Applicable Analysis, Vol. xx, pp. 1–xx
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THE GAUSS THEOREM FOR DOMAIN DECOMPOSITIONS IN SOBOLEV SPACES

Communicated by X. XXXXXXXXXXX

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AMS: 35A15, 35G15, 65N55

Abstract This paper belongs to a broad line of research leaded by Herrera, which encompasses a good number of numerical methods such as Localized Adjoint Method (LAM), Eulerian-Lagrangian LAM (ELAM) and Trefftz-Herrera Method. The results presented in this paper are required in order to incorporate Herrera's general theory in a Sobolev-space setting. In particular, this article introduces a class of partitions (or domain decompositions) whose internal boundaries belong to a category of manifolds with corners, here also presented. Then a version of Gauss (or divergence) theorem, in a wider sense, is established and an explicit integral formula is associated for any given linear partial differential operator $\mathcal{L}$, its adjoint and concomitant. The structure of the bilinear concomitant induced by $\mathcal{L}$ is first determined. Then the required formula is given over that class of domain decompositions. Finally, an integral formula well on the way of the Green–Herrera formula is settled.

KEY WORDS: Linear differential operator, formal adjoint, concomitant, manifold with corners, Sobolev space, traces, domain decomposition, Gauss theorem, Green Formula.

(Received for Publication — — —)

1. Introduction

A usual strategy of solution, when a boundary value problem for a linear partial differential operator is posed, is first to consider a weak formulation of the problem. In this way, the modeler may reduce to some extent the differentiability of the spaces of trial and test functions, design approximation methods and enlarge the scope of physical applications.

A further step is taken when Green formulas are applied, because then powerful indirect variational principles come into action. The general formulation given by Lions and Magenes [13] is well known but, in spite its level of abstraction and generality, presents some limitations in applications to numerical methods (see the Appendix below). In particular, Herrera [3–5,8] has developed an approach in which, instead of splines, simultaneous use is made of fully discontinuous trial and test functions, increasing in this manner theoretical versatility, applicability and algorithmic resources.

However, in that situation some conditions required in Lions and Magenes formulation are violated (see Appendix and [13], pp. 114–115). On the other hand, the kind of Green formula introduced by Herrera [3,4,6,8,9,11] here referred as Green–Herrera, overcome this difficulty. Green–Herrera formulas exhibit explicitly the information – about the sought solution– contained in an approximate one. This yields the following interpretation of usual finite–element formulations: the test functions used determine the information about the sought solution contained in an approximate one, while the base functions interpolate (or extrapolate) that information. An strategy which is optimal [7], in some sense, is to obtain enough information to define well posed problems locally and then use as base functions, local solutions of the differential equation. This leads to a *domain decomposition* strategy (see [12]).

Although Herrera's theory is very appealing, its theoretical foundations had not been fully developed until recently. In particular, Green-Herrera formulas had not been formulated in a Sobolev-space setting, as it is standard in theoretical numerical analysis. This weakness, has been overcome in a paper due to appear soon [11], but a part of the theory presented there, uses results that to our knowledge have not been published previously, in spite of the fact that they have interest in themselves. Thus, the purpose of the present paper is two-fold: to give a rigorous proof of them, in order to complete the theoretical foundations of Herrera's approach, and to make them available to other scientists working in this area of research.

In the present work we begin the setting of a general framework, abstract enough to be sufficiently encompassing but, at the same time, very concrete and algorithmic. It starts by giving an explicit formula, expressed as a divergence, for the bilinear concomitant of an arbitrary linear operator. As a by product, the non-uniqueness of the Green formula offered in [13] is clarified and the common prejudice that "integration by parts is needed" is erradicated.

The category of *manifolds with corners* is geometrically attractive, simple to grasp and suitable for most applications. Therefore, a general definition of a domain decomposition sitting on this concept is very convenient and has clear advantages when compared with the smooth and Lipschitz categories of [13] and Marti [15] respectively.

Traces of functions in Sobolev spaces over manifolds with corners are briefly discussed. Then, a Gauss theorem over domain decompositions is stated. Finally, an integral formula well on the way of a general Green formula in the sense of the algebraic theory of Herrera [3–6,8,9,11] is settled.

2. Multi-indices, strings and operators

In any discussion of functions of n variables an ordered n-tuple $t = (t_1, t_2, \ldots, t_n)$ of nonnegative integers is known as a *multi-index* of order $|t| = t_1 + t_2 + \cdots + t_n$ and is usually associated with the differential operator

$$D^t = {D_1}^{t_1} {D_2}^{t_2} \cdots {D_n}^{t_n} = \left(\frac{\partial}{\partial x_1}\right)^{t_1} \left(\frac{\partial}{\partial x_2}\right)^{t_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{t_n}$$

Suppose Ω is open in $\mathbb{R}^n$, N a positive integer, $f_t \in C^\infty(\Omega)$ for every multi-index t of order less or equal N, and at least one f_t with $|t| = N$ not identically zero. Then

$$\mathcal{L} = \sum_{|t| \leq N} f_t \cdot D^t$$

is a linear differential operator with smooth coefficients of order N.

Let K be a positive integer and let $\underline{\alpha} = \alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_K = (\alpha_1, \alpha_2, \ldots, \alpha_K)$ be a K-tuple of positive integers not exceeding n. Call $\underline{\alpha}$ a *string of length* $\lambda(\underline{\alpha}) = K$ (in the alphabet $\{1, 2, \ldots, n\}$).

If $1 \leq \beta \leq n$ is an integer, define $\underline{\alpha} \diamond \beta$ to be $\alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_K \diamond \beta$. Let the empty set $\emptyset$ be a string; actually the only with length equal to nought, and define $\emptyset \diamond \beta = \beta$.

As with multi-indices, for every string $\underline{\alpha}$ there is a differential operator associated with it, namely

$$\mathcal{L}_{\underline{\alpha}} = \begin{cases} Id = identity & \text{if } \underline{\alpha} = \emptyset, \\ D_{\alpha_1} D_{\alpha_2} \cdots D_{\alpha_K} & \text{otherwise} \end{cases}$$

Recursively, we are saying that $\mathcal{L}_\emptyset = Id$ and $\mathcal{L}_{\underline{\alpha}\diamond\beta} = \mathcal{L}_{\underline{\alpha}} D_\beta$.

As an illustration, consider the string $\underline{\alpha} = 2 \diamond 3 \diamond 4 \diamond 3 \diamond 2 \diamond 2$ and the corresponding multi-index $t = (0, 3, 2, 1)$. Now note that they produce the same operator (here $n = 4$). Obviously, any permutation of $\underline{\alpha}$ does not change the corresponding operator (nor the associated multi-index). That is, $\underline{\alpha}$ represents not only an operator, but also a determined order in which partial derivatives ought to be computed. This remark will prove to be of consequence in Corollary 10. In what follows, it will be important to distinguish well strings from multi-indices.

3. Definitions

Let $\mathcal{D}(\Omega)$ be the space of all compactly supported functions in $C^\infty(\Omega)$ endowed with the (real) scalar product $\prec u, w \succ = \int_\Omega u \cdot w d\mathbf{x}$ and let $\mathcal{L}$ be a linear differential operator on Ω as before. By restricting its action, $\mathcal{L}$ may be thought as an operator of $\mathcal{D}(\Omega)$ in itself. Since $\mathcal{D}(\Omega)$ is not a Hilbert space and $\mathcal{L}$ is not continuous under the scalar product norm (unless $\mathcal{L} \equiv 0$) , it is not obvious that there should exist a linear transformation $\mathcal{L}^*$ on $\mathcal{D}(\Omega)$ such that $\prec \mathcal{L}u, w \succ = \prec u, \mathcal{L}^* w \succ \;\; \forall\, u, w \in \mathcal{D}(\Omega)$. Nevertheless, it is a simple and well known fact that $\mathcal{L}^*$ always exists and that it is actually a differential operator. Of crucial importance for us here is also the correspondence

$$(u, w) \longmapsto w\mathcal{L}u - u\mathcal{L}^* w \qquad \forall\, u, w \in \mathcal{D}(\Omega)$$

which defines a bilinear differential operator $\mathfrak{F}[\mathcal{L}] = \mathfrak{F} : \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \longrightarrow \mathcal{D}(\Omega)$ sometimes refered to as the *bilinear concomitant of* $\mathcal{L}$.

The following paragraphs exhibit the structure of $\mathfrak{F}$ and, as a by-product, the existence and structure of the so-called *formal adjoint* $\mathcal{L}^*$ as well.

It is important to keep in mind that much of the power and flexibility of the theory, not to say its ability to circumvent difficulties, relies on the multiple domains over which $\mathcal{L}$ can effectively act. By way of example, we just saw how $\mathcal{D}(\Omega)$ proved to be an effective choice in order to make the concepts of formal adjoint and concomitant meaningful.

4. Lemma

Let $\mathcal{L}$ be a differential operator with formal adjoint $\mathcal{L}^*$ and concomitant $\mathfrak{F}[\mathcal{L}]$ so

$$w\mathcal{L}u = u\mathcal{L}^* w + \mathfrak{F}[\mathcal{L}](u, w) \qquad \forall\, u, w \in \mathcal{D}(\Omega) \qquad (a)$$

Then, for any nonnegative integer β,

$$[\mathcal{L}D_\beta]^* = -D_\beta \mathcal{L}^*$$

and

$$\mathfrak{F}[\mathcal{L}D_\beta](u, w) = \mathfrak{F}[\mathcal{L}](D_\beta\, u, w) + D_\beta(u\mathcal{L}^* w) \qquad (b)$$

Proof. By Leibniz's rule we can write

$$\begin{aligned} w\mathcal{L}D_\beta\, u &= (D_\beta\, u)\mathcal{L}^* w + \mathfrak{F}[\mathcal{L}](D_\beta\, u, w) \\ &= -uD_\beta \mathcal{L}^* w + D_\beta(u\mathcal{L}^* w) + \mathfrak{F}[\mathcal{L}](D_\beta\, u, w) \end{aligned}$$

5. Remark

The rules $(\mathcal{L}^*)^* = \mathcal{L}$, $(\mathcal{L}f)^* = f\mathcal{L}^*$ (where $[\mathcal{L}f](u)$ means $\mathcal{L}(fu)$), $(f\mathcal{L})^* = \mathcal{L}^*f$, $(\mathcal{L}_1 + \mathcal{L}_2)^* = {\mathcal{L}_1}^* + {\mathcal{L}_2}^*$, and $(\mathcal{L}_1 \circ \mathcal{L}_2)^* = {\mathcal{L}_2}^* \circ {\mathcal{L}_1}^*$ (operator composition) are easily derived (via formula (4.a)) and are the basis of a simple calculus of formal adjoints. Actually, the first part of the previous lemma is an elementary instance of the operator composition rule.

There is also a formal calculus of concomitants: namely, $\mathfrak{F}[\mathcal{L}^*](u,w) = -\mathfrak{F}[\mathcal{L}](w,u)$, $\mathfrak{F}[f\mathcal{L}](u,w) = \mathfrak{F}[\mathcal{L}](u,fw)$, $\mathfrak{F}[\mathcal{L}_1+\mathcal{L}_2] = \mathfrak{F}[\mathcal{L}_1]+\mathfrak{F}[\mathcal{L}_2]$, $\mathfrak{F}[\mathcal{L}_1\circ\mathcal{L}_2](u,w) = \mathfrak{F}[\mathcal{L}_1](\mathcal{L}_2u,w)+\mathfrak{F}[\mathcal{L}_2](u,\mathcal{L}_1^*w)$ (cf. (4.b)).

6. Examples

a) If $\mathcal{L} = \mathcal{L}_\emptyset = Id$, then $\mathcal{L}^* = \mathcal{L}$ and $\mathfrak{F} = 0$.

b) If $\mathcal{L} = \mathcal{L}_\beta = D_\beta$, then $\mathcal{L}^* = -\mathcal{L}$ and $\mathfrak{F}(u,w) = D_\beta(uw)$. This follows from Lemma 4 applied to the example above.

c) By Lemma 4, if $\underline{\alpha} = \alpha_1 \diamond \alpha_2$ and $\mathcal{L} = \mathcal{L}_{\underline{\alpha}} = \mathcal{L}_{\alpha_1}D_{\alpha_2}$, then $\mathcal{L}^* = (-1)^2\mathcal{L} = \mathcal{L}$ and $\mathfrak{F}(u,w) = D_{\alpha_1}[(\mathcal{L}_{\alpha_2}u)w] - D_{\alpha_2}[u\mathcal{L}_{\alpha_1}w]$.

d) An induction, over the order of a multi-index t and using example (b) as induction basis, shows that, if $\mathcal{L} = D^t$ then $\mathcal{L}^* = (-1)^{|t|}D^t$.

e) Finally, let $\mathcal{L} = \sum_{|t|\le N} f_t \cdot D^t$. The calculus sketched in Remark 5 in addition to example (d) imply

$$\mathcal{L}^* = \sum_{|t|\le N} (f_t \cdot D^t)^* = \sum_{|t|\le N} (D^t)^* \cdot f_t = \sum_{|t|\le N} (-1)^{|t|} D^t \cdot f_t$$

Now, in order to phrase a general result on the structure of $\mathfrak{F}[\mathcal{L}]$, we need to define what is meant for *initial* and *final segments* of a given string.

7. Definitions

Let $\underline{\alpha} = \alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_K$ be a string of length K, and let $k \in \{1,2,\ldots,K\}$. Define

$$A(k) = A(\underline{\alpha},k) = \begin{cases} \emptyset & \text{if } k = 1, \\ \alpha_1 \diamond \alpha_2 \diamond \cdots \diamond \alpha_{k-1} & \text{if } k > 1 \end{cases}$$

$$Z(k) = Z(\underline{\alpha},k) = \begin{cases} \alpha_{k+1} \diamond \alpha_{k+2} \diamond \cdots \diamond \alpha_K & \text{if } k < K, \\ \emptyset & \text{if } k = K \end{cases}$$

8. Proposition

Let $\underline{\alpha}$ be a string of length K. Then

$$\mathcal{L}_{\underline{\alpha}}{}^* = (-1)^K \mathcal{L}_{\underline{\alpha}} \qquad \text{(cf. Example (6.d))}$$

$$\mathfrak{F}_{\underline{\alpha}}(u,w) = \sum_{k=1}^{K} (-1)^{k+1} D_{\alpha_k}\{(\mathcal{L}_{Z(k)}\ u)(\mathcal{L}_{A(k)}\ w)\}$$

Proof. The proof is by induction and its basis is contained in Example (6.b). So let us suppose the result is true for every string of length K and let $\underline{\alpha} \diamond \alpha_{K+1}$ be a typical string of length $K+1$.

Since $\mathcal{L}_{\underline{\alpha}\diamond\alpha_{K+1}} = \mathcal{L}_{\underline{\alpha}} D_{\alpha_{K+1}}$ and due to Lemma 4, the induction hypothesis and the commutativity of partial derivation, we have that

$$\begin{aligned}
\mathcal{L}^*_{\underline{\alpha}\diamond\alpha_{K+1}} &= (\mathcal{L}_{\underline{\alpha}} D_{\alpha_{K+1}})^* \;=\; -D_{\alpha_{K+1}}\mathcal{L}^*_{\underline{\alpha}} \\
&= -D_{\alpha_{K+1}}(-1)^K\mathcal{L}_{\underline{\alpha}} = \;(-1)^{K+1}\mathcal{L}_{\underline{\alpha}} D_{\alpha_{K+1}} \\
&= (-1)^{K+1}\mathcal{L}^*_{\underline{\alpha}\diamond\alpha_{K+1}}
\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{F}_{\underline{\alpha}\diamond\alpha_{K+1}}(u,w) &= \mathfrak{F}_{\underline{\alpha}}(D_{\alpha_{K+1}}u,w) + D_{\alpha_{K+1}}(u\mathcal{L}^*_{\underline{\alpha}}w) \\
&= \sum_{k=1}^{K}(-1)^{k+1}D_{\alpha_k}\big\{(\mathcal{L}_{Z(\underline{\alpha},k)}D_{\alpha_{K+1}}\ u)(\mathcal{L}_{A(\underline{\alpha},k)}\ w)\big\} + \\
&+ (-1)^K D_{\alpha_{K+1}}(u\mathcal{L}_{\underline{\alpha}}\ w) \\
&= \sum_{k=1}^{K}(-1)^{k+1}D_{\alpha_k}\big\{(\mathcal{L}_{Z(\underline{\alpha}\diamond\alpha_{K+1},k)}\ u)(\mathcal{L}_{A(\underline{\alpha}\diamond\alpha_{K+1},k)}\ w)\big\} + \\
&+ (-1)^{K+2}D_{\alpha K+1}\big\{(\mathcal{L}_{Z(\underline{\alpha}\diamond\alpha_{K+1},K+1)}\ u)(\mathcal{L}_{A(\underline{\alpha}\diamond\alpha_{K+1},K+1)}\ w)\big\} \\
&= \sum_{k=1}^{K+1}(-1)^{k+1}D_{\alpha_k}\big\{(\mathcal{L}_{Z(\underline{\alpha}\diamond\alpha_{K+1},k)}\ u)(\mathcal{L}_{A(\underline{\alpha}\diamond\alpha_{K+1},k)}\ w)\big\}
\end{aligned}$$

9. Example

Let $\underline{\alpha} = i\diamond j\diamond k\diamond l$. Then

$$\begin{aligned}
\mathfrak{F}_{\underline{\alpha}}(u,w) \;&= D_i\big((D_jD_kD_lu)w\big) - D_j\big((D_kD_lu)D_iw\big) + \\
&+ D_k\big((D_lu)D_iD_jw\big) - D_l\big(uD_iD_jD_kw\big)
\end{aligned}$$

This clearly shows that the actual expression for $\mathfrak{F}_{\underline{\alpha}}$ depends, in most cases, on the order in which partial derivatives are to be computed.

The next corollary exhibits a non-trivial consequence of this seemingly formal trifle.

10. Corollary

Given a string $\underline{\alpha}$ in the alphabet $\{1,2,\ldots,n\}$, there exists a uniquely determined vector field, say

$$\underline{\mathfrak{D}}_{\underline{\alpha}}(u,w) = \big(\mathfrak{D}_{\underline{\alpha},1}(u,w), \mathfrak{D}_{\underline{\alpha},2}(u,w),\ldots,\mathfrak{D}_{\underline{\alpha},n}(u,w)\big),$$

such that

$$\mathfrak{F}_{\underline{\alpha}}(u,w) \;=\; \nabla\bullet\underline{\mathfrak{D}}_{\underline{\alpha}}(u,w)$$

Proof. Define, for each $1\le j\le n$, $\underline{\alpha}^{-1}(j) = \{1\le k\le K \mid \alpha_k = j\}$. Then, the subsets $\underline{\alpha}^{-1}(j)$ are a disjoint collection whose union is $\{1,2,\ldots,K\}$. Therefore,

$$\begin{aligned}
\mathfrak{F}_{\underline{\alpha}}(u,w) &= \sum_{j=1}^{n}\sum_{k\in\underline{\alpha}^{-1}(j)}(-1)^{k+1}D_j\big\{(\mathcal{L}_{Z(\underline{\alpha},k)}\ u)(\mathcal{L}_{A(\underline{\alpha},k)}\ w)\big\} \\
&= \sum_{j=1}^{n}D_j\sum_{k\in\underline{\alpha}^{-1}(j)}(-1)^{k+1}\big\{(\mathcal{L}_{Z(\underline{\alpha},k)}\ u)(\mathcal{L}_{A(\underline{\alpha},k)}\ w)\big\}
\end{aligned}$$

Hence the result follows by putting

$$\mathfrak{D}_{\underline{\alpha},j}(u,w) = \sum_{k\in\underline{\alpha}^{-1}(j)} (-1)^{k+1}\big\{(\mathcal{L}_{Z(\underline{\alpha},k)}\ u)(\mathcal{L}_{A(\underline{\alpha},k)}\ w)\big\}$$

for all $j = 1, 2, \ldots, n$, where $\mathfrak{D}_{\underline{\alpha},j} \equiv 0$ if $\underline{\alpha}^{-1}(j) = \emptyset$.

11. Example

If $\underline{\beta}$ is a permutation of a string $\underline{\alpha}$ then it is true that $\mathfrak{F}_{\underline{\alpha}} = \mathfrak{F}_{\underline{\beta}}$ as operators but, in most cases, $\underline{\mathfrak{D}}_{\underline{\alpha}}$ and $\underline{\mathfrak{D}}_{\underline{\beta}}$ turn out to be different fields. For example, let $\underline{\alpha} = 3 \diamond 1 \diamond 3 \diamond 4$ and $n = 4$. Then

$$\mathfrak{D}_{\underline{\alpha}}(u,w) = \big(-(D_3D_4u)D_3w, 0, (D_1D_3D_4u)w + (D_4u)D_3D_1w, -uD_3D_1D_3w\big)$$

If $\underline{\beta} = 4 \diamond 3 \diamond 3 \diamond 1$, then

$$\mathfrak{D}_{\underline{\beta}}(u,w) = \big(-uD_4D_3D_3w, 0, (D_1u)D_4D_3w - (D_3D_1u)D_4w, (D_3D_3D_1u)w\big)$$

12. Theorem

Let $\mathcal{L} = \sum_{\lambda(\underline{\alpha})\leq K} f_{\underline{\alpha}} \cdot \mathcal{L}_{\underline{\alpha}}$ be a differential operator. Here, the summation runs over a subset of strings (not multi-indices as is customary) whose length does not exceed K. Then we have the explicit formula

$$w\mathcal{L}u - u\mathcal{L}^*w = \nabla \bullet \underline{\mathfrak{D}}(u,w) \qquad \forall\ u, w \in C^\infty(\Omega)$$

where

$$\mathcal{L}^* = \sum_{\underline{\alpha}} (-1)^{\lambda(\underline{\alpha})} \mathcal{L}_{\underline{\alpha}} f_{\underline{\alpha}} \qquad \text{(cf. Example (6.e))}$$

and

$$\underline{\mathfrak{D}}(u,w) = \sum_{\underline{\alpha}} \underline{\mathfrak{D}}_{\underline{\alpha}}(u, f_{\underline{\alpha}}\ w)$$

Observe that, if $\mathcal{L} = \Delta$ is the Laplacian, the theorem gives Green's second identity before integration.

13. Manifolds with corners

In order to state a divergence theorem and other integral formulas in the wider sense, it is necessary to go through the construction of traces and jumps in Sobolev spaces over manifolds with corners. So let us start our discussion with this particular class of manifolds.

Let Ω be an open set in $\mathbb{R}^n$. We say that Ω is a *manifold with corners* or a *domain with almost regular boundary* (see L. Loomis and S. Sternberg [14]) if, for every $\mathbf{x} \in \mathbb{R}^n$, there is an open set U containing $\mathbf{x}$ and a diffeomorphism (called a *chart* or *coordinate system*) $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ such that one of the following possibilities holds:

a) $U \cap \Omega = \emptyset$,

b) $U \subset \Omega$,

c) $\varphi(U \cap \Omega) = \varphi(U) \cap \{\mathbf{y} \in \mathbb{R}^n \mid y_k \geq 0, y_{k+1} \geq 0, \ldots, y_n \geq 0\}$ for some $1 \leq k \leq n$ depending on $\mathbf{x}$.

Fig. 1

The definition does not cover all cases that can arise. Neither the plane cusp of Figure (2.a) nor the quadrilateral of Figure (2.b) qualify as manifolds with corners. However, the quadrilateral can be expressed as the union of two triangles for which the definition is valid (consider linear coordinate systems φ that convert a typical angle $< \pi$ into a right one).

(a) No φ can open up this tangential cusp

(b) No φ can convert this angle $> \pi$ into a right angle

Fig. 2

Denote the boundary of Ω by $\partial\Omega$ and let $\widetilde{\partial\Omega}$ be the set of boundary points for which condition (c) holds with $k = n$. These are called the *regular boundary* points of Ω. It is easy to see that $\widetilde{\partial\Omega}$ is a smooth manifold of dimension $n-1$ which is dense in $\partial\Omega$ and of complement of null $(n-1)$-dimensional measure. Let $\overline{\Omega} = \Omega \cup \partial\Omega$ be the topological closure of Ω in $\mathbb{R}^n$.

The manifolds

$$\mathbb{E}^n_k = \{\mathbf{y} \in \mathbb{R}^n \mid y_k > 0, y_{k+1} > 0, \ldots, y_n > 0\} \qquad (1 \leq k \leq n)$$

are the standard models for corners and, from them, all other manifolds with corners are locally modeled.

If j is such that $k \leq j \leq n$ then

$$\partial_j \mathbb{E}_k^n = \left\{ \mathbf{y} \in \overline{\mathbb{E}}_k^n \mid y_j = 0 \right\}$$

is called the *j-th face of the k-th standard corner.*

Fig. 3 $\partial \mathbb{E}_k^3$ is shaded.

14. Sobolev Spaces for nonnegative integers

Let m be a nonnegative integer and G an arbitrary open subset in $\mathbb{R}^n$. Let the *Sobolev norm* be given by

$$\|f\|_m = \left(\sum_{|t| \leq m} \int_\Omega \mid D^t f \mid^2 \, d\mathbf{x} \right)^{1/2} \qquad \forall f \in C^\infty(G)$$

(the sum running over multi-indices.)

The *Sobolev space* $\mathbb{H}^m(G)$ is then the completion of the pre-Hilbert space

$$\left\{ f \in C^\infty(G) \;\middle|\; \|f\|_m < \infty \right\}$$

For $m = 0$, $\mathbb{H}^0(G)$ is (isomorphic to) $\mathbf{L}^2(G)$, the space of square integrable functions.

In what follows, every result not proven here is standard, and can be found in at least one of R. Adams [1], J.L. Lions and E. Magenes [13], J.T. Marti [15], M. Reed and B. Simon [17], or W.Rudin [18]. The chosen construccions specified below make the sequel of assertions straightforward to prove.

Let $\mathcal{L} : V_1 \to V_2$ be a bounded linear transformation between normed spaces. Then $\mathcal{L}$ can be uniquely extended to a bounded linear transformation with the same bound, from the completion of V_1 to the completion of V_2. The same symbol $\mathcal{L}$ is used for the extension.

15. Fourier Transforms

Let s be an arbitrary real number and let $C_0^\infty(\mathbb{R}^n)$ be the space of smooth functions with compact support. Define the norm

$$\|f\|_{F,s} = \left[\int_{\mathbb{R}^n} (1+|\mathbf{y}|^2)^s \; |\mathcal{F}f(\mathbf{y})|^2 \; d\mathbf{y}\right]^{1/2} \qquad \forall\; f \in C_0^\infty(\mathbb{R}^n)$$

where

$$\mathcal{F}f(\mathbf{y}) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} exp(-i\mathbf{x}\bullet\mathbf{y})f(\mathbf{x})d\mathbf{x} \qquad \forall\;\mathbf{y}\in\mathbb{R}^n$$

is the Fourier transform of the compactly supported function f. The completion of $C_0^\infty(\mathbb{R}^n)$ under $\|f\|_{F,s}$ is $\mathbb{H}^s(\mathbb{R}^n)$. When s is equal to a nonnegative integer m the norm just defined is equivalent to the Sobolev norm defined in (14) and the resulting spaces are isomorphic. Hence there is no ambiguity in using the same notation for both constructions.

It is desirable, when possible, to exhibit "concrete" realizations of this abstract process of completion. In the present situation, first observe that, for $s_0 < s_1$, we have the (continuous) inclusion $\mathbb{H}^{s_1}(\mathbb{R}^n) \hookrightarrow \mathbb{H}^{s_0}(\mathbb{R}^n)$. Hence, for $s > 0$, $\mathbb{H}^s(\mathbb{R}^n)$ is realized as a "concrete" subspace of (square integrable) functions. On the other hand, for $s < 0$, it is necessary to go into the realm of the theory of distributions to exhibit a definite description of $\mathbb{H}^s(\mathbb{R}^n)$ as a subspace of tempered distributions.

From now on, we shall restrict ourselves to the case of a nonnegative, but otherwise arbitrary, parameter s. The resulting theory is somewhat simpler and sufficient to our purposes.

16. Sobolev spaces for a nonnegative real

Let $s \geq 0$ and let G be an arbitrary open set in $\mathbb{R}^n$. Define $\mathbb{H}^s(G)$ as the space of restrictions of the functions in $\mathbb{H}^s(\mathbb{R}^n)$ to G, and let

$$\|f\|_s = Inf\left\{\|\tilde{f}\|_{F,s} \;\middle|\; \tilde{f}\,|_G = \; f \;,\; \tilde{f}\in\mathbb{H}^s(\mathbb{R}^n)\right\}$$

This norm amounts to providing $\mathbb{H}^s(G)$ with the norm of the quotient of $\mathbb{H}^s(\mathbb{R}^n)$ by the closed subspace of functions vanishing almost everywhere on G. Therefore $\mathbb{H}^s(G)$ is already complete.

17. Sobolev spaces for boundaries

Let Ω be a relatively compact manifold with corners. Before proceeding into the intended definition, we want to make two observations:

(a) As with any arbitrary open set, the space

$$R_0(\Omega) = \left\{f:\Omega\to\mathbb{R} \;\middle|\; f \text{ is the restriction } \tilde{f}\,|_\Omega \text{ of a function } \tilde{f}\in C_0^\infty(\mathbb{R}^n)\right\}$$

is dense in $\mathbb{H}^s(\Omega)$.

(b) Let

$$\left\{\; (\varphi_i, U_i) \;\middle|\; \varphi_i : U_i \to \mathbb{R}^n \supset \mathbb{E}^n_{k(i)} \;\right\}_{i=1}^{\sharp}$$

be a finite number of charts (of type (13.c)) covering $\partial\Omega$ and add $(\varphi_0 = Id\ ,\ U_0 = \Omega)$ as a valid chart to complete a covering of $\overline{\Omega}$.

Fig. 4

Let

$$\left\{ \lambda_i : U_i \to [0,1] \right\}_{i=0}^{\sharp}$$

be a partition of unity subordinated to $\{U_i\}_{i=0}^{\sharp}$. Then $f \in \mathbb{H}^s(\Omega)$ if and only if $(\lambda_i f) \circ \varphi_i^{-1} \in \mathbb{H}^s(\mathbb{E}^n_{k(i)})\ \ \forall i = 1, 2, \ldots \sharp$.

These observations suggest the following construction: Let $R_0(\partial\Omega)$ be the set of elements of $C_0^\infty(\mathbb{R}^n)$ restricted to $\partial\Omega$. Define, for $s > 0$,

$$\|g\|_s = \left(\sum_{i=1}^{\sharp} \sum_{j=k(i)}^{n} \left\| \ (\lambda_i g) \circ \varphi^{-1} \Big|_{\partial_j \mathbb{E}^n_{k(i)}} \right\|_s^2 \right)^{1/2} \qquad \forall\ g \in R_0(\partial\Omega)$$

Define $\mathbb{H}^s(\partial\Omega)$ as the completion of $R_0(\partial\Omega)$ under this norm.

It is not difficult to convince oneself that this metric is independent, up to equivalence of Banach spaces, of the chosen covering $\{U_i\}_{i=1}^{\sharp}$ and of the partition of unity.

Finally, for any not necessarily relatively compact manifold Ω, define the following important restriction mapping or *trace operator*

$$\mathbf{T} : R_0(\Omega) \to R_0(\partial\Omega)$$

such that, for every $f \in R_0(\Omega)$ and any smooth extension $\tilde{f}$ of f,

$$\mathbf{T}f(\mathbf{x}) = \tilde{f}(\mathbf{x}) \qquad \forall\ \mathbf{x} \in \partial\Omega$$

18. Lemma

Let $\mathbb{E}^n_n$ be the n-th dimensional standard n-th corner, so $\partial\mathbb{E}^n_n$ is just $\mathbb{R}^{(n-1)}$, and let $s > 1/2$. Then, there is a constant $K > 0$ such that

$$\|\mathbf{T}f\|_{s-1/2} \leq K\|f\|_s \qquad \forall\ f \in R_0(\mathbb{E}^n_n)$$

19. Theorem

Let Ω be a relatively compact manifold with corners. Then, the trace operator $\mathbf{T}$ can be extended continuously to

$$\mathbf{T} : \mathbb{H}^s(\Omega) \longrightarrow \mathbb{H}^{s-1/2}(\partial\Omega)$$

Proof. By Lemma 18, $\mathbf{T} : R_0(\Omega) \to R_0(\partial\Omega)$ is continuous locally for each face. Then the discussion in (17) makes it clear that $\mathbf{T}$ is continuous globally. By density, it then follows that $\mathbf{T}$ can be extended meaningfully to the desired Sobolev spaces.

20. A generalized Gauss theorem over domain decompositions

Let $\mathfrak{F}$ be a smooth vector field defined on $\overline{\Omega}$. Then the divergence theorem, valid for compact manifolds with corners, asserts that

$$\int_\Omega \nabla \bullet \mathfrak{F}\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{T}\mathfrak{F} \bullet \mathbf{n}\, d\mathbf{S}$$

where, of course, $\mathbf{n}$ is only defined over $\widetilde{\partial\Omega}$. Now, by the process of completion and extension sketched in (14) and Theorem 19, the above relation remains valid for all vector fields $\mathfrak{F} \in \mathbb{H}^1(\Omega)^n$.

Let $\{\Omega_1, \Omega_2, \ldots, \Omega_E\}$ be a *domain decomposition* of Ω; that is, a pairwise disjoint family of manifolds with corners such that the union of its closures is $\Omega \cup \partial\Omega$.

Fig. 5

Define the *inner boundary* Σ to be the closed complement of $\partial\Omega$ in $\bigcup_i \partial\Omega_i$.

Define the *inner regular boundary* to be $\widetilde{\Sigma} = \left(\bigcup_{i\neq j} \widetilde{\partial\Omega_i} \cap \widetilde{\partial\Omega_j}\right)$ and observe that each of its points belong to exactly two of the $\widetilde{\partial\Omega_i}$'s. By compactness, $\widetilde{\Sigma}$ consists of a finite number of connected components, all manifolds of dimension $n-1$, which are to be called the *inner faces (of the domain decomposition)*. The set of inner faces is denoted by $\lfloor\widetilde{\Sigma}\rfloor$.

Similarly, the set of regular points in $\partial\Omega$ not in Σ is an $(n-1)$-dimensional submanifold of $\widetilde{\partial\Omega}$ with finitely many connected components. The set of these components is denoted by $\lfloor \widetilde{\partial\Omega} \rfloor$ and may be called the collection of *regular outer faces.* Note that, in general, an outer face may be partitioned by several regular outer faces (see Figure 5).

A smooth choice of a unit normal vector field on $\widetilde{\Sigma}$, denoted again by $\mathbf{n}$, makes precise the idea of the *positive* and the *negative sides* for inner faces in much the same way that the unit normal vector over $\partial\Omega$ captures the idea of *inside and outside.*

With these preliminaries and the aid of the trace operator, now it is possible to define as well the *"field of jumps* $[\cdot]$ *of* $\mathfrak{F}$ *over* Σ*"* in order to get the relation

$$\int_{\Omega} \nabla \bullet \mathfrak{F}\, d\mathbf{x} \;=\; \int_{\partial\Omega} \mathbf{T}\mathfrak{F} \bullet \mathbf{n}\, d\mathbf{S} \;-\; \int_{\Sigma} [\mathfrak{F}] \bullet \mathbf{n}\, d\mathbf{S}$$

valid for all $\mathfrak{F} = \oplus_i \mathfrak{F}_i \in \mathcal{D}^n = \bigoplus_i \mathbb{H}^1(\Omega_i)^n$, where $\mathcal{D} = \mathbb{H}^1(\Omega_1) \oplus \mathbb{H}^1(\Omega_2) \oplus \cdots \oplus \mathbb{H}^1(\Omega_E)$ (cf. Herrera [2]).

The following discussion makes precise the analytic and algebraic details behind the above formula.

21. Some calculations

Let $\mathfrak{F} \in \mathcal{D}^n$. Then $\mathfrak{F}$ may be thought as a family $\oplus_i \mathfrak{F}_i$ of vector fields. This means that there is one vector field, say $\mathfrak{F}_i = (\mathfrak{F}_{i,1}, \mathfrak{F}_{i,2}, \cdots, \mathfrak{F}_{i,n})$ defined on Ω_i for each $i = 1, 2, \ldots, E$. So let $\nabla \bullet \mathfrak{F} = \bigoplus_i \nabla \bullet \mathfrak{F}_i$. Since there is a natural inclusion $\bigoplus_i \mathbf{L}^2(\Omega_i) \hookrightarrow \mathbf{L}^2(\Omega)$ such that $\oplus_i g_i \;\mapsto \sum_i \mathcal{X}_{\Omega_i} g_i$, where $\mathcal{X}_{\Omega_i}$ denotes the characteristic function of Ω_i in Ω, then $\bigoplus_i \nabla \bullet \mathfrak{F}_i$ belongs to $\mathbf{L}^2(\Omega)$. Now let $\mathbf{T}\mathfrak{F}_i = (\mathbf{T}\mathfrak{F}_{i,1}, \mathbf{T}\mathfrak{F}_{i,2}, \cdots, \mathbf{T}\mathfrak{F}_{i,n}) \in \mathbb{H}^{1/2}(\partial\Omega_i)^n$.

Define the following combinatorial mappings:

$$\begin{aligned} &\iota \colon \lfloor \widetilde{\partial\Omega} \rfloor \longrightarrow \{1, 2, \ldots, E\} \;\; \text{with} \\ &\iota(F) \;=\; \text{index } i \text{ such that } F \text{ is a regular outer face of } \Omega_i \end{aligned}$$

$$\begin{aligned} &\iota_+ \colon \lfloor \widetilde{\Sigma} \rfloor \longrightarrow \{1, 2, \ldots, E\} \;\; \text{with} \\ &\iota_+(F) \;=\; \text{index } i \text{ such that } \Omega_i \text{ lies on the positive side of } F \end{aligned}$$

$$\begin{aligned} &\iota_- \colon \lfloor \widetilde{\Sigma} \rfloor \longrightarrow \{1, 2, \ldots, E\} \;\; \text{with} \\ &\iota_-(F) \;=\; \text{index } i \text{ such that } \Omega_i \text{ lies on the negative side of } F \end{aligned}$$

Define the boundary traces,

$$\begin{aligned} &\mathbf{T}\mathfrak{F} \in \mathbf{L}^2(\partial\Omega)^n \quad \text{such that} \\ &\mathbf{T}\mathfrak{F}\Big|_F \;=\; \mathbf{T}\mathfrak{F}_{\iota(F)}\Big|_F \qquad \forall\, F \in \lfloor \widetilde{\partial\Omega} \rfloor \end{aligned}$$

and analogous traces for the inner boundary,

$$\begin{aligned} &\mathbf{T}_+\mathfrak{F}\;,\;\mathbf{T}_-\mathfrak{F} \in \mathbf{L}^2(\Sigma)^n \quad \text{such that} \\ &\mathbf{T}_+\mathfrak{F}\Big|_F \;=\; \mathbf{T}\mathfrak{F}_{\iota_+(F)}\Big|_F \\ &\mathbf{T}_-\mathfrak{F}\Big|_F \;=\; \mathbf{T}\mathfrak{F}_{\iota_-(F)}\Big|_F \qquad \forall\, F \in \lfloor \widetilde{\Sigma} \rfloor \end{aligned}$$

Finally, define the field of jumps as

$$[\mathfrak{F}] \;=\; \mathbf{T}_+\mathfrak{F} \;-\; \mathbf{T}_-\mathfrak{F}$$

With these notions, now the relation at the end of (20) is easy to prove.

22. An integral formula for $\mathcal{L}$

Let $\{ f_{\underline{\alpha}} \}_{\underline{\alpha}} \subset R_0(\Omega)$ be the coefficients of a linear differential operator $\mathcal{L}$ (recall that the $\underline{\alpha}'s$ are in a subset of strings of length less or equal K). Applying the above result to the formula of Theorem 12 we arrive to the expression

$$\int_\Omega \{w\mathcal{L}u - u\mathcal{L}^*w\}\, d\mathbf{x} \;=\; \int_{\partial\Omega} \mathbf{T}\underline{\mathfrak{D}}(u,w) \bullet \mathbf{n}\, d\mathbf{S} \;-\; \int_\Sigma [\underline{\mathfrak{D}}(u,w)] \bullet \mathbf{n}\, d\mathbf{S}$$

valid for all $(u,v) \in \mathcal{D} \times \mathcal{D}$, where $\mathcal{D} = \mathbb{H}^s(\Omega_1) \oplus \mathbb{H}^s(\Omega_2) \oplus \cdots \oplus \mathbb{H}^s(\Omega_E)$ and s is any real not less than the order of the operator.

The following transitional definitions are necesary: Let $\underline{\beta}$ be a string, $s \geq \lambda(\underline{\beta})$ and $\mathcal{L}_{\underline{\beta}}$ the elementary operator defined in paragraph (2). Define

$$\mathbf{T}_{\underline{\beta}} \;=\; \mathbf{T} \circ \mathcal{L}_{\underline{\beta}} : \mathbb{H}^s(\Omega_i) \longrightarrow \mathbf{L}^2(\partial\Omega_i)$$

where the dependence on the index i or, more conceptually the element Ω_i, is omitted.

Let $\{ f_{\underline{\alpha}} \}_{\underline{\alpha}} \subset R_0(\Omega)$ be the coefficients of the operator $\mathcal{L}$ (recall that the $\underline{\alpha}'s$ are in a subset of strings of length less or equal K). Define

$$\mathbf{T}\mathfrak{D}_j(u_i, w_i) = \sum_{\underline{\alpha}} \sum_{k\in\underline{\alpha}^{-1}(j)} (-1)^{k+1}\{(\mathbf{T}_{Z(\underline{\alpha},k)}\, u_i)(\mathbf{T}_{A(\underline{\alpha},k)}\, f_{\underline{\alpha}}\, w_i)\}$$

where $u_i, w_i \in \mathbb{H}^s(\Omega_i)$ (cf. Corollary 10 and Theorem 12) .

The vectorial version of the above traces over a given domain decomposition element is given by

$$\mathbf{T}\underline{\mathfrak{D}}(u_i, w_i) \;=\; \big(\mathbf{T}\mathfrak{D}_1(u_i,w_i), \mathbf{T}\mathfrak{D}_2(u_i,w_i), \ldots, \mathbf{T}\mathfrak{D}_n(u_i,w_i)\big)$$

Let the boundary traces, inner boundary traces and field of jumps be given by,

$$\mathbf{T}\underline{\mathfrak{D}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbf{L}^1(\partial\Omega)^n$$

$$\mathbf{T}\underline{\mathfrak{D}}(u,w)\Big|_F \;=\; \mathbf{T}\underline{\mathfrak{D}}(u_{\iota(F)}, w_{\iota(F)})\Big|_F \qquad \forall\, F \in \lfloor\widetilde{\partial\Omega}\rfloor$$

$$\mathbf{T}_+\underline{\mathfrak{D}}\;,\; \mathbf{T}_-\underline{\mathfrak{D}} : \mathcal{D} \times \mathcal{D} \longrightarrow \mathbf{L}^1(\Sigma)^n$$

$$\mathbf{T}_+\underline{\mathfrak{D}}(u,w)\Big|_F \;=\; \mathbf{T}\underline{\mathfrak{D}}(u_{\iota_+(F)}, w_{\iota_+(F)})\Big|_F$$

$$\mathbf{T}_-\underline{\mathfrak{D}}(u,w)\Big|_F \;=\; \mathbf{T}\underline{\mathfrak{D}}(u_{\iota_-(F)}, w_{\iota_-(F)})\Big|_F \qquad \forall\, F \in \lfloor\widetilde{\Sigma}\rfloor$$

$$[\underline{\mathfrak{D}}] \;=\; \mathbf{T}_+\underline{\mathfrak{D}} \;-\; \mathbf{T}_-\underline{\mathfrak{D}}$$

23. Appendix

Let $\mathcal{L}$ denote a linear differential operator, of order $2m > 0$, with infinitely differentiable coefficients and *elliptic* on $\overline{\Omega}$, where Ω is a bounded region with a smooth boundary $\Gamma = \partial\Omega$. Let a *system of boundary operators (of order at most 2m-1)* be a collection $\{\mathbf{G}_j\}_{j=0}^{m-1}$ of infinitely differentiable operators defined only on Γ, each of order not exceeding $2m-1$. If u is a function defined on Ω and $\mathbf{G}$ is a boundary operator, then $\mathbf{G}u$ is to be understood as $\mathbf{GT}u$, where $\mathbf{T}$ is the trace operator.

The theorem on Green's formula, as stated by J. L. Lions and E. Magenes in [13, pp. 114-115], starts by assuming that an elliptic operator $\mathcal{L}$ and a *normal* system of boundary operators $\{\mathbf{B}_j\}_{j=0}^{m-1}$ are given (the condition of normality is of no relevance for the present discussion). The first conclusion is that it is always possible to choose, non-uniquely, another system of normal boundary operators $\{\mathbf{S}_j\}_{j=0}^{m-1}$ such that the joint system of **B**'s and **S**'s is *Dirichlet* in a technical sense. Having made this choice, then a second conclusion is the unique existence of another two boundary systems $\{\mathbf{C}_j\}_{j=0}^{m-1}$ and $\{\mathbf{T}_j\}_{j=0}^{m-1}$, such that the following Green formula holds:

$$\int_\Omega w\mathcal{L}u \; d\mathbf{x} \; - \int_\Omega u\mathcal{L}^* w \; d\mathbf{x} \; = \; \sum_{j=0}^{m-1} \int_\Gamma \mathbf{S}_j u \; \mathbf{C}_j w \; d\sigma \; - \; \sum_{j=0}^{m-1} \int_\Gamma \mathbf{B}_j u \; \mathbf{T}_j w \; d\sigma$$

A natural restriction for this formula to make sense is that $u \in \mathbb{H}^s(\Omega)$ and $w \in \mathbb{H}^t(\Omega)$ with $s+t \geq 4m-1$. This is because the right hand side of the above relation is meaningful when $(s-1/2)-(2m-1) \; + \; (t-1/2)-(2m-1) \geq 0$. The left hand side is well defined also under this condition. A suitable simple choice is just $s = t = 2m$.

To override this limitation and establish formulas valid for general discontinuous functions belonging to spaces naturally embedded in $\mathbb{H}^0(\Omega)$, say $u \in \bigoplus_i \mathbb{H}^s(\Omega_i)$ and $w \in \bigoplus_i \mathbb{H}^t(\Omega_i)$ with s, t nonnegative and $s+t \geq 2m$, such as the Green-Herrera formulas, a process of extension of the distributional operators must be first developed. This is done in a paper to appear soon ([11], see also [3,4,6,8,9]).

24. References

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