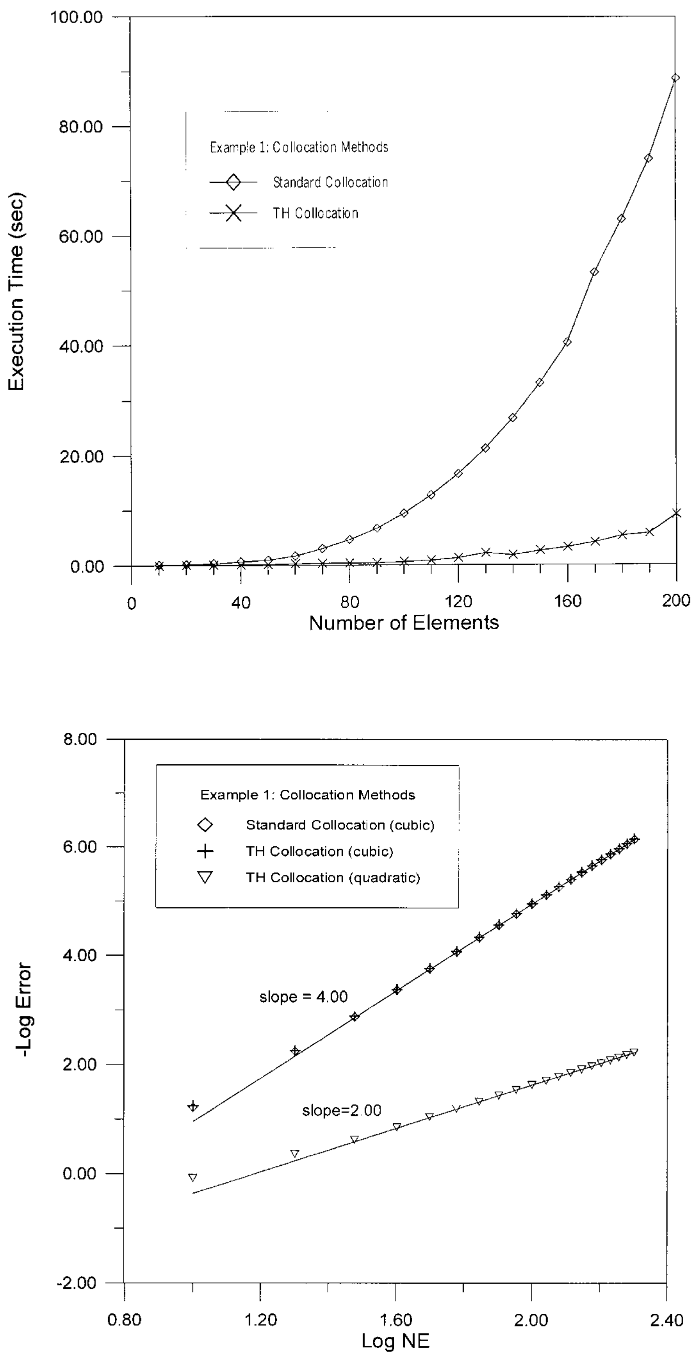


FIG. 2. (a) Example 1. Performance comparison between Standard and Trefftz–Herrera collocation. (b) Example 1: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions.

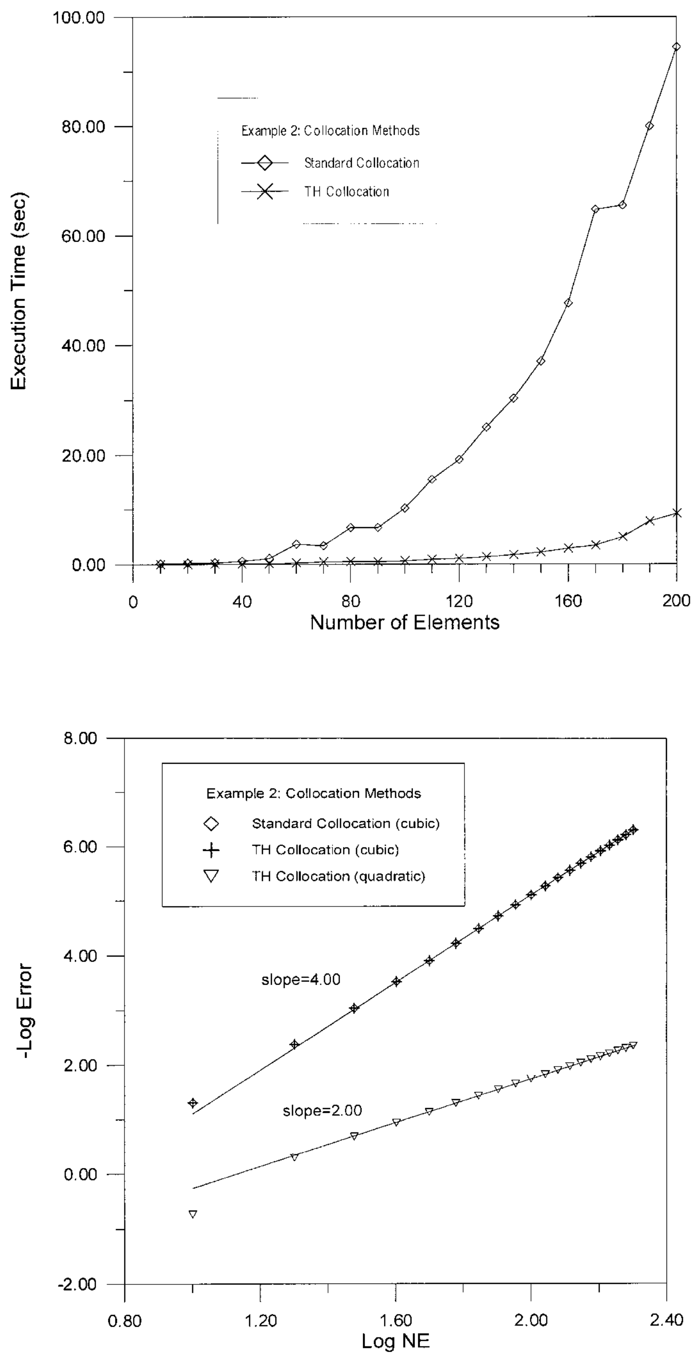


FIG. 3. (a) Example 2. Performance comparison between Standard and Trefftz–Herrera collocation. (b) Example 2: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions.

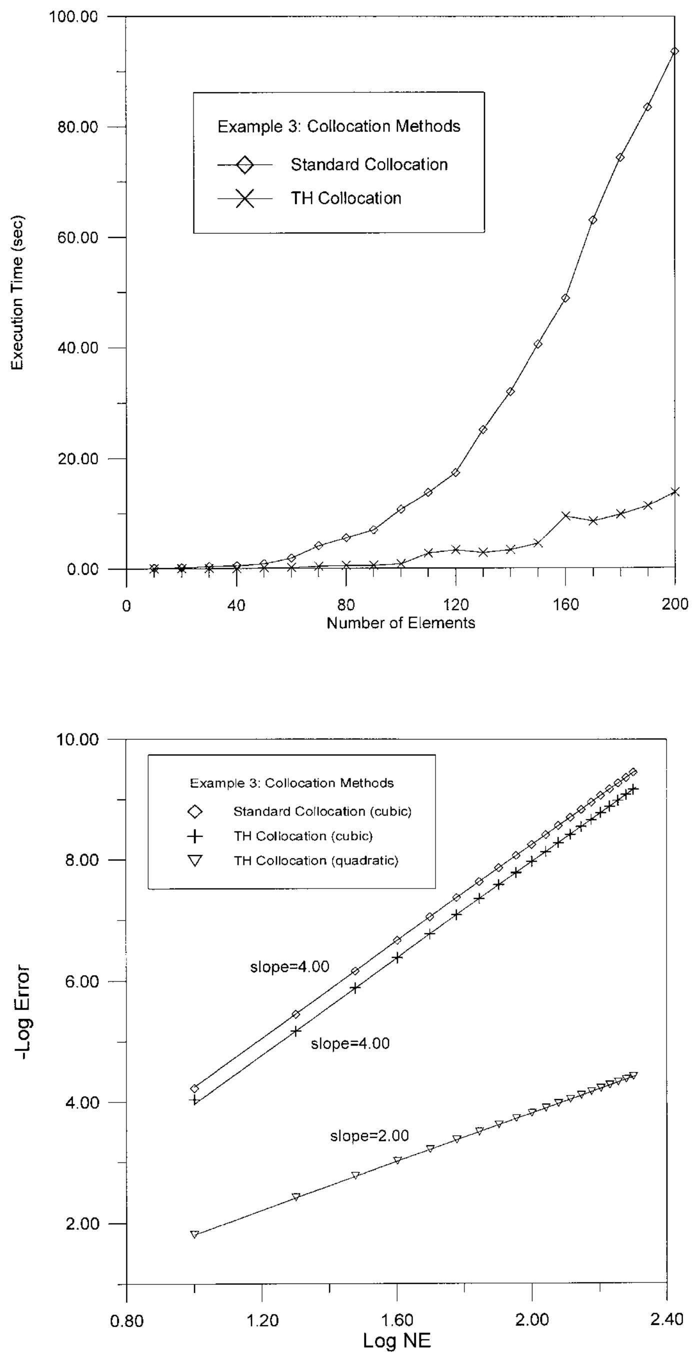
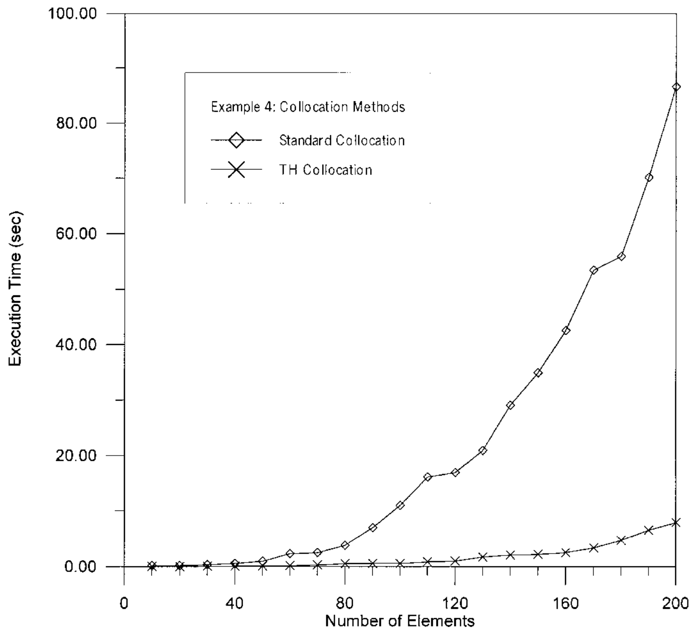


FIG. 4. (a) Example 3. Performance comparison between Standard and Trefftz–Herrera collocation. (b) Example 3: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions.



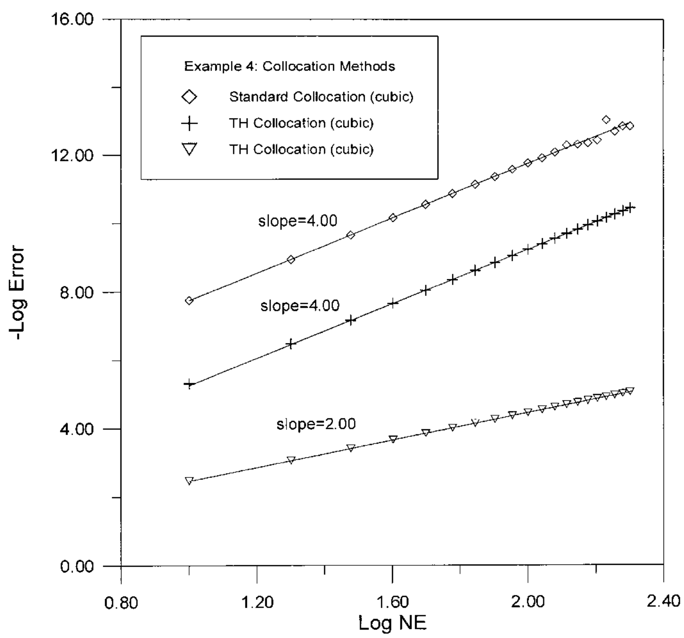


FIG. 5. (a) Example 4. Performance comparison between Standard and Trefftz–Herrera collocation. (b) Example 4: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions.

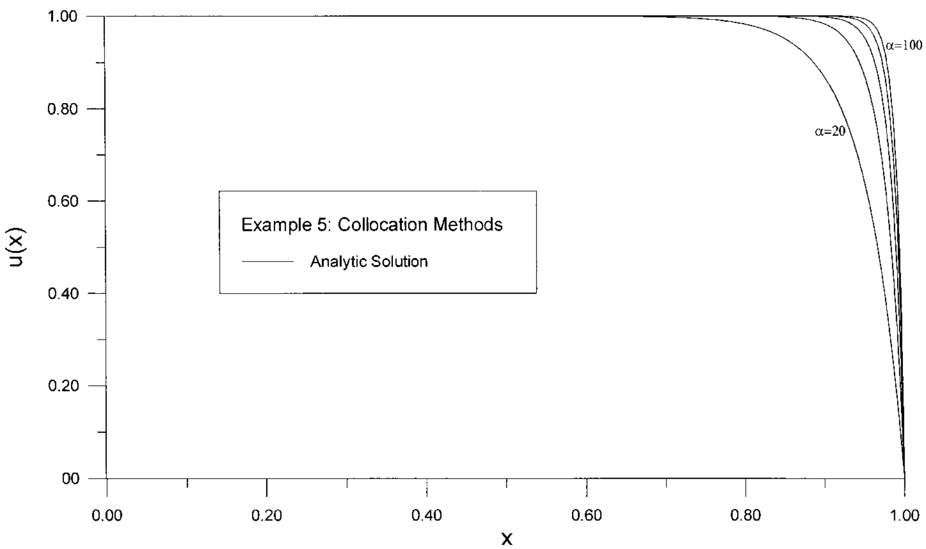


FIG. 6. Example 5: Graphic of analytical solution for $\alpha = 20, 40, 60, 80$, and 100.

spaces, since fully discontinuous functions are admissible, quadratic polynomials can be applied in this method, which is not possible in standard collocation. The results for quadratic polynomials are shown in part (b) of Figs. 2–5. The slope observed is 2 and agrees with the asymptotic behavior predicted theoretically. This result implies that the order of accuracy depends exclusively on the total number of collocation points used in the whole interval [0, 1] and not on the kind of polynomials that are applied.

Two main conclusions have been drawn from these numerical experiments. First, TH collocations offers considerable execution-time savings. Second, the experimental error for both TH and standard collocation, using cubic polynomials, is of the same order $-O(h^4)-$, and it is $O(h^2)$ when quadratic polynomials are used. These results seem to indicate that the asymptotic behavior of the error for TH-collocation, theoretically predicted in Section V, is correct but may be conservative in some instances, because it is actually $O(h^{2N})$ in all cases, even if

$$\frac{da}{dx} + b \neq 0.$$

VII. CONCLUSIONS

A nonstandard method of collocation (TH-collocation) has been presented. The peculiarity of this method is that in it collocation is used to construct specialized weighting functions with the property that they concentrate the information on the internal boundaries; i.e., the inter-element boundaries of the finite elements. In this sense, TH-collocation is an indirect method, because collocation is not used to construct the solution directly, but only the weighting functions. The method is quite general, because it can be applied to any partial differential equation that is linear or a system of such equations. A significant advantage of the present method, with respect to standard collocation methods, is that the continuity conditions imposed on the trial functions are relaxed. The procedure has been tested, applying it to the most general one-dimensional second-

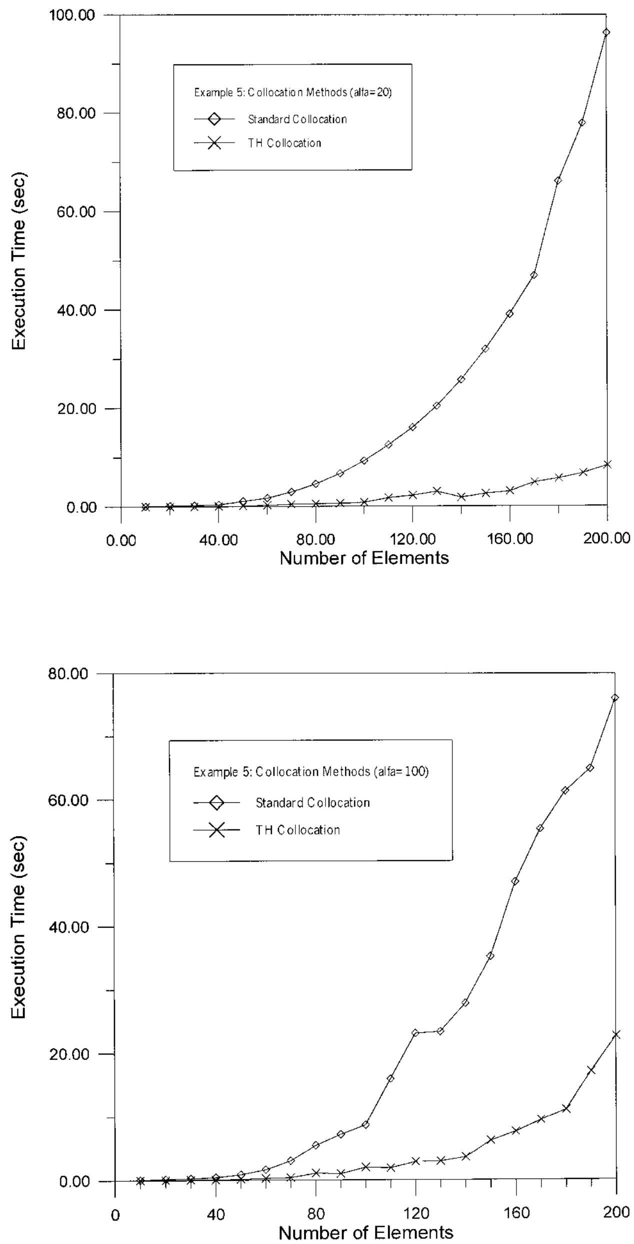
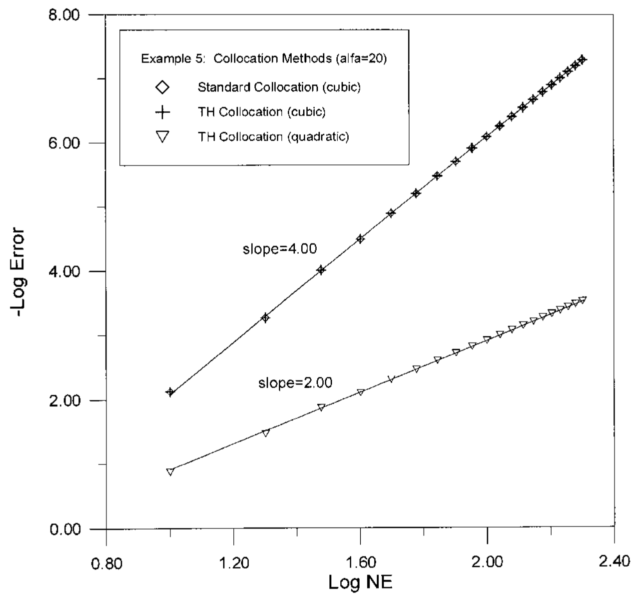


FIG. 7. (a) Example 5. Performance comparison between Standard and Trefftz–Herrera collocation ($\alpha = 20$). (b) Example 5. Performance comparison between Standard and Trefftz–Herrera collocation ($\alpha = 100$).



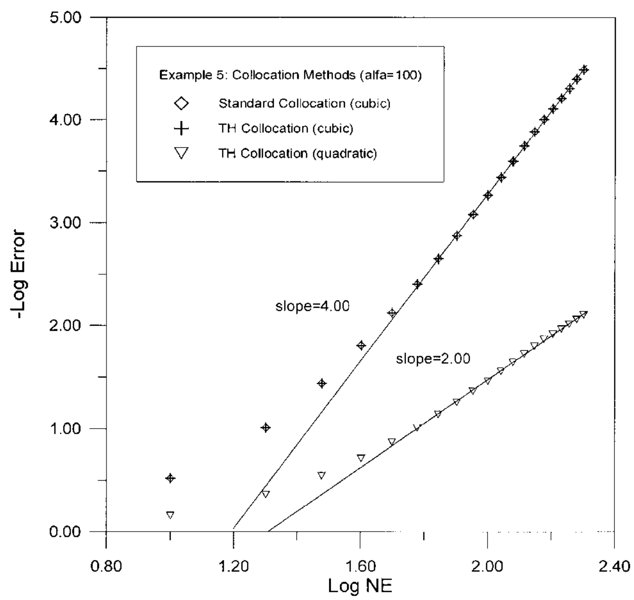


FIG. 7. (c) Example 5: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions (Case $\alpha = 20$). (d) Example 5: Convergence rate of Trefftz–Herrera collocation method using cubic and quadratic weighting functions comparing to Standard Collocation Method using cubic weighting functions (Case $\alpha = 100$).

order differential equation, and theoretical error bounds were derived for it. These error bounds have been tested by means of numerical experiments, and TH-collocation has been compared with standard collocation, reaching the conclusion that it is as accurate, but computationally far more efficient.

The results presented are a first step in a wide program of research devoted to explore thoroughly collocation methods. The full area of this search, collocation methods, has been divided into direct and indirect methods. Using another criterion for classifying them, two other broad categories are defined: nonoverlapping and overlapping; depending on whether the finite elements in which the region is divided are disjoint or overlapping, respectively. Thus, combining these two classifications the collocation methods for possible research are: direct-nonoverlapping; indirect-nonoverlapping; direct-overlapping; and indirect-overlapping. The class, more thoroughly studied in the literature thus far, is direct-nonoverlapping. The present method, TH-collocation, is an indirect-nonoverlapping method.

VIII. APPENDIX

A.1. Herrera's Algebraic Theory

Here, Herrera's abstract formulation is briefly explained. It is assumed that the spaces of trial and test functions constitute linear spaces, to be denoted by D_1 and D_2. In the developments presented in this article, it is assumed that $D = D_1 = D_2$. In addition, six bilinear functionals will be considered: $\langle Pu, w\rangle, \langle Bu, w\rangle, \langle Ju, w\rangle, \langle Qu, w\rangle, \langle Cu, w\rangle$, and $\langle Ku, w\rangle$, all of them defined on $D \times D$. They also define functional valued operators, which are linear. Thus, for example: P: $D \to D^*$, where D^* is the algebraic dual of D. For any $u \in D, Pu \in D$, is defined by $Pu(w) \equiv \langle Pu, w\rangle$. It must be stressed that elements $f \in D^*$, when D^* is the algebraic dual of D, are linear functionals that need not be continuous. In the theory, it is required that

$$P - B - J = (Q - C - K)^*. \tag{A.1}$$

Here as in what follows, a star is used to denote the transposition of a bilinear functional. Thus, for example: $\langle P^*u, w\rangle \equiv \langle Pw, u\rangle$.

Herrera's theory applies to ‘‘boundary value problems with prescribed jumps,’’ such that there is a ‘‘differential equation’’ (this may be a system of such equations), a ‘‘boundary condition,’’ and a ‘‘jump condition’’ to be satisfied by jump discontinuities of the sought solution. The differential equation, boundary condition, and jump condition are satisfied at the region Ω, the ‘‘external boundary,’’ $\partial\Omega$, and the ‘‘internal boundaries,’’ Σ, respectively. Then, it is required that the bilinear functionals mentioned before, be such that there exist functionals $f \in D^*, g \in D^*$, and $j \in D^*$, such that, given any functions $u_P, u_\partial, u_\Sigma \in D$, they fulfill the differential equation, the boundary condition, and the jump conditions, respectively, if and only if they satisfy

$$Pu_P = f, \quad Bu_\partial = g, \quad Ju_\Sigma = j. \tag{A.2}$$

Then, the boundary value problem with prescribed (BVPJ) is defined as follows: A function $u \in D$ is a solution of the BVPJ if and only if

$$Pu = f, \quad Bu = g, \quad Ju = j. \tag{A.3}$$

In addition, it is assumed that the operators P, B, and J are such that the single equation

$$(P - B - J)u = f - g - j$$

is equivalent to the three Eqs. (A.3). Clearly, this is a variational formulation of the ''boundary value problems with prescribed jumps,'' because this equation is an identity between linear functions. Thus, it means

$$\langle (P - B - J)u, w \rangle = \langle f - g - j, w \rangle \quad \forall w \in D. \tag{A.4a}$$

In the theory, it is assumed that the terms Q^*u, C^*u, and K^*u constitute the ''sought information.'' More specifically, Q^*u, C^*u, and K^*u are referred as the information sought in $\Omega, \partial\Omega$, and Σ, respectively. Equation (A.5a) is equivalent to

$$\langle (Q - C - K)^* u, w \rangle = \langle f - g - j, w \rangle \quad \forall w \in D, \tag{A.4b}$$

by virtue of Eq. (A.1). Thus, Eqs. (5.a) and (5.b) constitute two equivalent variational formulations of the BVPJ that are referred as the ''variational formulation in terms of the data'' and the ''variational formulation in terms of the sought information,'' respectively.

Let $N_Q \subset D$ and $N_C \subset D$ be the null subspaces of Q and C, respectively; i.e.,

$$N_Q = \{w \in D | Qw = 0\} \quad \text{and} \quad N_C = \{w \in D | Cw = 0\}. \tag{A.5}$$

Then, a necessary condition for $u \in D$ being a solution of the problem is that

$$-\langle K^* u, w \rangle = \langle f - g - j, w \rangle \quad \forall w \in N_Q \cap N_C \subset D. \tag{A.6}$$

However, this is not a sufficient condition for $u \in D$ being a solution, which is not difficult to see. Due to this fact, it is convenient to introduce the following definition: Let $u \in D$ be a solution of the ''boundary value problem with prescribed jumps,'' and $\hat{u} \in D$ be such that

$$K^* \hat{u} = K^* u. \tag{A.7}$$

Then, $\hat{u}$ is said to be a ''boundary solution.''

In many instances, Eq. (A.6) is a sufficient condition for $\hat{u}$ being a boundary solution and the following definition is useful: A subset $\mathcal{E}$ is said to be TH-complete when, for any $\hat{u} \in D$, one has that

$$-\langle K^* \hat{u}, w \rangle = \langle f - g - j, w \rangle \quad \forall w \in \mathcal{E} \subset D \tag{A.8}$$

implies that $\hat{u}$ is a boundary solution.

Frequently, the operators J and K can be expressed as the sum of several operators. Assume, in particular, that

$$J = J^0 + J^1; \quad \text{and} \quad K = K^0 + K^1. \tag{A.9}$$

In this case, for weighting functions $w \in N_Q \cap N_C \cap N_{K^1}$, the left-hand side of Eq. (A.8) reduces to $-\langle K^{0*} \hat{u}, w \rangle$. Due to this fact, a subset $\mathcal{E} \subset N_Q \cap N_C \cap N_{K^1}$ is said to be TH-complete when, for any $\hat{u} \in D$, one has

$$-\langle K^{0*} \hat{u}, w \rangle = \langle f - g - j, w \rangle \quad \forall w \in \mathcal{E} \subset N_Q \cap N_C \cap N_{K^1}, \tag{A.10}$$

which implies that $K^{0*}\hat{u} = K^{0*}u$, where $u \in D$ is a solution of the boundary value problem with prescribed jumps.

Using Eq. (A.2) again, if $u_\Sigma \in D$ is any function that fulfills the jump conditions, one can define $j^0 = J^0 u_\Sigma$ and $j^1 = J^1 u_\Sigma$, so that $j = j^0 + j^1$. Using this notation and taking weighting functions $w \in N_Q \cap N_C \cap N_{K^1}$, if $u_P \in D$ is such that

$$Pu_P = f; Bu_P = g; J^0 u_P = j^0 \quad \text{and} \quad K^{0*} u_P = 0. \tag{A.11}$$

One has

$$\langle (P-B-J)u_P, w\rangle = \langle (Q-C-K^0-K^1)^* u_P, w\rangle = \langle (Q-C-K^1)^* u_P, w\rangle$$
$$= \langle (Q-C-K^1)w, u_P\rangle = 0.$$

Therefore,

$$\langle f-g-j^0-J^1 u_P, w\rangle = 0 \tag{A.12a}$$

and

$$\langle f-g-j^0, w\rangle = \langle J^1 u_P, w\rangle. \tag{A.12b}$$

In view of (A.12b), Eq. (A.10) is equivalent to

$$-\langle K^{0*}\hat{u}, w\rangle = \langle J^1 u_P - j^1, w\rangle \quad \forall w \in \mathcal{E} \subset N_Q \cap N_C \cap N_{K^1}. \tag{A.13}$$

Herrera's theory has been applied using definitions of Eqs. (3.2)–(3.4) and

$$\langle Pu, w\rangle = \int_0^l w\mathcal{L}u\,dx; \quad \langle Qu, w\rangle = \int_0^l u\mathcal{L}^* w\,dx \tag{A.14a}$$

$$\langle Bu, w\rangle = u\left(a\frac{dw}{dx} + bw\right)\Big|_0^l; \quad \langle Cu, w\rangle = w\left(a\frac{du}{dx}\right)\Big|_0^l. \tag{A.14b}$$

It can be verified that, with these definitions, all the assumptions of the theory are satisfied.

Finally, it is shown that the matrix $\mathbf{M}$ of Eq. (3.25) is symmetric and positive definite, when so is the operator $\mathcal{L}$ and the specialized weighting functions are continuous. For this purpose, it will be shown that the bilinear form $-\langle K^* v, w\rangle$ is symmetric and positive definite whenever $v, w \in N_Q \cap N_C \cap N_{K^1}$. Indeed, observe that

$$0 = \int_0^l w\mathcal{L}v\,dx = \int_0^l \left(a\frac{dv}{dx}\frac{dw}{dx} + cvw\right)dx + \sum_{i=1}^{E-1}\left[va\frac{dw}{dx}\right]_i. \tag{A.15}$$

Therefore,

$$-\langle Kw, v\rangle = -\sum_{i=1}^{E-1} v\left[a\frac{dw}{dx}\right]_i = -\sum_{i=1}^{E-1}\left[va\frac{dw}{dx}\right]_i = \int_0^l \left(a\frac{dv}{dx}\frac{dw}{dx} + cvw\right)dx \tag{A.16}$$

is symmetric and positive definite, since $c \geq 0$.

A.2. Auxilliary Results for Error Estimation

Here, a result that was used in Section V, when establishing bounds for the errors of the approximate solution, is established. It is the following.

Theorem A.1. *Let v^i, be defined by Eq.* (5.1). *Then there exists a function $\epsilon_\Omega(x) \in \mathbf{H}^0(0, l)$, such that*

$$\int_0^l \epsilon_\Omega(\xi) w^i(\xi)\,d\xi = \langle K^*\hat{u}, v^i\rangle + \langle f-g-j, v^i\rangle, \tag{A.17}$$

and a generic constant $M > 0$, independent of h and i, with the property that

$$\|\epsilon_\Omega\|_\infty < Mh^{\lambda+2N}. \tag{A.18}$$

Here, $\lambda = 0$ *if* $b + \frac{da}{dx} = 0$ *or* $N \equiv G - 1 = 1$, *and* $\lambda = -1$ *otherwise.*

Proof. This Theorem follows from a sequence of lemmas to be shown next. To formulate the first one, consider Eq. (3.5a), written in the form:

$$\mathcal{L}^* w = \hat{a}\frac{d^2 w}{dx^2} + \hat{b}\frac{dw}{dx} + \hat{c}w = 0, \tag{A.19}$$

where

$$\hat{a} = -a; \hat{b} = -\left(b + \frac{da}{dx}\right); \hat{c} = c. \tag{A.20}$$

Let $w \in D$, be a function fulfilling Eq. (A.19) in a subinterval (x_{i-1}, x_i), vanishing identically outside it, and subjected to the boundary conditions:

$$w(x_{i-1}) = 0 \quad \text{and} \quad w(x_i) = 1. \tag{A.21}$$

On the other hand, let $\hat{w}(x)$ be the polynomial approximation of w, of degree $G \geq 2$, fulfilling the $G - 1$ orthogonal collocation conditions at the Gaussian points. Define the function $r(x)$ by

$$r(x) \equiv \mathcal{L}^* w(x) - \mathcal{L}^* \hat{w}(x) = -\mathcal{L}^* \hat{w}(x). \tag{A.22}$$

Then we have the following.

Lemma 1. *There is a bound for the function* $|hr(x)|$, *independent of* h *and* $i = 1, \ldots, E$. *Even more, the same is true for* $|r(x)|$ *when either* $b + \frac{da}{dx} \equiv 0$, *or* $G = 2$.

Proof. Let ξ be

$$\xi = \frac{x - x_{i-1}}{x_i - x_{i-1}}. \tag{A.23}$$

Observe that ξ fulfills the boundary conditions of Eq. (A.21). The polynomial expression of $\hat{w}(x)$ is

$$\hat{w}(x) \equiv \sum_{j=1}^{G} A_j \xi^j. \tag{A.24}$$

The coefficients A_j $(j = 1, \ldots, G)$ must satisfy the following conditions:

$$\sum_{j=1}^{G} \left\{ \hat{a} j(j-1)\frac{\xi^{j-2}}{h^2} + \frac{\hat{b}}{h} j\xi^{j-1} + \hat{c}\xi^j \right\} A_j = 0 \tag{A.25}$$

at the $G - 1$ collocation points, and also

$$\sum_{j=1}^{G} A_j = 1, \tag{A.26}$$

in view of the second boundary condition. More explicitly, Eq. (A.25) is

$$\sum_{j=2}^{G} \left\{ \hat{a} j(j-1)\frac{\xi^{j-2}}{h^2} + \frac{\hat{b}}{h} j\xi^{j-1} + \hat{c}\xi^j \right\} A_j + \left(\frac{\hat{b}}{h} + \hat{c}\xi\right) A_1 = 0. \tag{A.27}$$

From Eq. (A.26), it follows that

$$A_1 = 1 - \sum_{j=2}^{G} A_j, \tag{A.28}$$

and, therefore,

$$\sum_{j=2}^{G}\left\{\hat{a}j(j-1)\frac{\xi^{j-2}}{h^2}+\frac{\hat{b}}{h}(j\xi^{j-1}-1)+\hat{c}(\xi^j-\xi)\right\}A_j+\frac{\hat{b}}{h}+\hat{c}\xi=0. \qquad \text{(A.29)}$$

Observe that the coefficients A_j $(j=2,\ldots,G)$ may be determined by the condition that Eq. (A.29) be satisfied at the $G-1$ collocation points. Also, after multiplying by h^2, it may be seen that

$$h^2r(x)\equiv\sum_{j=2}^{G}\{\hat{a}j(j-1)\xi^{j-2}+\hat{b}h(j\xi^{j-1}-1)+\hat{c}h^2(\xi^j-\xi)\}A_j+\hat{b}h+\hat{c}h^2\xi, \qquad \text{(A.30)}$$

and at collocation points

$$\sum_{j=2}^{G}\{\hat{a}j(j-1)\xi^{j-2}+\hat{b}h(j\xi^{j-1}-1)+\hat{c}h^2(\xi^j-\xi)\}A_j=-\hat{b}h-\hat{c}h^2\xi. \qquad \text{(A.31)}$$

The only solution of this system of equations, when $h=0$, is $A_2=\cdots=A_G=0$, in which case $A_1=1$. Then, using this fact, it can be shown that there is a generic constant $M>0$, independent of h, such that $|A_i|<Mh$, for $i=2,\ldots,G$. Therefore, the function

$$hr(x)\equiv\sum_{j=2}^{G}\{\hat{a}j(j-1)\xi^{j-2}+\hat{b}h(j\xi^{j-1}-1)+\hat{c}h^2(\xi^j-\xi)\}\frac{A_j}{h}+\hat{b}+\hat{c}h\xi \qquad \text{(A.32)}$$

is bounded, since so is A_j/h, for every $j\geq 2$.

When $G=2$,

$$h^2r(x)=\{2\hat{a}+h\hat{b}(2\xi-1)+h^2\hat{c}(\xi^2-\xi)\}A_2+\hat{b}h+h^2\hat{c}\xi, \qquad \text{(A.33)}$$

and at the only collocation point ξ_C $(=1/2)$, this is,

$$h^2r(x)=\{2\hat{a}^*+h\hat{b}^*(2\xi_C-1)+h^2\hat{c}^*(\xi_C^2-\xi_C)\}A_2+\hat{b}^*h+h^2\hat{c}^*\xi_C=0, \qquad \text{(A.34)}$$

where $\hat{a}^*$, $\hat{b}^*$, and $\hat{c}^*$ are the values of $\hat{a}$, $\hat{b}$, and $\hat{c}$, at the collocation point, respectively. Subtracting this latter equation from (A.33), it can be shown that $r(x)$ is bounded, when $a(x)$ is Lipschitz continuous.

Finally, when $\hat{b}\equiv 0$, one has

$$h^2r(x)\equiv\sum_{j=2}^{G}\{\hat{a}j(j-1)\xi^{j-2}+\hat{c}h^2(\xi^j-\xi)\}A_j+\hat{c}h^2\xi, \qquad \text{(A.35)}$$

and it can be seen that $\frac{A_j}{h^2}$ is bounded for $j=2,\ldots,G$. Thus,

$$r(x)\equiv\sum_{j=2}^{G}\{\hat{a}j(j-1)\xi^{j-2}+\hat{c}h^2(\xi^j-\xi)\}\frac{A_j}{h^2}+\hat{c}\xi \qquad \text{(A.36)}$$

is also bounded.

Recalling the definition of λ given in Theorem A.1, this Lemma can be summarized, by the equation:

$$\|r\|_\infty=O(h^\lambda). \qquad \text{(A.37)}$$

Lemma 2. *Let $v^i \equiv w^i - \hat{w}^i$, as in Section V, then*

$$\langle K^* \hat{u}, v^i \rangle + \langle f - g - j, v^i \rangle = O(h^{\lambda+2N+1}). \tag{A.38}$$

Proof. Let $G^i(x, \xi)$, for each $i = 1, \ldots, E$, be the Green's function for the interval (x_{i-1}, x_i), fulfilling the homogeneous Eq. (A.19), and the boundary conditions

$$G^i(x_{i-1}, \xi) = G^i(x_i, \xi) = 0. \tag{A.39}$$

It can be shown that there is a generic constant $M' \geq 0$, such that

$$|G^i(x, \xi)| \leq M'h \quad \text{and} \quad |G^{i+1}(x, \xi)| \leq M'h \tag{A.40a}$$

and

$$\left| \frac{dG^i}{dx}(x, \xi) \right| \leq M' \quad \left| \frac{dG^{i+1}}{dx}(x, \xi) \right| \leq M'. \tag{A.40b}$$

Define $r^i \equiv \mathcal{L}^* v^i$ and observe that the support of $v^i \equiv w^i - \hat{w}^i$, as well as that of r^i is the subinterval (x_{i-1}, x_{i+1}). In addition, v^i vanishes at the nodes; i.e.,

$$v^i(x_{i-1}) = v^i(x_i) = v^i(x_{i+1}) = 0. \tag{A.41}$$

Using Eqs. (A.40) and Lemma 1, it may be shown that (see [48], p. 307)

$$v^i(x) = \int_{x_{i-1}}^{x_i} G^i(x, \xi) r^i(\xi)\, d\xi = O(h^{\lambda+2N+2}) \quad \text{for} \quad x \in (x_{i-1}, x_i), \tag{A.42a}$$

while

$$\frac{dv^i}{dx}(x) = \int_{x_{i-1}}^{x_i} \frac{dG^i}{dx}(x, \xi) r^i(\xi)\, d\xi = O(h^{\lambda+2N+1}) \quad \text{for} \quad x \in (x_{i-1}, x_i), \tag{A.42b}$$

and similar relations hold for $x \in (x_i, x_{i+1})$. Therefore,

$$\langle f, v^i \rangle = \int_{x_{i-1}}^{x_{i+1}} v^i f_\Omega\, dx = O(h^{\lambda+2N+2}) \tag{A.43a}$$

$$\langle g, v^i \rangle = u_l \left(a \frac{dv^i}{dx} \right)_l - u_0 \left(a \frac{dv^i}{dx} \right)_0 = O(h^{\lambda+2N+1}) \tag{A.43b}$$

$$\langle j^0, v^i \rangle = -\sum_{k=i-1}^{k=i+1} j_k^0 \left(a \frac{dv^i}{dx} + bv^i \right)_k = O(h^{\lambda+2N+1}) \tag{A.43c}$$

$$\langle j^1, v^i \rangle = -\sum_{k=i-1}^{k=i+1} j_k^1 v_k^i = O(h^{\lambda+2N+1}) \tag{A.43d}$$

$$\langle K^0 v^i, \hat{u} \rangle = \sum_{k=i-1}^{k=i+1} \dot{\hat{u}}_k \left[a \frac{dv^i}{dx} + bv^i \right]_k = O(h^{\lambda+2N+1}) \tag{A.44a}$$

and

$$\langle K^1 v^i, \hat{u}\rangle = -\sum_{k=i-1}^{k=i+1} \overbrace{\left(a\frac{d\hat{u}^i}{dx}\right)_k}^{\cdot} [v^i]_k = 0. \tag{A.44b}$$

Therefore,

$$\langle K^* \hat{u}, v^i\rangle + \langle f - g - j, v^i\rangle = O(h^{\lambda+2N+1}). \tag{A.45}$$

Lemma 3. *Using the notation of Section IV, write the specialized weighting functions w^i in the form*

$$w^i(x) = l_{i,i-1}(x) + s^i(x);\ x_{i-1} < x < x_i \tag{A.46a}$$

and

$$w^i(x) = l_{i,i+1}(x) + s^i(x);\ x_i < x < x_{i+1}. \tag{A.46b}$$

Then, there is a number $M > 0$ such that

$$\|s^i\|_\infty \leq Mh. \tag{A.47}$$

Proof. In the interval (x_{i-1}, x_i), the function s^i, fulfills

$$\hat{a}\frac{d^2 s^i}{dx^2} + \hat{b}\frac{ds^i}{dx} + \hat{c}s^i = -\frac{\hat{b}}{h} - \hat{c}l_{i,i-1} \tag{A.48}$$

and

$$s^i(x_{i-1}) = s^i(x_i) = 0. \tag{A.49}$$

Hence,

$$hs^i(x) = -\int_{x_{i-1}}^{x_i} \{\hat{b} + h\hat{c}l_{i,i-1}\} G^i(x,\xi)\, d\xi;\ x \in (x_{i-1}, x_i), \tag{A.50}$$

and, therefore,

$$h|s^i(x)| \leq M'\|\hat{b}\|_\infty h^2 + M''\|\hat{c}\|_\infty h^3;\ x \in (x_{i-1}, x_i), \tag{A.51}$$

where M' and M'' are suitable generic constants. Clearly, a similar relation holds in the interval (x_i, x_{i+1}), and Lemma 3 follows.

Lemma 4. *There is a number $M > 0$, independent of h, such that, for any given system of numbers q^i $(i = 1, \ldots, E-1)$, there exists a function $\epsilon_\Omega(x) \in H^0(0, l)$ such that, for each $i = 1, \ldots, E-1$, one has*

$$\int_0^l \epsilon_\Omega(x) w^i(x)\, dx = q^i \tag{A.52}$$

and

$$h\|\epsilon_\Omega\|_\infty \leq M \underbrace{\max}_{i} |q^i|. \tag{A.53}$$

Proof. Actually, it can be seen that, when such an $M > 0$ exists, then there are many functions belonging to $H^0(0,l)$ that satisfy Eq. (A.52) and the restriction (A.53). Thus, the Lemma is shown by exhibiting one such function. The following notation is used: for any pair of functions $p, s \in H^0(0,l)$, write

$$(p,s) = \int_0^l p(\xi)s(\xi)\,d\xi \tag{A.54a}$$

$$(p,s)_i = \int_{x_{i-1}}^{x_i} p(\xi)s(\xi)\,d\xi. \tag{A.54b}$$

Then, for each $i = 1,\ldots,E$, auxiliary functions $\tilde{w}^i(x) \in H^0(0,l)$ are introduced, which are defined for each $x \in (x_{i-1}, x_i)$ by

$$\tilde{w}^1(x) = w^1(x); \quad \tilde{w}^E(x) = 0, \tag{A.55}$$

and, when $i = 2,\ldots,E-1$, by

$$\tilde{w}^i(x) = w^i(x) + \rho^i w^{i-1}(x), \tag{A.56}$$

where

$$\rho^i = -\frac{(w^{i-1}, w^i)_i}{(w^{i-1}, w^{i-1})_i}. \tag{A.57}$$

In addition, for each $i = 1,\ldots,E, \tilde{w}^i(x)$ vanishes identically outside the interval (x_{i-1}, x_i). Thus, observe that the support of $\tilde{w}^i$ is (x_{i-1}, x_i), while that of w^i is (x_{i-1}, x_{i+1}). Also, that $(\tilde{w}^i, w^{i-1})_i = 0$.

Define

$$\epsilon_\Omega(x) = A^i\tilde{w}^i(x); \quad \text{where } A^i = \mu_i q^i \quad \text{and} \quad x_{i-1} < x < x_i, \tag{A.58a}$$

with

$$\mu_i = \frac{1}{(\tilde{w}^i, w^i)_i}. \tag{A.58b}$$

It is not difficult to verify that, with this definition, $\epsilon_\Omega(x)$ fulfills Eq. (A.17). Indeed, for each $i = 1,\ldots,E-1$,

$$\begin{aligned}\int_0^l \epsilon_\Omega(x)w^i(x)\,dx &= \sum_{j=1}^{E}(\epsilon_\Omega, w^i)_j = (\epsilon_\Omega, w^i)_i + (\epsilon_\Omega, w^i)_{i+1}\\ &= A^i(\tilde{w}^i, w^i)_i + A^{i+1}(\tilde{w}^{i+1}, w^i)_{i+1} = A^i(\tilde{w}^i, w^i)_i = q^i.\end{aligned} \tag{A.59}$$

In view of Eqs. (A.58), it is clear that

$$\|\epsilon_\Omega\|_\infty \le (\underbrace{\max}_{i}|\mu_i|)(\underbrace{\max}_{i}\|\tilde{w}^i\|)(\underbrace{\max}_{i}|q^i|). \tag{A.60}$$

Now, using Lemma 3, it can be seen that

$$\underbrace{\max}_{i}\|\tilde{w}^i\|_\infty \le 1 + O(h) \tag{A.61}$$

and

$$h\mu_i = 4 + O(h). \tag{A.62}$$

Hence,

$$h\|\epsilon_\Omega\| \leq \underbrace{\max}_{i} |q^i|\{4 + O(h)\}. \tag{A.63}$$

From which the lemma is clear.

Going back to Theorem A.1, it is implied by Lemmas 2 and 4, together, in a straightforward manner.

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