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On Jirousek method and its generalizations

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Professor Jirousek has been a very important driving force in the modern development of Trefftz method, contributing to its application in many different fields such as elasticity, shells and plates theory, Poisson equation and transient heat analysis. This article is dedicated to him. The focus of the paper is to incorporate Jirousek method into a very general framework of Trefftz method which has been introduced by Herrera. Usually finite element methods are developed using splines, but a more general point of view is obtained when they are formulated in spaces of fully discontinuous functions — i.e., spaces in which the functions together with their derivatives may have jump discontinuities — and in the general context of boundary value problems with prescribed jumps. Two broad classes of Trefftz methods are obtained: direct (Trefftz–Jirousek) and indirect (Trefftz–Herrera) methods. In turn, each one of them can be divided into overlapping and non-overlapping.

1. Introduction

This paper is dedicated to Professor Jirousek who has been a very important driving force in the modern development of Trefftz method, contributing to its application in many different fields such as plates and shells theories [41–45, 49, 54, 55, 70], elasticity [48, 50, 71, 72], and transient heat analysis [46]. Jirousek and his collaborators have carried out the developments which are necessary for applying his approach in a reliable and adaptive manner [47, 51, 52]. In this respect, an important feature is the possibility of applying h and p convergence (see [53, 56] for recent surveys).

Taking as a starting point a precise and quite general definition of the procedure originally introduced by Trefftz [68], the method that has been developed by Jirousek together with a related method due to Herrera [5–7, 19–29, 32, 33, 36–38, 67], are revised in this framework and it is shown that when Trefftz methods are conceptualized in this manner they include many of the basic problems considered in numerical methods for partial differential equations, becoming fundamental for that subject. One avenue of this approach includes domain decomposition methods, but many other aspects may be illuminated.

Trefftz methods can be classified firstly into two broad categories: direct and indirect methods. The first one is essentially Jirousek's method in which the local solutions are used directly as bricks to build the global solution. However, this procedure is more general if one is not restricted to use analytical methods for the production of the local solutions, but instead resorts to numerical methods, as well. The second category is constituted by Trefftz–Herrera methods, in which local solutions of the adjoint differential equations are used, in an indirect manner, as specialized test functions which have the property of concentrating the information about the sought solution in the internal boundaries defined by the partition.

From another point of view, just as domain decomposition methods [8, 9, 15–17, 57, 58, 66], they can be classified into two very broad classes: overlapping and disjoint (or non-overlapping) methods. This terminology derives from the corresponding properties of the partitions. Since these two classifications are independent of each other, they can be combined to yield four categories of Trefftz methods: direct-overlapping, direct-non-overlapping, indirect-overlapping and indirect-non-overlapping, In this paper, these four categories are outlined and of some their main features discussed.

2. Jirousek method

In 1977 [41, 44], Jirousek started the development of a generalization of Trefftz method [68], in which non conforming elements are assumed to fulfill the governing equations 'a priori' and the inter-element continuity and the boundary conditions are then enforced in some weighted residual or point wise sense. As in the case of Trefftz, Jirousek in his early work used variational principles related to the differential equations considered. However, their use is not essential — collocation and least-squares, for example, are also suitable [73] — and many alternative formulations can be applied in order to generate "Trefftz-type" finite elements, which in more recent work have been referred to as T-elements [53].

This method — Jirousek's Method — has been quite successful because of its generality and efficiency. Recent states of the art are available [53, 56] from which we draw. Jirousek's method has been applied to the biharmonic equation [41, 44], plane elasticity [48] and Kirchhoff plates [45, 48, 63]. Later the approach was further extended to thin shells [70], moderately thick Reissner–Mindlin plates [54, 55, 63], thick plates [65], general 3-D solid mechanics [64], axisymmetric solid mechanics [71, 72], Poisson equation [74] and transient heat conduction analysis [52].

Just as in FEM, in Jirousek's method one has h-convergence and p-convergence. Thus, this leads to developing the h-version and the p-version of T-elements, as was first suggested by Jirousek and Teodorescu in 1982 [48], and implemented and studied several years later [42, 49]. According to Jirousek [53], the superiority of this version over the h-version has been so overwhelming that most of the new developments refer to the p-version. One of its most important advantages has been the facility with which a simple a posteriori stress error estimator [50] can be developed [52] and, using it, derive a procedure for adaptive reliability assurance [47, 51, 52].

3. Preliminary notions and notations

In what follows a region $\Omega \subset \mathbb{R}^n$ will be considered and $\{\Omega_1, \Omega_2, \dots, \Omega_E\}$ will be a *partition* (or *domain decomposition*) of Ω (Fig. 1); more precisely, this will be a pair-wise disjoint family of *manifolds with corners* [4, 62], such that the union of its closures is the closure of Ω. The inner boundary Σ, is defined to be the closed complement of $\partial\Omega$ in $\bigcup_i \partial\Omega_i$. In addition, the following notations will also be used in the sequel:

$$\partial_i\Omega \equiv (\partial\Omega) \cap (\partial\Omega_i), \qquad \Sigma_i \equiv \Sigma \cap (\partial\Omega_i), \qquad \text{and} \qquad \Sigma_{ij} \equiv \Sigma_i \cap \Sigma_j . \tag{1}$$

A unit normal vector $\underline{n}$, pointing outwards, is defined almost everywhere in $\partial\Omega$, in the standard manner. Similarly, a unit normal vector $\underline{n}$, will be defined almost everywhere on Σ_{ij} — this is unique, except for the sense which is chosen arbitrarily.

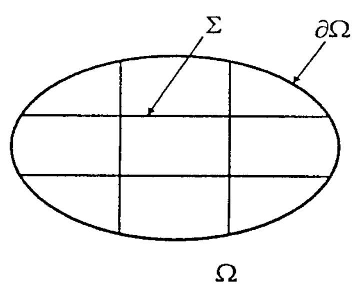

Fig. 1. The region Ω

Two linear spaces of functions defined in Ω, $D_1(\Omega)$ and $D_2(\Omega)$, will be considered. For every i ($i = 1, \dots, E$) and $\alpha = 1, 2$ let $D_\alpha(\Omega)$ be the space whose elements are the restrictions to Ω_i, of functions belonging to $D_\alpha(\Omega)$. Then

$$\hat{D}_\alpha(\Omega) = D_\alpha(\Omega_1) \oplus \cdots \oplus D_\alpha(\Omega_N); \qquad \alpha = 1, 2. \tag{2}$$

In view of this definition, with every function $v \in \hat{D}_\alpha(\Omega)$, $\alpha = 1, 2$ there is a finite sequence of functions $\{v^1, v^2, \dots, v^E\}$ such that for each $i = 1, 2, \dots, E$, v^i is defined in Ω_i. It will be assumed that for every $v \equiv \{v^1, v^2, \dots, v^E\} \in \hat{D}_\alpha(\Omega)$, $\alpha = 1, 2$, the trace on Σ of v^i ($i = 1, 2, \dots, E$), is well defined. However, on each $\Sigma_{ij} \equiv \Sigma_i \cap \Sigma_j$ two traces are defined — one corresponding to v^i and the other one to v^j — and in order to distinguish them the following notation is here introduced,

$$v_+ \equiv \text{trace of } (v^i) \tag{3}$$

when Ω_i lies on the positive side of and

$$v_- \equiv \text{trace of } (v^i) \tag{4}$$

otherwise. The *jump* of u across Σ is defined by

$$[v] \equiv v_+ - v_- \tag{5}$$

and the *average* by

$$\dot{v} \equiv \frac{1}{2}(v_+ + v_-). \tag{6}$$

More generally, whenever such a sequence of functions is associated to a function defined in Ω, it is possible to define two traces on Σ and the notations of Eqs. (5), (6) will be used in such cases. Observe that the average, $\dot{v}$, of a function and the product, $[v]\underline{n}$, are not dependent on the sense chosen for the unit normal vector $\underline{n}$.

4. Boundary value problem with prescribed jumps

To formulate this problem some additional notation is here introduced. The symbols $\mathcal{L}$ and $\mathcal{L}^*$ will stand for a linear differential operator and its formal adjoint, respectively. Also, $\mathcal{B}(v, w)$ and $\mathcal{C}(w, v)$ will be bilinear functions defined point-wise on $\partial\Omega$, for every $v \in \hat{D}_1(\Omega)$ and $w \in \hat{D}_2(\Omega)$. In a similar fashion, $\mathcal{J}(v, w)$ and $\mathcal{K}(w, v)$ will be bilinear functions defined point-wise, on Σ. When dealing with bilinear functions and functionals, a star on top will be used to denote its transpose; thus, for example,

$$\mathcal{C}^*(v, w) \equiv \mathcal{C}(w, v) \qquad \text{and} \qquad \mathcal{K}^*(v, w) \equiv \mathcal{K}(w, v). \tag{7}$$

In addition, $g_\partial(\cdot)$ and $j_\Sigma(\cdot)$ are linear functionals defined point-wise on $\partial\Omega$ and Σ, respectively, whose values at any $w \in \hat{D}_2(\Omega)$ will be written as $g_\partial(w)$ and $j_\Sigma(w)$. Given any function $v \in \hat{D}_1(\Omega)$, $\mathcal{B}(v, \cdot)$ and $\mathcal{J}(v, \cdot)$ will denote the linear functionals whose values at any $w \in \hat{D}_2(\Omega)$ are $\mathcal{B}(v, w)$ and $\mathcal{J}(v, w)$, respectively.

The *general boundary value problem with prescribed jumps* (BVPJ), to be considered is defined by

$$\mathcal{L}u^i = f_\Omega^i \equiv \mathcal{L}u_\Omega^i \qquad \text{in } \Omega_i, \quad i = 1, 2, \dots, E, \tag{8}$$

$$\mathcal{B}(u, \cdot) = g_\partial(\cdot) \equiv \mathcal{B}(u_\partial, \cdot) \qquad \text{in } \partial\Omega, \tag{9}$$

and

$$\mathcal{J}(u, \cdot) = j_\Sigma(\cdot) \equiv \mathcal{J}(u_\Sigma, \cdot) \qquad \text{in } \Sigma, \tag{10}$$

where $u_\Omega \in \hat{D}_1(\Omega)$, $u_\partial \in \hat{D}_1(\Omega)$, $u_\Sigma \in \hat{D}_1(\Omega)$ and f_Ω^i $(i = 1, 2, \dots, E)$ are given functions, while $g_\partial(\cdot)$ and $j_\Sigma(\cdot)$ are given linear functionals. They constitute the data of the problem and it is assumed that they fulfill Eqs. (8)–(10).

An important property is that, in applications, such functions can be constructed solving local problems, if necessary. For simplicity, in what follows it will be assumed that the BVPJ poses a unique solution fulfilling Eqs. (8)–(10) and the notation $u \in \hat{D}_1(\Omega)$ will be reserved for it.

As an illustration consider the general elliptic equation of second order. It will be assumed that the coefficients of the differential operator may have jump discontinuities across the internal boundary Σ. Then, the boundary value problem with prescribed jumps to be considered is:

$$\mathcal{L}u^i \equiv -\nabla \cdot \left(\underline{\underline{a}} \cdot \nabla u^i\right) + \nabla \cdot \left(\underline{b}u^i\right) + cu^i = f_\Omega^i \qquad \text{in } \Omega_i, \quad i = 1, \dots, E, \tag{11}$$

subjected to Dirichlet boundary conditions

$$u = u_\partial \qquad \text{on } \partial\Omega \tag{12}$$

and jump conditions

$$[u] = [u_\Sigma] \qquad \text{and} \qquad [\underline{a}_n \cdot \nabla u] = [\underline{a}_n \cdot \nabla u_\Sigma] \qquad \text{on } \Sigma. \tag{13}$$

Here $\underline{a}_n = \underline{\underline{a}} \cdot \underline{n}$. When the coefficients of the differential operator are continuous, it may be seen that the conditions of Eq. (13), are equivalent to prescribing the jump of the function and its normal derivative. Define the bilinear functions

$$\mathcal{B}(u, w) \equiv u(\underline{a}_n \cdot \nabla w + b_n w), \tag{14}$$

$$\mathcal{J}(u, w) \equiv \dot{w}[\underline{a}_n \cdot \nabla u] - [u]\overline{(\underline{a}_n \cdot \nabla w + b_n w)}, \tag{15}$$

and the linear functions $g(\cdot)$ and $j(\cdot)$ by $g(\cdot) \equiv \mathcal{B}(u_\partial, \cdot)$ together with $j(\cdot) \equiv \mathcal{J}(u_\Sigma, \cdot)$. Then, the BVPJ of Eqs. (11)–(13) take the form given by Eqs. (8)–(10).

5. General Variational Formulations

By the definition of formal adjoint, there exists a vector valued-bilinear function $\underline{\mathcal{D}}(u, w)$ which satisfies

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla \cdot \mathcal{D}(u, w). \tag{16}$$

It will also be assumed that

$$\underline{\mathcal{D}}(u, w) \cdot \underline{n} = \mathcal{B}(u, w) - \mathcal{C}(w, u) \qquad \text{on } \partial\Omega, \tag{17}$$

$$-[\underline{\mathcal{D}}(u, w)] \cdot \underline{n} = \mathcal{J}(u, w) - \mathcal{K}(w, u) \qquad \text{on } \Sigma. \tag{18}$$

Applying the generalized divergence theorem [4, 62], this implies the following Green–Herrera formula [25, 30, 32, 35]:

$$\int_\Omega w\mathcal{L}u \, dx - \int_{\partial\Omega} \mathcal{B}(u, w) \, dx - \int_\Sigma \mathcal{J}(u, w) \, dx = \int_\Omega u\mathcal{L}^* w \, dx - \int_{\partial\Omega} \mathcal{C}^*(u, w) \, dx - \int_\Sigma \mathcal{K}^*(u, w) \, dx. \tag{19}$$

A weak formulation of the BVPJ is

$$\int_\Omega w\mathcal{L}u \, dx - \int_{\partial\Omega} \mathcal{B}(u, w) \, dx - \int_\Sigma \mathcal{J}(u, w) \, dx$$
$$= \int_\Omega w\mathcal{L}u_\Omega \, dx - \int_{\partial\Omega} \mathcal{B}(u_\partial, w) \, dx - \int_\Sigma \mathcal{J}(u_\Sigma, w) \, dx \qquad \forall w \in \hat{D}_2(\Omega) \tag{20}$$

in which in view of Eq. (19), is equivalent to

$$\int_\Omega u\mathcal{L}^* w \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{C}^*(u,w)\,\mathrm{d}x - \int_\Sigma \mathcal{K}^*(u,w)\,\mathrm{d}x$$
$$= \int_\Omega w\mathcal{L}u_\Omega \,\mathrm{d}x - \int_{\partial\Omega} \mathcal{B}(u_\partial,w)\,\mathrm{d}x - \int_\Sigma \mathcal{J}(u_\Sigma,w)\,\mathrm{d}x \qquad \forall w \in \hat{D}_2(\Omega). \tag{21}$$

Equations (20)–(21) supply two alternative and equivalent variational formulations of the BVPJ. The first one is referred as the *variational formulation in terms of the data of the problem*, while the second one is referred as the *variational formulation in terms of the sought information.*

Introduce the following notation,

$$\langle Pu, w\rangle = \int_\Omega w\mathcal{L}u\,\mathrm{d}x, \qquad \langle Q^*u, w\rangle = \int_\Omega u\mathcal{L}^*w\,\mathrm{d}x, \tag{22}$$

$$\langle Bu, w\rangle = \int_{\partial\Omega} \mathcal{B}(u,w)\,\mathrm{d}x, \qquad \langle C^*u, w\rangle = \int_{\partial\Omega} \mathcal{C}^*(u,w)\,\mathrm{d}x, \tag{23}$$

$$\langle Ju, w\rangle = \int_\Sigma \mathcal{J}(u,w)\,\mathrm{d}x, \qquad \langle K^*u, w\rangle = \int_\Sigma \mathcal{K}^*(u,w)\,\mathrm{d}x. \tag{24}$$

With these definitions, each one of P, B, J, Q^*, C^* and K^*, are real-valued bilinear functionals defined on $\hat{D}_1(\Omega) \times \hat{D}_2(\Omega)$ and a more brief expression for Eq. (19) is the identity

$$P - B - J \equiv Q^* - C^* - K^* \tag{25}$$

When the definitions

$$f \equiv Pu_\Omega\,, \qquad g \equiv Bu_\partial\,, \qquad j \equiv Ju_\Sigma \tag{26}$$

are adopted, Eqs. (20), (21) can also be written as equalities between linear functionals,

$$(P - B - J)u = f - g - j \tag{27}$$

and

$$(Q - C - K)^*u = f - g - j. \tag{28}$$

Notice that Eqs. (27), (28) may be written as

$$\langle (P - B - J)u, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in D_2\,, \tag{29}$$

and

$$\langle (Q - C - K)^*u, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in D_2\,, \tag{30}$$

respectively. These equations exhibit more clearly their variational character.

Generally, the definitions of $\mathcal{B}$, $\mathcal{C}$, $\mathcal{J}$ and $\mathcal{K}$, depend on the kind of boundary conditions and the *smoothness criterion* of the specific problem. However, for the case when the **coefficients of the differential operators are continuous**, Herrera [30, 32, 36] has given very general formulas for $\mathcal{J}$ and $\mathcal{K}$, they are:

$$\mathcal{J}(u,w) \equiv -\underline{\mathcal{D}}([u], \dot{w}) \cdot \underline{n} \qquad \text{and} \qquad \mathcal{K}(w,u) \equiv \underline{\mathcal{D}}(\dot{u}, [w]) \cdot \underline{n}. \tag{31}$$

The fact that they fulfill Eq. (18) is easy to verify, when use is made of the algebraic identity

$$[\underline{\mathcal{D}}(u,w)] \equiv \underline{\mathcal{D}}([u], \dot{w}) + \underline{\mathcal{D}}(\dot{u}, [w]). \tag{32}$$

The case when $\hat{D}_1(\Omega) = \hat{D}_2(\Omega) = \hat{D}(\Omega)$, the differential operator $\mathcal{L}$ is formally symmetric and, in addition, $B = C$ and $J = K$, will be referred as *the symmetric case*. Since $\mathcal{L}^* = \mathcal{L}$ and therefore

$P = Q$, it is seen that the bilinear functional $P - B - J$ is symmetric; i.e., $P - B - J \equiv (P - B - J)^*$, and $P - B - J \equiv Q - C - K$ by virtue of Eq. (25). Using these facts, it is clear that in the symmetric case the variational principles of Eqs. (29), (30) are derivable from the potential

$$X(\hat{u}) \equiv \left\langle \frac{1}{2}(P - B - J)\hat{u} - (f - g - j)\,,\,\hat{u} \right\rangle \equiv \left\langle \frac{1}{2}(Q - C - K)\hat{u} - (f - g - j)\,,\,\hat{u} \right\rangle \tag{33}$$

where $\hat{u}$ is any function belonging to $\hat{D}(\Omega)$. More precisely, Eqs. (29), (30) can be written as

$$\langle X'(u), w \rangle = 0 \qquad \forall w \in D_2(\Omega) \tag{34}$$

where $X'(u)$ is the derivative of the functional $X(u)$; or more briefly, as $X'(u) = 0$. In particular, when $P - B - J \equiv Q - C - K$ is positive definite, in a subspace $N \subset \hat{D}(\Omega)$ such that $u \in N$, then the functional $X(\hat{u})$ yields a minimum principle for the BVPJ; i.e., $X(\hat{u})$ attains a minimum at $\hat{u} \in N \subset \hat{D}(\Omega)$, if and only if, $\hat{u} = u$.

In the case of the general elliptic equation of second order, in which the differential operator $\mathcal{L}$ is given by Eq. (11), one has

$$\mathcal{L}^* w \equiv -\nabla \cdot \left(\underline{\underline{a}} \cdot \nabla w\right) - \underline{b} \cdot \nabla w + cw \tag{35}$$

for the formal adjoint, and Eq. (16) is fulfilled with

$$\underline{\mathcal{D}}(u, w) \equiv \underline{\underline{a}} \cdot (u \nabla w - w \nabla u) + \underline{b} w, \tag{36}$$

and therefore, Eq. (31) yields

$$\begin{aligned} \mathcal{J}(u, w) &\equiv \dot{w}[\underline{a}_n \cdot \nabla u] - [u]\overline{\dot{(\underline{a}_n \cdot \nabla w + \mathbf{b}_n w)}}, \\ \mathcal{K}(w, u) &\equiv \dot{u}[\underline{a}_n \cdot \nabla u] - [w]\overline{\dot{(\underline{a}_n \cdot \nabla u - \mathbf{b}_n u)}}. \end{aligned} \tag{37}$$

Using these functions one can apply the previous definitions and obtain the two equivalent weak formulations of Eqs. (29), (30). In particular when $\mathbf{b} = \mathbf{0}$, a symmetric case is obtained, because the differential operator $\mathcal{L}$ is formally symmetric and in addition $B = C$ and $J = K$. Even more, let $N \subset \hat{D}(\Omega)$ be the subset of functions which satisfy $v = 0$ on $\partial\Omega$ and $[v] = 0$ on Σ, then it can be shown that $P - B - J \equiv Q - C - K$ is positive definite in $N \subset \hat{D}(\Omega)$. Thus, a minimum principle is applicable when the sought solution is continuous across Σ and vanishes on $\partial\Omega$.

6. SCOPE

The generality of the methodologies presented in this article is great, since they are applicable to any partial differential equation or system of such equations, which are linear, independently of its type. The coefficients of the operators can also be discontinuous across the internal boundary Σ. To illustrate the wide applicability of theory the following cases are next presented: the general elliptic equation of second order, the biharmonic equation, Stokes problem and the equations of equilibrium of linear elasticity.

6.1. Second order elliptic operators

The formulas here presented are applicable when the coefficients of the differential operators are discontinuous across the internal boundary Σ.

a) $\mathcal{L}u \equiv -\nabla \cdot \left(\underline{\underline{a}} \cdot \nabla u\right) + \nabla \cdot (\underline{b}u) + cu$, while $\mathcal{L}^* w \equiv -\nabla \cdot \left(\underline{\underline{a}} \cdot \nabla w\right) + \nabla \cdot (\underline{b}w) + cw$

b) $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega)$

c) $\hat{D}_1 \equiv \hat{D}_2 \equiv \hat{D} \equiv H^2(\Omega_1) \oplus H^2(\Omega_2) \oplus \cdots \oplus H^2(\Omega_E)$

d) $\underline{\mathcal{D}}(u,w) \equiv \underline{\underline{a}} \cdot (u \nabla w - w \nabla u) + \underline{\mathbf{b}} uw$

e) $\mathcal{B}(u,w) \equiv u(\underline{\mathbf{a}}_n \cdot \nabla w + \mathbf{b}_n w)$ and $\mathcal{C}(u,w) \equiv w \underline{\mathbf{a}}_n \cdot \nabla u$, where $\underline{\mathbf{a}}_n = \underline{\underline{a}} \cdot \underline{n}$ and $\mathbf{b}_n = \underline{\mathbf{b}} \cdot \underline{n}$

f) $\mathcal{J}(u,w) \equiv \dot{w}[\underline{\mathbf{a}}_n \cdot \nabla u] - [u]\overline{(\underline{\mathbf{a}}_n \cdot \nabla w + \mathbf{b}_n w)}^{\cdot}$ and $\mathcal{K}(w,u) \equiv \dot{u}[\underline{\mathbf{a}}_n \cdot \nabla u] - [w]\overline{(\underline{\mathbf{a}}_n \cdot \nabla u - \mathbf{b}_n u)}^{\cdot}$

g) Boundary conditions $u = u_\partial$

h) Jump conditions $[u] = [u_\Sigma]$ and $[\underline{\mathbf{a}}_n \cdot \nabla u] = [\underline{\mathbf{a}}_n \cdot \nabla u_\Sigma]$

i) Data on the external boundary $u = u_\partial$

j) Data on the internal boundary $[u_\Sigma]$ and $[\underline{\mathbf{a}}_n \cdot \nabla u_\Sigma]$

k) Sought information on the external boundary $\underline{\mathbf{a}}_n \cdot \nabla u$

l) Sought information on the internal boundary $\dot{u}$ and $\overline{(\underline{\mathbf{a}}_n \cdot \nabla u)}^{\cdot}$.

6.2. Biharmonic equation

a) $\mathcal{L}u \equiv \Delta^4 u$ and $\mathcal{L}^* w = \Delta^4 w$

b) $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^4(\Omega)$

c) $\hat{D}_1 \equiv \hat{D}_2 \equiv \hat{D} \equiv H^4(\Omega_1) \oplus H^4(\Omega_2) \oplus \cdots \oplus H^4(\Omega_E)$

d) $\underline{\mathcal{D}}(u,w) \equiv w \nabla \Delta u - u \nabla \Delta w + \Delta w \nabla u - \Delta u \nabla w$

e) $\mathcal{B}(u,w) \equiv \Delta w \dfrac{\partial u}{\partial n} - u \dfrac{\partial \Delta w}{\partial n}$

f) $\mathcal{C}(w,u) \equiv \Delta u \dfrac{\partial w}{\partial n} - w \dfrac{\partial \Delta u}{\partial n}$

g) $\mathcal{J}(u,w) \equiv [u] \overline{\dfrac{\partial \Delta w}{\partial n}}^{\cdot} - \dot{w} \left[\dfrac{\partial \Delta u}{\partial n}\right] + [\Delta u] \overline{\dfrac{\partial w}{\partial n}}^{\cdot} - \overline{\Delta w}^{\cdot} \left[\dfrac{\partial u}{\partial n}\right]$

h) $\mathcal{K}(w,u) \equiv [w] \overline{\dfrac{\partial \Delta u}{\partial n}}^{\cdot} - \dot{u} \left[\dfrac{\partial \Delta w}{\partial n}\right] + [\Delta w] \overline{\dfrac{\partial u}{\partial n}}^{\cdot} - \overline{\Delta u}^{\cdot} \left[\dfrac{\partial w}{\partial n}\right]$

i) Data on the external boundary u, $\partial u / \partial n$

j) Data on the internal boundary $[u]$, $[\partial u/\partial n]$, $[\Delta u]$ and $[\partial \Delta u / \partial n]$

k) Sought information on the external boundary Δu and $\partial \Delta u / \partial n$

l) Sought information on the internal boundary $\dot{u}$, $\overline{\partial u/\partial n}^{\cdot}$, $\overline{\Delta u}^{\cdot}$ and $\overline{\partial \Delta u / \partial n}^{\cdot}$.

6.3. Stokes problems

The system of equations to be considered is

$$-\Delta \underline{u} + \nabla p = \mathbf{0}; \qquad \nabla \cdot u = 0.$$

a) Let $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega) \oplus H^1(\Omega)$, and adopt the notation $\tilde{u} \equiv (\underline{u}, p)$ whenever $\tilde{u} \in D(\Omega)$

b) Define the vector valued differential operator $\underline{\mathcal{L}}$ by $\underline{\mathcal{L}} \cdot \tilde{u} \equiv (-\Delta \underline{u} + \nabla p = 0, -\nabla \cdot \underline{u} = 0)$

c) Then $\underline{\mathcal{L}}$ is self adjoint and, writing $\tilde{w} \equiv (\underline{w}, q)$, one has $\tilde{w} \cdot \underline{\mathcal{L}} \cdot \tilde{u} - \tilde{u} \cdot \underline{\mathcal{L}} \cdot \tilde{w} \equiv \nabla \cdot (\underline{u} \cdot \nabla \underline{w} - \underline{w} \cdot \nabla \underline{u} + p\underline{w} - q\underline{u})$

d) Thus $\underline{\mathcal{D}}(\tilde{u}, \tilde{w}) \equiv \underline{u} \cdot (\nabla \underline{w} - q) - \underline{w} \cdot (\nabla \underline{u} - p)$

e) $\mathcal{B}(\tilde{u}, \tilde{w}) \equiv \underline{u} \cdot \left(\dfrac{\partial \underline{w}}{\partial n} - q\underline{n} \right)$

f) $\mathcal{B}(\tilde{w}, \tilde{u}) \equiv \underline{w} \cdot \left(\dfrac{\partial \underline{u}}{\partial n} - p\underline{n} \right)$

g) $\mathcal{J}(\tilde{u}, \tilde{w}) \equiv \dot{\underline{w}} \cdot \left[\dfrac{\partial \underline{u}}{\partial n} - p\underline{n} \right] - [\underline{u}] \cdot \overline{\dfrac{\partial \underline{w}}{\partial n} - q\underline{n}}^{\cdot}$

h) $\mathcal{K}(\tilde{u}, \tilde{w}) \equiv \dot{\underline{u}} \cdot \left[\dfrac{\partial \underline{w}}{\partial n} - q\underline{n} \right] - [\underline{w}] \cdot \overline{\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}}^{\cdot}$

i) Data on the external boundary $\underline{u}$

j) Data on the internal boundary $[\underline{u}]$ and $\left[\dfrac{\partial \underline{u}}{\partial n} - p\underline{n} \right]$

k) Sought information at the external boundary $\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}$

l) Sought information at the internal boundary $\dot{\underline{u}}$ and $\overline{\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}}^{\cdot}$.

6.4. Equations of elasticity

Let $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega) \oplus H^2(\Omega) \oplus H^2(\Omega)$, and define for every $u \equiv (u_1, u_2, u_3) \in D(\Omega)$: $t_{ij}(\underline{u}) \equiv C_{ijpq} \frac{\partial u_p}{\partial x_q}$, where as usual, it is assumed that the elastic tensor possesses the following symmetries: $C_{ijpq} = C_{ijpq} = C_{ijqp}$

a) Define the vector valued differential operator $\underline{\mathcal{L}}$ by $\underline{\mathcal{L}} \cdot \underline{u} \equiv -\nabla \cdot \underline{t}(\underline{u})$, whose adjoint is $\underline{\mathcal{L}}^* \cdot \underline{w} \equiv -\nabla \cdot \underline{t}(\underline{w})$

b) $\underline{D}(\underline{u}, \underline{w}) \equiv \underline{u} \cdot \underline{t}(\underline{w}) - \underline{w} \cdot \underline{t}(\underline{u})$

c) $\underline{B}(\underline{u}, \underline{w}) \equiv \underline{u} \cdot \underline{t}(\underline{w}) \cdot \underline{n}$

d) $\underline{C}(\underline{u}, \underline{w}) \equiv \underline{w} \cdot \underline{t}(\underline{u}) \cdot \underline{n}$

e) $\mathcal{J}(\underline{u}, \underline{w}) \equiv \dot{\underline{w}} \cdot [\underline{t}(\underline{u})] \cdot \underline{n} - [\underline{u}] \cdot \overline{\underline{t}(\underline{w})}^{\cdot} \cdot \underline{n}$

f) $\mathcal{K}(\underline{w},\underline{u}) \equiv \dot{\underline{u}} \cdot [\underline{t}(\underline{w})] \cdot \underline{n} - [\underline{w}] \cdot \dot{\overline{\underline{t}(\underline{u})}} \cdot \underline{n}$

g) Data on the external boundary $\underline{u}$

h) Data on the internal boundary $[\underline{u}]$ and $[\underline{t}(\underline{u})] \cdot \underline{n}$

i) Sought information at the external boundary $\underline{t}(\underline{u}) \cdot \underline{n}$

j) Sought information at the internal boundary $\dot{\underline{u}}$ and $\dot{\overline{\underline{t}(\underline{u})}} \cdot \underline{n}$.

7. Trefftz methods

As mentioned in the Introduction, the method proposed originally by Trefftz, in 1926 [68], has been generalized very much, and to be precise the following definition is proposed.

Definition 1. *Let $\Pi = \{\Omega_1, \ldots, \Omega_E\}$ be a partition and for every $i = 1, \ldots, E$, let $\mathcal{H}_i$ be defined by the condition that $u^i_H \in \mathcal{H}_i$, if and only if $u^i_H \in \hat{D}(\Omega_i)$ and $\mathcal{L}u^i_H = 0$ in Ω_i. In addition, let $\mathcal{H} \equiv \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_E$. Then the problem of finding $u^i_H \in \mathcal{H}_i$, $i = 1, \ldots, E$, such that*

$$u = \sum_{i=1}^{E} u^i_\Omega + \sum_{i=1}^{E} u^i_H = u_\Omega + u_H \tag{38}$$

is the solution of the Boundary Value Problem with Prescribed Jumps, will be referred as "Trefftz Problem".

Observe that the 'solution of Trefftz problem', $u_H \equiv \sum_{i=1}^{E} u^i_H$, is unique necessarily, because $u_H = u - u_\Omega$ and, by assumption the solution $u \in \hat{D}(\Omega)$ is unique, while $u_\Omega \in \hat{D}(\Omega)$ is a datum. The notation $u_H \in \mathcal{H}$, will be reserved for it. Notice, however, that the definition of $u_H \in \mathcal{H}$, will change if the function u_Ω, used to specify the right-hand side of the differential equation is modified.

Two approaches for constructing the solution of Trefftz problem will be considered; methods derived from one or the other will be referred as *direct (Trefftz–Jirousek) method* and *indirect (Trefftz–Herrera) methods*, respectively. In the direct approach the local solutions are put together in such a way that the boundary conditions and prescribed jumps on Σ, are fulfilled, and the search for u_H is guided by such requirements. In Trefftz–Herrera method, on the other hand, special test or weighting functions are applied to obtain enough information on the internal boundary Σ, so as to define well posed problems in each one of the subregions Ω_i, $i = 1, \ldots, E$. This condition assures that the solution can be reconstructed locally, from the information available.

A second point of view for classifying Trefftz methods, which is independent of the first one, yields other two wide groups: overlapping and non-overlapping methods; i.e., the same classes that are considered when studying domain decomposition methods [8, 9, 15–17, 57, 58, 66]. Since these two points of view are independent of each other, they may be combined to give four types of methods: direct-non-overlapping, direct -overlapping, indirect-non-overlapping and indirect-overlapping.

For numerical applications, it is relevant to observe that the number of degrees of freedom is minimal when superfluous information is eliminated; i.e., when only information that is essential for defining local well-posed problems is retained. Generally, in order to eliminate superfluous information and handle essential information only, in both Trefftz–Jirousek and Trefftz–Herrera methods, it is necessary to resort to overlapping methods, as it will be seen in the next Sections.

8. Variational formulations of Trefftz methods

In what follows, $\hat{u}_H$ will stand for any function belonging to $\mathcal{H} \equiv N_P \subset \hat{D}_1(\Omega)$. For direct methods, a basic variational formulation, derived from Eq. (29), is that a function $\hat{u}_H \in N_P$ is solution of Trefftz problem if and only if

$$-\langle (B+J)\hat{u}_H\,,\,w\rangle = \langle (B+J)u_\Omega\,,\,w\rangle - \langle g+j;w\rangle, \qquad \forall w \in D_2\,. \tag{39}$$

The condition $\forall w \in D_2$ may be relaxed. Indeed, generally it is enough to require that Eq. (39) be satisfied for $\forall w \in N_Q \subset D_2(\Omega)$.

For the symmetric case, discussed in Section 5, one can define the functional

$$Y(\hat{u}_H) \equiv -\frac{1}{2}\langle (B+J)\hat{u}_H\,,\,\hat{u}_H\rangle + \langle g+j-(B+J)u_\Omega\,,\,\hat{u}_H\rangle \tag{40}$$

where $\hat{u}_H$ is any function belonging to $N_P \equiv N_Q \subset \hat{D}(\Omega)$. Then, $\hat{u}_H \equiv \sum_{i=1}^{E} \hat{u}_H^i$ is solution of Trefftz problem, if and only if $Y'(\hat{u}_H) = 0$. When the bilinear functional $-(B+J)$ is positive definite in $N_P \equiv N_Q \subset \hat{D}(\Omega)$, the functional $Y(\hat{u}_H)$ yields a maximum principle. Observe that a sufficient condition for $-(B+J)$ to be positive definite in $N_P \equiv N_Q \subset \hat{D}(\Omega)$, is that $P-B-J$ be positive definite in $\hat{D}(\Omega)$. More generally, when $N \subset N_P \equiv N_Q \subset \hat{D}(\Omega)$, is a subspace in which $-(B+J)$ is positive definite and $\hat{u}_H \in N$, then $Y(\hat{u}_H)$ attains a maximum at $\hat{u}_H \in N$, if and only if, $\hat{u}_H = u_H$.

9. Trefftz–Jirousek methods

The application of direct methods to one dimensional problems is relatively straight-forward [40]. However, their application in several dimension is considerably more complicated. The search for the solution of Trefftz problem, $u_H \in H$, can be done in several manners. In his pioneering work, Jirousek [41, 44] applied variational principles that were specific for the differential equations considered; they are particular cases of the general variational principles of Section 8. However, other procedures can be, and have been used — for example, collocation in the internal boundaries [73]. The application of least-squares possesses also great generality and has the additional advantage of yielding symmetric and positive definite matrices [53, 56].

In the case of direct overlapping methods, it is possible to apply two different approaches; one which is more direct and the other one which is less direct. In this latter one, the base functions are used to impose a *compatibility condition* from which the global system of equations is derived [40]. In addition, in this manner information about the sought solution is obtained, which is enough to formulate well-posed local problems. This procedure handles only essential information, so that the number of degrees of freedom is minimal.

In the first and more direct of the overlapping methods, the reduction in the number of degrees of freedom, is achieved using base functions which fulfilled some of the jump conditions, such as continuity conditions, from the start. This kind of weighting functions are easy to construct if numerical methods are used to build them, but this is not feasible, in most cases, when systems of analytical solutions are applied. The construction of TH-complete systems of weighting functions will be discussed and illustrated in Section 11.

Consider, as an example, the BVPJ for the general elliptic equation of second order, defined by Eqs. (11)–(13). When a direct method is applied, one can use the variational principle in terms of the data of the problem of Eq. (20), with the help of Eqs. (14), (15). Other possibility is to apply least squares to the quantities $[\hat{u} - u_\partial]$, on $\partial\Omega$, together with $[\hat{u} - u_\Sigma]$ and $[\underline{a}_n \cdot \nabla\hat{u} - \underline{a}_n \cdot \nabla u_\Sigma]$, on Σ, where $\hat{u} \in D$ is any trial function. When the coefficients of the differential operator are continuous, it is simpler, to replace this latter quantity by $[\partial\hat{u}/\partial n - \partial u_\Sigma/\partial n]$. In addition, the following observation must be made: when the numerical method that is applied to solve the local problems is collocation [1], the boundary condition $\hat{u} = u_\partial$, on $\partial\Omega$, can be fulfilled by the trial tions from the start, so that the least squares on $[\hat{u} - u_\partial]$, need not be applied. Also, when lapping methods are used, it is easy to construct trial functions which fulfill the condition $u_\Sigma]$, on Σ (see, Section 12), and this reduces the number of degrees of freedom of the matrices ıe global system of equations. As has already been mentioned, this is not possible when analytical tions are applied.

To illustrate the alternative overlapping procedure [40], which in some sense is only *semi-direct*, sider the equation $\mathcal{L}u = 0$ in an interval of the real line, where $\mathcal{L}$ is a second order differen- operator. Let $x_i \in (x_{i-1}, x_{i+1})$, then $u(x_i)$ depends linearly on $u(x_{i-1})$ and $u(x_{i+1})$. Indeed, $_i) = \varphi_i^-(x_i)\, u(x_{i-1}) + \varphi_i^+(x_i)\, u(x_{i+1})$, and this equation constitutes a three-diagonal system of ations, whose coefficients can be obtained solving locally, by collocation, a pair of boundary value blems in the interval (x_{i-1}, x_{i+1}): $\mathcal{L}\varphi_i^- = \mathcal{L}\varphi_i^+ = 0$, subjected to $\varphi_i^-(x_{i-1}) = \varphi_i^+(x_{i+1}) = 1$ and $(x_{i+1}) = \varphi_i^+(x_{i-1}) = 0$.

The generalization of this method to more complicated problems and to several dimensions will presented in [40]. In particular, it will be shown that this is the basic procedure which is behind well-known Schwarz alternating method [61].

. Trefftz–Herrera methods

e indirect Trefftz methods have been introduced and developed by Herrera and his collabora- s [5, 7, 7, 19–29, 32, 33, 36–38, 67]. They stem from the following observation [25]: when the thod of weighted residuals is applied — and this includes the Finite Element Method (FEM) — : information about the sought solution contained in an approximate one, is determined by the stem of weighting functions that are applied and it is independent of the base functions that are ed. A convenient strategy is to apply test functions of a special kind — *specialized test functions* , with the property of yielding information in the boundaries $\partial\Omega$ and Σ, exclusively. In order to lve Trefftz problem — i.e., in order to recover u_{II}^i, $i = 1, \ldots, E$ — it is necessary to have enough formation on Σ for defining well posed problems in each one of the subregions Ω_i $(i = 1, \ldots, E)$, ıce this will determine the functions u_{II}^i. In addition, Herrera's algebraic theory of boundary value oblems supplies a very effective framework for guiding the construction of such test functions [29].

The point of view just mentioned yields the following interpretation of FEM formulations: the stem of test functions that are applied determine the information about the sought solution con- ined in an approximate one, while the base functions interpolate (or extrapolate) such information. strategy, which in some sense is optimal [31], is to obtain enough information to define well-posed oblems locally and then use the solutions of these local problems, instead of base functions, to tending the information that is available, since this is the most efficient way of performing this ınction. Some times the specialized test functions have been referred as Optimal Test Functions [6] ıd the extension of the information by means of the solution of the local boundary value problems, s Optimal Interpolation [31].

By inspection of Eqs. (22)–(24), it can be recognized that the information about the solution $\in D$ is given by Q^*u, in the interior of the subregions Ω_i $(i = 1, \ldots, E)$; it is given by C^*u, in the uter boundary $\partial\Omega$; and it is given by K^*u, in the internal boundary Σ. Jirousek [53], refers to $\Sigma \cup \partial\Omega$ s the '*generalized boundary*'. A first step to derive Trefftz–Herrera procedures, is to manipulate he variational formulation in terms of the sought information of Eq. (30), in such a way as to ave information in the generalized boundary, exclusively. This requires eliminating Q^*u in that quation, and can be achieved by taking special weighting functions such that $Qw = 0$. This yields

$$-\langle (C + K)^* u, w \rangle = \langle f - g - j; w \rangle, \qquad \forall w \in N_Q \subset \hat{D}_1(\Omega). \tag{41}$$

Generally, one is interested only in part of the information contained in $(C + K)^*u$; so, it is useful o introduce a decomposition of the bilinear functional $C + K$ and write

$$C + K \equiv S + R \tag{42}$$

where S is chosen so that S^*u is precisely "*the sought information*".

Definition 2. *Given R and S which fulfill Eq. (42), let $\tilde{u} \in \hat{D}_1(\Omega)$ be such that there exists a solution, $u \in \hat{D}_1(\Omega)$, of the BVPJ with the property that $S^*\tilde{u}$ is the sought information; i.e.,*

$$S^*\tilde{u} = S^*u. \tag{43}$$

Then $\tilde{u} \in \hat{D}_1(\Omega)$ is said to contain "the sought information".

In what follows, the symbol $\tilde{u} \in D$ is reserved for functions which contain the sought information. Let $N_Q \subset \hat{D}_2(\Omega)$ and $N_R \subset \hat{D}_2(\Omega)$ be the null subspaces of Q and R respectively. In order to formulate a necessary and sufficient condition for a function for $\hat{u} \in D_1$ to contain the sought information, it will be necessary to define a concept of completeness, similar to that introduced by the author in 1976 [20] and which has been very effective in the study of complete families [2].

Definition 3. *A subset of weighting functions, $\mathcal{E} \subset N_Q \cap N_R$, is said to be TH-complete for S^*, when for any $\hat{u} \in \hat{D}_1(\Omega)$, one has*

$$\langle S^*\hat{u}, w\rangle = 0, \qquad \forall w \in \mathcal{E} \Longrightarrow S^*\hat{u} = 0. \tag{44}$$

Clearly, a necessary and sufficient condition for the existence of TH-complete systems, is that $N_Q \cap N_R$ be, itself, a TH-complete system.

Theorem 1. *Let $\mathcal{E} \subset N_Q \cap N_R$ be a system of weighting functions, TH-complete for S^*, and assume that there exists $u \in \hat{D}_1(\Omega)$ a solution of the BVPJ. Then, a necessary and sufficient condition for $\hat{u} \in D_1$ to contain the sought information, is that*

$$-\langle S^*\hat{u}, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in \mathcal{E}. \tag{45}$$

Proof. The necessity of this condition can be derived using Eqs. (41) and (42). To prove the sufficiency, observe that the necessary condition just mentioned, implies that for the solution $u \in \hat{D}_1(\Omega)$, whose existence is assumed, one has

$$-\langle S^*u, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in \mathcal{E}. \tag{46}$$

Therefore, if $\hat{u} \in D_1$ fulfills Eq. (45), then Eqs. (46) and (45) together, imply:

$$\langle S^*\hat{u}, w\rangle = \langle S^*u, w\rangle, \qquad \forall w \in \mathcal{E}. \tag{47}$$

Hence Eq. (43), since is TH-complete. ■

In numerical applications of indirect methods, Theorem 1 yields the basic system of equations whose solution is sought. To obtain a formulation which is suitable for both elliptic and time-dependent problems, it is necessary, in addition, to introduce decompositions of the bilinear functionals C and K. These will be

$$C = C^S + C^C \qquad \text{and} \qquad K \equiv K^S + K^C. \tag{48}$$

When time dependent problems are considered, Ω is a space-time region and the final state of the system that is modeled by the partial differential equation, lies in the outer boundary, $\partial\Omega$. Thus, a suitable choice of C^S permits handling this situation. In applications to elliptic problems, on the other hand, it is frequently convenient defining $S \equiv K^S$, so that $R \equiv C + K^C$. In this case the information on the external boundary is eliminated and the sought information $S^*u \equiv K^{S*}u$ contains information in the internal boundary, exclusively. The choice $K^C = 0$ leads to non-overlapping indirect methods, while $K^C \neq 0$ corresponds to overlapping indirect methods.

A corollary of Theorem 1, is that when $u_P \in \hat{D}_1(\Omega)$ is such that $Pu_P = f$ and $Bu_P = g$, then Eq. (45) can be replaced by

$$-\langle K^{S*}\hat{u}, w\rangle = -\langle K^{S*}u_P, w\rangle + \langle J(u_P - u_\Sigma), w\rangle, \qquad \forall w \in \mathcal{E}, \tag{49}$$

in that theorem. In applications, this result may be used to replace an expression involving integrals in the interior of the subregions Ω_i, $(i = 1, \ldots, E)$, by one which involves integrals over the internal boundary, only. In our discussions, it has been assumed that $u_P \in \hat{D}_1(\Omega)$ is a datum. Generally, when this is not available from the start, its construction requires solving local boundary value problems in each one of the subregions Ω_i, $(i = 1, \ldots, E)$, exclusively.

When K^{S*} is symmetric in $N \equiv N_Q \cap N_P$ the variational principles of Eqs. (45) and (49) can be derived from the potentials

$$Z(\hat{u}) \equiv -\frac{1}{2}\langle K^S \hat{u}, \hat{u}\rangle - \langle f - g - j, \hat{u}_H\rangle \tag{50}$$

and

$$\tilde{Z}(\hat{u}) \equiv -\frac{1}{2}\langle K^S \hat{u}, \hat{u}\rangle + \langle K^{S*} u_P\,, w\rangle - \langle J(u_P - u_\Sigma), w\rangle, \tag{51}$$

respectively. When it is positive definite on $N \equiv N_Q \cap N_R$, then a minimum principle holds, in addition.

As in Section 5, let us illustrate the TH-method by applying it to the elliptic BVPJ of second order of Eqs. (11)–(13). Since in the case of elliptic problems a convenient strategy is to concentrate all the sought information on Σ, a first possibility is to set $S \equiv K$; i.e., $K^C \equiv 0$ and $R \equiv C$, so that the test functions are required to fulfill $\mathcal{L}^* w = 0$, in each one of the subregions separately, together with $w = 0$, on $\partial\Omega$. Observe that no matching condition between the subregions is imposed. Thus, in this case the method is non-overlapping. The sought information is $\dot{u}$ and $\overline{\dot{\partial u/\partial n}}$, on Σ. This information is excessive, in the sense that when it is used to define local boundary value problems, they turn out to be over-determined.

Indeed, it would be enough, for example, to prescribe $\dot{u}$ on Σ, to have a well posed problem, if that information is complemented with the data, on $\partial\Omega$ (see Fig. 1). Thus, one strategy which permits handling information which is essential only, is to concentrate all the information in $\dot{u}$, on Σ. This is achieved if one sets

$$\langle K^C w, u\rangle = -\int_\Sigma [w]\overline{\dot{(\underline{\mathbf{a}}_n \cdot \nabla u - \mathbf{b}_n u)}}\,\mathrm{d}x \tag{52}$$

in Eq. (48), together with $C^S \equiv 0$ (and $C^C \equiv C$). In this case, the requirement $w \in N_R$ implies the condition $[w] = 0$, on Σ, in addition to the previous conditions. Thus, such functions must be continuous across Σ. The construction, by collocation, of test functions fulfilling these conditions is not difficult, but requires putting together several subregions. Thus, diminishing the information that has to be handled, and so the degrees of freedom, leads to an overlapping method. In Sections 11 and 12, TH-complete systems of functions and procedures for their construction are presented.

11. TH-COMPLETE SYSTEMS

The application of Trefftz methods requires to have available systems of functions which are complete for the space $H \equiv H_1 \oplus \cdots \oplus H_E$. A criterion of completeness which has permitted applying the function theoretic approach as an effective means to solving boundary value problems [2], is due to Herrera [20] and an extension of that concept was given in Section 10; it will be referred to as TH-completeness (Trefftz–Herrera completeness; it has also been referred as C-completeness or T-completeness). This Section is devoted to discuss briefly the methods available for developing such systems of functions, which can be grouped into two broad categories: analytical and numerical.

The classical approach is based on analytical methods and a thorough account may be found in a book by Begehr and Gilbert [2]. The function theoretic method was pioneered by Bergman [3] and Vekua [69], and further developed by Colton [10–12], Gilbert [13, 14], Kracht–Kreyszig [59], Lanckau [60] and others. The author has supplied such systems for Stokes problem [39], Helmholtz equation (in [67] it is shown that a system of plane waves posses that property) and biharmonic equation [18]. Other means of constructing them are using fundamental solutions and spectral methods, among others (see [2]).

The most general procedures for constructing TH-complete systems are, by far, numerical methods. Any such method can be applied, but collocation is quite suitable [1]. One has to construct families of solutions which span suitable spaces of boundary conditions, as it is illustrated in the next Section, in the case of the general elliptic equation of second order.

12. Construction of TH-complete systems by collocation

Consider again the BVPJ for the general elliptic equation of second order. For simplicity, a rectangular region will be considered and the subregions of the partition, will be rectangles (Fig. 2a).

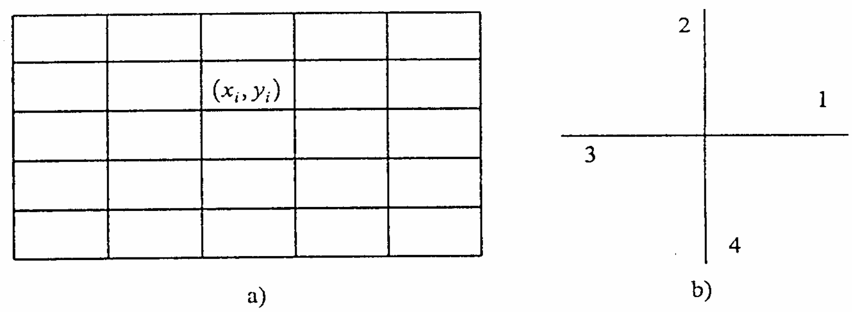

Fig. 2. a) Rectangular domain decomposition of Ω; b) numbering of internal boundaries

What is required, for a system of functions to be TH-complete is that, for each subregion Ω_i, the traces of its members span $H^0(\partial\Omega_i)$. When collocation methods are used in the construction of TH-complete systems, one may choose a system of functions which spans $H^0(\partial\Omega_i)$ and then solve a family of boundary value problems taking as boundary conditions each one of the members of such system. A convenient choice for the system of functions that spans $H^0(\partial\Omega_i)$, is a system of piece-wise polynomials. A linear basis of such system of polynomials may be obtained taking the four bilinear polynomials which have the property of assuming the value 1 at one corner of each given quadrilateral and vanishing at all the other three corners, together with all the piecewise polynomials defined on $\partial\Omega_i$, which vanish identically at three sides of the quadrilateral.

For constructing a TH-complete system, fulfilling a continuity condition, collocation methods are also quite suitable. With each internal node (x_i, y_i) a region Ω_{ij}, which is the union of the four rectangles of the original partition that surround that node, is associated. Then, the system of subregions $\{\Omega_{ij}\}$ is overlapping. The boundary of Ω_{ij} is $\partial\Omega_{ij}$, while that part of Σ laying in the interior of Ω_{ij} will be denoted by Σ_{ij} (Fig. 2b); it is constituted by four segments which will be numbered as indicated in Fig. 2b and form a cross. Given any sub-region Ω_{ij}, a system of functions which fulfill $\mathcal{L}^*w = 0$ in its interior and vanish on $\partial\Omega_{ij}$, is developed. Using the numbering already introduced, with each interior node (x_i, y_i) five groups of weighting functions are constructed, which are identified by the conditions satisfied on Σ_{ij}:

Group 0. This group is made of only one function, which is linear in each one of the four segments of Σ_{ij} and $w_{ij}(x_i, y_j) = 1$

For $N = 1, \dots, 4$, they are defined by:

Group N. The restriction to interval "N", of Fig. 1b, is a polynomial in x which vanishes at the end points of interval "N". For each degree$\geq$2, there is only one such polynomial.

The support of the test function of Group "0", is the whole square, while those weighting functions associated with Groups "1" to "4", have as support rectangles which can be obtained from each other by rotation, as it is shown in Fig. 3.

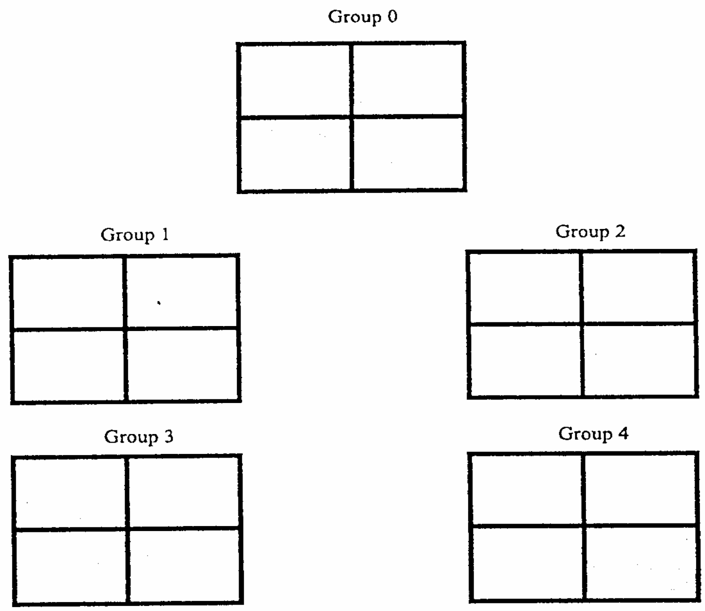

Fig. 3. The five groups of weighting functions, according to their supports

Of course, when developing numerical algorithms for the solution of boundary value problems, only a few terms of these Th-complete systems are taken; it could be only one (see [5]). Generally, the order of precision of the resulting scheme will depend on the number of terms taken.

13. Conclusions

A large class of numerical methods has been formulated, whose research thus far has been quite incomplete. The conclusion is drawn that a lot of work should be done on them, because they have great potential in the theory and practice of numerical methods for partial differential equations. The framework here presented would valuable for this purpose. In particular, collocation methods could be improved bvery much along these lines [34].

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