Trefftz Method: A General Theory

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A precise definition of Trefftz method is proposed and, starting with it, a general theory is briefly explained. This leads to formulating numerical methods from a *domain decomposition* perspective. An important feature of this approach is the systematic use of ''fully discontinuous functions'' and the treatment of a general boundary value problem with prescribed jumps. Usually finite element methods are developed using splines, but a more general point of view is obtained when they are formulated in spaces in which the functions together with their derivatives may have jump discontinuities and in the general context of boundary value problems with prescribed jumps. Two broad classes of Trefftz methods are obtained: direct (Trefftz–Jirousek) and indirect (Trefftz–Herrera) methods. In turn, each one of them can be divided into overlapping and nonoverlapping. The generality of the resulting theory is remarkable, because it is applicable to any partial (or ordinary) differential equation or system of such equations, which is linear. The article is dedicated to Professor Jiroslav Jirousek, who has been a very important driving force in the modern development of Trefftz method. Numer Methods Partial Differential Eq 16: 561–580, 2000

I. INTRODUCTION

In 1926 [1], Trefftz introduced an approach to solving partial differential equations that has received considerable attention in recent years. This method was generalized by several authors. In particular, Jirousek [2, 3] expanded its applicability by introducing partitions of the region of interest and using families of analytical solutions in each one of the subregions. The global solution is then constructed using, as bricks, the solutions defined in different subregions and fitting them together according to a suitable criterion.

Another related approach was proposed and developed by Herrera [4–23], in which local solutions of the adjoint differential equations are used to obtain information about the sought solution at internal boundaries, to define well-posed problems in the subregions. In this manner, it is possible to reconstruct the solution in the whole region, solving local problems exclusively. Both of these approaches are generally referred to as Trefftz methods, although the original method of Trefftz was far more restricted.

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Actually, to our knowledge, Trefftz Methods have not received a precise definition, although this terminology has had wide acceptance. In this article, a very broad definition of what is meant by a Trefftz Method is proposed; roughly, this is:

Given a region of an Euclidean space of some partitions of that region, a "Trefftz Method" is any procedure for solving boundary value problems of partial differential equations or systems of such equations, on such region, using solutions of that differential equation or its adjoint, defined in its subregions.

When Trefftz Method is conceptualized in this manner, it includes many of the basic problems considered in numerical methods for partial differential equations and becomes a fundamental concept of that subject. One avenue of this approach includes domain decomposition methods, but many other aspects of numerical methods may be illuminated using a Trefftz Method perspective.

When formulating numerical methods for partial differential equations, it is necessary to decompose the domain of the problem into smaller ones, although this process is not always carried out explicitly. Applying a Trefftz approach, many numerical methods can be derived. A first division separates them into two broad categories: *direct* and *indirect* methods. The first one is essentially Jirousek's method, in which the local solutions are used directly as *bricks* to build the global solution. But this procedure is far more general, if one is not restricted to use analytical methods for the production of the local solutions, as Jirousek originally did, but instead resorts to a greater diversity of procedures, such as numerical methods, to obtain them. The second category is that of indirect Trefftz method or Trefftz–Herrera method [4–23], in which local solutions of the adjoint differential equations are used as specialized test functions to concentrate the information about the sought solution in the internal boundaries associated with the partition.

From another point of view, just as in domain decomposition methods [24–31], these kinds of numerical methods can be classified into two very broad classes: overlapping and disjoint (or nonoverlapping) methods. This terminology derives from the corresponding properties of the partitions. Thus, by combining these classifications, four groups of Trefftz methods are derived: direct overlapping, direct nonoverlapping, direct overlapping and direct nonoverlapping.

Above, as it was already mentioned in connection with direct Trefftz methods, the procedures applied to construct the local solutions may be any. When numerical methods are used to obtain them, one is lead to four different versions of the corresponding numerical procedure and generally they are nonconventional. If the numerical method chosen is collocation, for example, as Herrera [32] has done, the four categories of collocation methods that were mentioned before are obtained: two of them are subclasses of direct collocation, and the other two are subclasses of indirect collocation (or Trefftz–Herrera collocation [20–22, 32]. Thus far, the research that has been carried out on these methods has been quite insufficient and must be expanded.

This article is devoted to presenting some basic concepts for a ''General Theory of Trefftz Method'' and its scope is indicated. An important feature of the approach here followed is the systematic use of ''fully discontinuous functions,'' which possesses many advantages over the more customary use ''splines,'' when formulating numerical methods for partial differential equations, such as finite elements and collocation. Herrera has supplied an extensive fundamental theory on which the application of fully discontinuous functions will be based. The generality of the method is also outstanding, since it is applicable to any partial (or ordinary) differential equation or system of such equations, which is linear.

The case of collocation methods, which are known as efficient and highly accurate numerical solution procedures for partial differential equations, is quite suitable for illustrating some of the advantages of Trefftz approach. Indeed, when they are seen from this more general perspective, the standard formulations of collocation methods using splines turn out to be particular cases,

which are obtained when some special strategies for solving the final system of equations are followed and, as it will be shown, they are far from being the most efficient strategies. Also, when using splines, domain decomposition is accomplished by means of partitions of unity, as Babuska and Melenk have done, and this introduces complications into the problem, which are avoided when fully discontinuous functions are applied.

The article is dedicated to Professor Jirousek, who has been a very important driving force in the modern development of Trefftz method, contributing to its application in many different fields such as plates and shells theories [2, 3, 32–39], elasticity [40–43], and transient heat analysis [44]. Jirousek and his collaborators have carried out the developments that are necessary for applying his approach in a reliable and adaptive manner [45–47]. In this respect, an important feature is the possibility of applying h and p convergence (see [48, 49] for recent surveys).

II. JIROUSEK METHOD

In 1977 [2, 3], Jirousek started the development of a generalization of Trefftz method [1], in which nonconforming elements are assumed to fulfill the governing equations *a priori* and the inter-element continuity and the boundary conditions are then enforced in some weighted residual or pointwise sense. As in the case of Trefftz, Jirousek in his early work used variational principles related to the differential equations considered. However, their use is not essential–-collocation and least-squares, for example, are also suitable [50]–-and many alternative formulations can be applied to generate ''Trefftz-type'' finite elements, which in more recent work have been referred to as T-elements [48].

This method, Jirousek's Method, has been quite successful because of its generality and efficiency. Recent states of the art are available [48, 49] from which we draw. Jirousek's method has been applied to the biharmonic equation [2, 3], plane elasticity [40], and Kirchhoff plates [34, 40, 51]. Later the approach was further extended to thin shells [39], moderately thick Reissner–Mindlin plates [35, 36, 51], thick plates [52], general 3-D solid mechanics [53], axisymmetric solid mechanics [42, 43], Poisson equation [54], and transient heat conduction analysis [45].

Just as in FEM, in Jirousek's method one has h-convergence and p-convergence. Thus, this leads to developing the h-version and the p-version of T-elements, as was first suggested by Jirousek and Teodorescu in 1982 [40], and implemented and studied several years later [37, 38]. According to Jirousek [48], the superiority of this version over the h-version has been so overwhelming that most of the new developments refer to the p-version. One of its most important advantages has been the facility with which a simple *a posteriori* stress error estimator [41] can be developed [45] and, using it, derive a procedure for adaptive reliability assurance [45–47].

III. PRELIMINARY NOTIONS AND NOTATIONS

In what follows, a region $\Omega \subset R^n$ is considered and $\{\Omega_1, \Omega_2, \ldots, \Omega_E\}$ is a *partition* (or *domain decomposition*) of Ω (Fig. 1); more precisely, this is a pairwise disjoint family of *manifolds with corners* [55, 56], such that the union of its closures is the closure of Ω. The inner boundary Σ, is defined to be the closed complement of $\partial\Omega$ in $\cup_i \partial\Omega_i$. In addition, the following notations are also used in the sequel:

$$\partial_i\Omega \equiv (\partial\Omega) \cap (\partial\Omega_i), \Sigma_i \equiv \Sigma \cap (\partial\Omega_i) \text{ and } \Sigma_{ij} \equiv \Sigma_i \cap \Sigma_j. \tag{3.1}$$

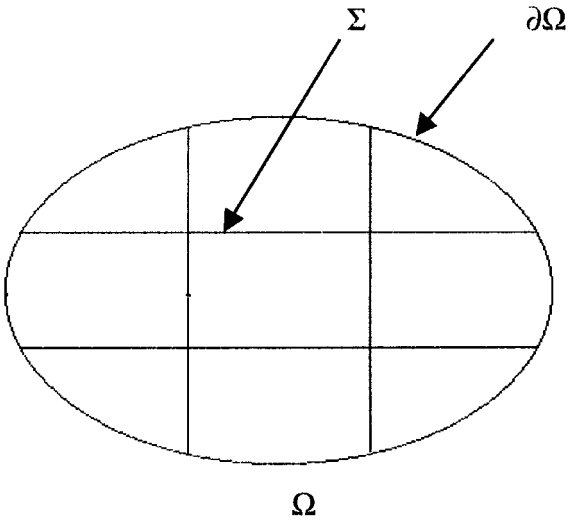

FIG. 1. The region Ω.

A unit normal vector $\underline{n}$, pointing outwards, is defined almost everywhere in $\partial\Omega$, in the standard manner. Similarly, a unit normal vector $\underline{n}$, is defined almost everywhere on Σ_{ij}. This is unique, except for the sense that it is chosen arbitrarily.

Two linear spaces of functions defined in Ω, $D_1(\Omega)$, and $D_2(\Omega)$, are considered. For every $i = 1, \ldots, E$ and $\alpha = 1, 2$, let $D_\alpha(\Omega_i)$ be the space whose elements are the restrictions to Ω_i, of functions belonging to $D_\alpha(\Omega_i)$. Then

$$\hat{D}_\alpha(\Omega) \equiv D_\alpha(\Omega_1) \oplus \cdots \oplus D_\alpha(\Omega_N); \alpha = 1, 2. \tag{3.2}$$

In view of this definition, with every function $v \in \hat{D}_\alpha(\Omega), \alpha = 1, 2$, there is a finite sequence of functions $\{v^1, v^2, \ldots, v^E\}$ such that, for each $i(= 1, 2, \ldots, E)$, v^i is defined in Ω_i. It is assumed that, for every $v \equiv \{v^1, \ldots, v^E\} \in \hat{D}_\alpha(\Omega), \alpha = 1, 2$, the trace on Σ of v^i $(i = 1, 2, \cdots E)$, is well defined. However, on each $\Sigma_{ij} \equiv \Sigma_i \cap \Sigma_j$ two traces are defined—one corresponding to v^i and the other one to v^j–-and in order to distinguish them, the following notation is here introduced:

$$v_+ \equiv \text{Trace of } (v^i), \tag{3.3a}$$

when Ω_i lies on the positive side of Σ_{ij}, and

$$v_- \equiv \text{Trace of } (v^i) \tag{3.3b}$$

otherwise. The *jump* of u across Σ is defined by

$$[v] \equiv v_+ - v_-, \tag{3.4a}$$

and the *average* by

$$\dot{v} \equiv \tfrac{1}{2}(v_+ + v_-). \tag{3.4b}$$

More generally, whenever such a sequence of functions is associated to a function defined in Ω, it is possible to define two traces on Σ and the notations of Eqs. (3.4) are used in such cases. Observe that the average, $\dot{v}$, of a function and the product, $[v]\underline{n}$, are not dependent on the sense chosen for the unit normal vector $\underline{n}$.

IV. BOUNDARY VALUE PROBLEM WITH PRESCRIBED JUMPS

To formulate this problem some additional notation is here introduced. The symbols $\mathcal{L}$ and $\mathcal{L}^*$ stand for a linear differential operator and its formal adjoint, respectively. Also, $\mathcal{B}(v,w)$ and $\mathcal{C}(w,v)$ are bilinear functions defined pointwise on $\partial\Omega$, for every $v \in \hat{D}_1(\Omega)$ and $w \in \hat{D}_2(\Omega)$. In a similar fashion, $\mathcal{J}(w,v)$ and $\mathcal{K}(w,v)$ are bilinear functions defined pointwise, on Σ. When dealing with bilinear functions and functionals, a star on top is used to denote its transpose; thus, for example:

$$\mathcal{C}^*(v,w) \equiv \mathcal{C}(w,v) \text{ and } \mathcal{K}^*(v,w) \equiv \mathcal{K}(w,v). \tag{4.1}$$

In addition, $g_\partial(\cdot)$ and $j_\Sigma(\cdot)$ are linear functionals defined pointwise on $\partial\Omega$ and Σ, respectively, whose values at any $w \in \hat{D}_2(\Omega)$ are written as $g_\partial(w)$ and $j_\Sigma(w)$. Given any function $v \in \hat{D}_1(\Omega)$, $\mathcal{B}(v,\cdot)$ and $\mathcal{J}(v,\cdot)$ denote the linear functionals whose values at any $w \in \hat{D}_2(\Omega)$ are $\mathcal{B}(v,w)$ and $\mathcal{J}(v,w)$, respectively.

The general *boundary value problem with prescribed jumps* (BVPJ) to be considered is defined by

$$\mathcal{L}u^i = f^i_\Omega \equiv \mathcal{L}u^i_\Omega; \text{ in } \Omega_i, i = 1,\ldots,E, \tag{4.2a}$$

$$\mathcal{B}(u,\cdot) = g_\partial(\cdot) \equiv \mathcal{B}(u_\partial,\cdot); \text{ in } \partial\Omega, \tag{4.2b}$$

and

$$\mathcal{J}(u,\cdot) = j_\Sigma(\cdot) \equiv \mathcal{J}(u_\Sigma,\cdot); \text{ in } \Sigma, \tag{4.2c}$$

where f^i_Ω $(i = 1,\ldots,E)$, and the linear functionals $g_\partial(\cdot)$ and $j_\Sigma(\cdot)$ are given. They constitute the data of the problem and may be defined by means of some auxiliary functions: $u_\Omega \in \hat{D}_1(\Omega), u_\partial \in \hat{D}_1(\Omega), u_\Sigma \in \hat{D}_1(\Omega)$. An important property is that, in applications, such functions can be constructed solving local problems, if necessary. For simplicity, in what follows it is assumed that the BVPJ possess a unique solution fulfilling Eqs. (4.2), and the notation $u \in \hat{D}_1(\Omega)$ is reserved for it.

As an illustration, consider the general elliptic equation of second order. It is assumed that the coefficients of the differential operator may have jump discontinuities across the internal boundary Σ. Then, the boundary value problem with prescribed jumps to be considered is

$$\mathcal{L}u^i \equiv -\nabla\cdot(\underline{\underline{\mathbf{a}}}\cdot\nabla u^i) + \nabla\cdot(\underline{\mathbf{b}}u^i) + \mathbf{c}u^i = f^i_\Omega, \text{ in } \Omega_i, i = 1,\ldots,E, \tag{4.3a}$$

subjected to Dirichlet boundary conditions

$$u = u_\partial, \text{ on } \partial\Omega \tag{4.3b}$$

and jump conditions

$$[u] = [u_\Sigma] \text{ and } [\underline{\mathbf{a}}_n\cdot\nabla u] = [\underline{\mathbf{a}}_n\cdot\nabla u_\Sigma], \text{ on } \Sigma. \tag{4.3c}$$

Here $\underline{\mathbf{a}}_n \equiv \underline{\underline{\mathbf{a}}}\cdot\underline{n}$. When the coefficients of the differential operator are continuous, it may be seen that the conditions of Eq. (4.3c), are equivalent to prescribing the jump of the function and its normal derivative. Define the bilinear functions

$$\mathcal{B}(u,w) \equiv u(\underline{\mathbf{a}}_n\cdot\nabla w + \mathbf{b}_n w) \text{ and } \mathcal{J}(u,w) \equiv \dot{w}[\underline{\mathbf{a}}_n\cdot\nabla u] - [u](\overline{\underline{\mathbf{a}}_n\cdot\nabla w + b_n w}), \tag{4.4}$$

and the linear functions $g(\cdot)$ and $j(\cdot)$ by $g(\cdot) \equiv \mathcal{B}(u_\partial,\cdot)$ together with $j(\cdot) \equiv \mathcal{J}(u_\Sigma,\cdot)$. Then, the BVPJ of Eqs. (4.3) take the form given by Eqs. (4.2).

V. GENERAL VARIATIONAL FORMULATIONS

By the definition of formal adjoint, a vector valued-bilinear function $\underline{\mathcal{D}}(u,w)$ exists, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^*w \equiv \nabla\cdot\underline{\mathcal{D}}(u,w). \tag{5.1}$$

It will also be assumed that

$$\underline{\mathcal{D}}(u,w)\cdot\underline{n} = \mathcal{B}(u,w) - \mathcal{C}(w,u), \text{ on } \partial\Omega \tag{5.2a}$$

$$-[\underline{\mathcal{D}}(u,w)]\cdot\underline{n} = \mathcal{J}(u,w) - \mathcal{K}(w,u), \text{ on } \Sigma. \tag{5.2b}$$

Applying the generalized divergence theorem [55, 56], this implies the following Green–Herrera formula [13, 20, 57, 58]:

$$\int_\Omega w\mathcal{L}u\,dx - \int_{\partial\Omega}\mathcal{B}(u,w)\,dx - \int_\Sigma \mathcal{J}(u,w)\,dx = \int_\Omega u\mathcal{L}^*w\,dx - \int_{\partial\Omega}\mathcal{C}^*(u,w)\,dx - \int_\Sigma \mathcal{K}^*(u,w)\,dx.$$

A weak formulation of the BVPJ is

$$\int_\Omega w\mathcal{L}u\,dx - \int_{\partial\Omega}\mathcal{B}(u,w)\,dx - \int_\Sigma \mathcal{J}(u,w)\,dx = \int_\Omega w\mathcal{L}u_\Omega\,dx - \int_{\partial\Omega}\mathcal{B}(u_\partial,w)\,dx - \int_\Sigma \mathcal{J}(u_\Sigma,w)\,dx, \forall w\in\hat{D}_2(\Omega),$$

which, in view of Eq. (5.3), is equivalent to

$$\int_\Omega u\mathcal{L}^*w\,dx - \int_{\partial\Omega}\mathcal{C}^*(u,w)\,dx - \int_\Sigma \mathcal{K}^*(u,w)\,dx = \int_\Omega w\mathcal{L}u_\Omega\,dx - \int_{\partial\Omega}\mathcal{B}(u_\partial,w)\,dx - \int_\Sigma \mathcal{J}(u_\Sigma,w)\,dx, \forall w\in\hat{D}_2(\Omega).$$

Eqs. (5.4) supply two alternative and equivalent variational formulations of the BVPJ. The first one is referred as the "*variational formulation in terms of the data of the problem*," while the second one is referred as the "*variational formulation in terms of the sought information*."

Introduce the following notation:

$$\langle Pu,w\rangle = \int_\Omega w\mathcal{L}u\,dx; \langle Q^*u,w\rangle = \int_\Omega u\mathcal{L}^*w\,dx \tag{5.5a}$$

$$\langle Bu,w\rangle = \int_{\partial\Omega}\mathcal{B}(u,w)\,dx; \langle C^*u,w\rangle = \int_{\partial\Omega}\mathcal{C}^*(u,w)\,dx \tag{5.5b}$$

$$\langle Ju,w\rangle = \int_\Sigma \mathcal{J}(u,w)\,dx; \langle K^*u,w\rangle = \int_\Sigma \mathcal{K}^*(u,w)\,dx. \tag{5.5c}$$

With these definitions, each one of P, B, J, Q^*, C^*, and K^* are real-valued bilinear functionals defined on $\hat{D}_1(\Omega)X\hat{D}_2(\Omega)$, and a more brief expression for Eq. (5.3) is the identity

$$P - B - J \equiv Q^* - C^* - K^*. \tag{5.6}$$

When the definitions

$$f \equiv Pu_\Omega; \cdots g \equiv Bu_\partial; \cdots j \equiv Ju_\Sigma \tag{5.7}$$

are adopted, Eqs. (5.4) can also be written as equalities between linear functionals:

$$(P - B - J)u = f - g - j; \tag{5.8a}$$

and

$$(Q - C - K)^* u = f - g - j. \tag{5.8b}$$

Notice that Eqs. (5.8) may be written as

$$\langle (P - B - J)u, w\rangle = \langle f - g - j; w\rangle; \cdots \forall w \in D_2 \tag{5.9a}$$

and

$$\langle (Q - C - K)^* u, w\rangle = \langle f - g - j; w\rangle; \cdots \forall w \in D_2, \tag{5.9b}$$

respectively. These equations exhibit more clearly their variational character.

Generally, the definitions of $\mathcal{B}, \mathcal{C}, \mathcal{J}$, and $\mathcal{K}$ depend on the kind of boundary conditions and the "*smoothness criterion*" of the specific problem. However, for the case when the coefficients of the differential operators are continuous, Herrera [18, 20, 57] has given very general formulas for $\mathcal{J}$ and $\mathcal{K}$. They are

$$\mathcal{J}(u, w) \equiv -\underline{\mathcal{D}}([u], \dot{w}) \cdot \underline{n}, \text{ and } \mathcal{K}(w, u) \equiv \underline{\mathcal{D}}(\dot{u}, [w]) \cdot \underline{n}. \tag{5.10}$$

The fact that they fulfill Eq. (5.2b) is easy to verify, when use is made of the algebraic identity:

$$[\underline{\mathcal{D}}(u, w)] \equiv \underline{\mathcal{D}}([u], \dot{w}) + \underline{\mathcal{D}}(\dot{u}, [w]). \tag{5.11}$$

The case when $\hat{D}_1(\Omega) = \hat{D}_2(\Omega) \equiv \hat{D}(\Omega)$, the differential operator $\mathcal{L}$ is formally symmetric and, in addition, $B = C$ and $J = K$ are referred to as the *symmetric case*. Since $\mathcal{L}^* = \mathcal{L}$ and, therefore, $P = Q$, it is seen that the bilinear functional P-B-J is symmetric; i.e., $P - B - J \equiv (P - B - J)^*$, and $P - B - J \equiv Q - C - K$ by virtue of Eq. (5.6). Using these facts, it is clear that in the symmetric case the variational principles of Eqs. (5.9) are derivable from the potential

$$X(\hat{u}) \equiv \langle \tfrac{1}{2}(P - B - J)\hat{u} - (f - g - j), \hat{u}\rangle \equiv \langle \tfrac{1}{2}(Q - C - K)\hat{u} - (f - g - j), \hat{u}\rangle, \tag{5.12}$$

where $\hat{u}$ is any function belonging to $\hat{D}(\Omega)$. More precisely, Eqs. (5.9) can be written as

$$\langle X'(u), w\rangle = 0; \forall w \in D_2(\Omega), \tag{5.13}$$

where $X'(u)$ is the derivative of the functional $X(u)$, or more briefly, as $X'(u) = 0$. In particular, when $P - B - J \equiv Q - C - K$ is positive definite, in a subspace $N \subset \hat{D}(\Omega)$ such that $u \in N$, then the functional $X(\hat{u})$ yields a minimum principle for the BVPJ; i.e., $X(\hat{u})$ attains a minimum at $\hat{u} \in N \subset \hat{D}(\Omega)$ if and only if $\hat{u} = u$.

In the case of the general elliptic equation of second order, in which the differential operator $\mathcal{L}$ is given by Eq. (4.3a), one has

$$\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{\mathbf{a}}} \cdot \nabla w) - \underline{\mathbf{b}} \cdot \nabla w + \mathbf{c}w, \tag{5.14}$$

for the formal adjoint, and Eq. (5.1) is fulfilled with

$$\underline{\mathcal{D}}(u, w) \equiv \underline{\underline{\mathbf{a}}} \cdot (u\nabla w - w\nabla u) + \underline{b}uw, \tag{5.15}$$

and, therefore, Eq. (5.10) yields

$$\mathcal{J}(u,w) \equiv \dot{w}[\underline{\mathbf{a}}_n \cdot \nabla u] - [u](\dot{\overline{\underline{\mathbf{a}}_n \cdot \nabla w + b_n w}}),$$
$$\mathcal{K}(w,u) \equiv \dot{u}[\underline{\mathbf{a}}_n \cdot \nabla u] - [w](\dot{\overline{\underline{\mathbf{a}}_n \cdot \nabla u - b_n u}}). \tag{5.16}$$

Using these functions, one can apply the previous definitions and obtain the two equivalent weak formulations of Eqs. (5.9). In particular, when $\underline{\mathbf{b}} \equiv 0$ a symmetric case is obtained, because the differential operator $\mathcal{L}$ is formally symmetric and, in addition, $B = C$ and $J = K$. Even more, let $N \subset \hat{D}(\Omega)$ be the subset of functions that satisfy $v \equiv 0$, on $\partial\Omega$, and $[v] \equiv 0$, on Σ, then it can be shown that $P - B - J \equiv Q - C - K$ is positive definite in $N \subset \hat{D}(\Omega)$. Thus, a minimum principle is applicable when the sought solution is continuous across Σ and vanishes on $\partial\Omega$.

VI. SCOPE

The generality of the methodologies presented in this article is great, because they are applicable to any partial differential equation or system of such equations, which are linear, independently of its type. The coefficients of the operators can also be discontinuous across the internal boundary Σ. To illustrate the wide applicability of theory, the following cases are next presented: the general elliptic equation of second order, the biharmonic equation, the Stokes problem, and the equations of equilibrium of linear elasticity.

A. Second-Order Elliptic Operators

The formulas here presented are applicable when the coefficients of the differential operators are discontinuous across the internal boundary Σ:

a) $\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{\mathbf{a}}} \cdot \nabla u) + \nabla \cdot (\underline{b} u) + cu$, while $\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{\mathbf{a}}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw$.
b) $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega)$,
c) $\hat{D}_1 \equiv \hat{D}_2 \equiv \hat{D} \equiv H^2(\Omega_1) \oplus H^2(\Omega_2) \oplus \cdots \oplus H^2(\Omega_E)$;
d) $\underline{\mathcal{D}}(u,w) \equiv \underline{\underline{\mathbf{a}}} \cdot (u\nabla w - w\nabla u) + \underline{b} uw$
e) $\mathcal{B}(u,w) \equiv u(\underline{\mathbf{a}}_n \cdot \nabla w + \mathbf{b}_n w)$ and $\mathcal{C}(u,w) \equiv w\underline{\mathbf{a}}_n \cdot \nabla u$, where $\underline{\mathbf{a}}_n = \underline{\underline{\mathbf{a}}} \cdot \underline{n}$ and $\underline{\mathbf{b}}_n = \underline{\mathbf{b}} \cdot \underline{n}$.
f) $\mathcal{J}(u,w) \equiv \dot{w}[\underline{\mathbf{a}}_n \cdot \nabla u] - [u](\dot{\overline{\underline{\mathbf{a}}_n \cdot \nabla w + b_n w}})$, and $\mathcal{K}(w,u) \equiv \dot{u}[\underline{\mathbf{a}}_n \cdot \nabla w + \mathbf{b}_n w] - [w](\dot{\overline{\underline{\mathbf{a}}_n \cdot \nabla u}})$,
g) Boundary conditions $u = u_\partial$
h) Jump conditions $[u] = [u_\Sigma]$ and $[\underline{\mathbf{a}}_n \cdot \nabla u] = [\underline{\mathbf{a}}_n \cdot \nabla u_\Sigma]$
i) Data on the external boundary: $u = u_\partial$
j) Data on the internal boundary: $[u_\Sigma]$ and $[\underline{\mathbf{a}}_n \cdot \nabla u_\Sigma]$
k) Sought information on the external boundary: $\underline{\mathbf{a}}_n \cdot \nabla u$
l) Sought information on the internal boundary: $\dot{u}$ and $(\dot{\overline{\underline{\mathbf{a}}_n \cdot \nabla u}})$.

B. Biharmonic Equation

a) $\mathcal{L}u \equiv \Delta^2 u$ and $\mathcal{L}^* w \equiv \Delta^2 w$
b) $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^4(\Omega)$,

c) $\hat{D}_1 \equiv \hat{D}_2 \equiv \hat{D} \equiv H^4(\Omega_1) \oplus H^4(\Omega_2) \oplus \cdots \oplus H^4(\Omega_E)$;

d) $\underline{\mathcal{D}}(u,w) \equiv w\nabla\Delta u - u\nabla\Delta w + \Delta w\nabla u - \Delta u\nabla w$

e) $\mathcal{B}(u,w) \equiv \Delta w \dfrac{\partial u}{\partial n} - u\,\dfrac{\partial \Delta w}{\partial n}$

f) $\mathcal{C}(w,u) \equiv \Delta u \dfrac{\partial w}{\partial n} - w\dfrac{\partial \Delta u}{\partial n}$

g) $\mathcal{J}(u,w) \equiv [u]\overline{\dfrac{\partial \dot{\Delta} w}{\partial n}} - \dot{w}\left[\dfrac{\partial \Delta u}{\partial n}|\right] + [\Delta u]\overline{\dfrac{\dot{\partial w}}{\partial n}} - \overline{\dot{\Delta} w}\left[\dfrac{\partial u}{\partial n}|\right]$,

h) $\mathcal{K}(w,u) \equiv [w]\overline{\dfrac{\partial \dot{\Delta} u}{\partial n}} - \dot{u}\left[\dfrac{\partial \Delta w}{\partial n}|\right] + [\Delta w]\overline{\dfrac{\partial \dot{u}}{\partial n}} - \overline{\dot{\Delta} u}\left[\dfrac{\partial w}{\partial n}|\right]$,

i) Data on the external boundary: $u, \partial u/\partial n u$

j) Data on the internal boundary: $[u], [\partial u/\partial n], [\Delta u]$ and $[\partial \Delta u/\partial n]$

k) Sought information on the external boundary: Δu and $\partial \Delta u/\partial n$

l) Sought information on the internal boundary: $\dot{\tilde{u}}, \overline{\dot{\partial u/\partial n}}, \overline{\dot{\Delta} u}$ and $\overline{\dot{\partial \Delta u/\partial n}}$.

C. Stokes Problems

The system of equations to be considered is

$$-\Delta \underline{u} + \nabla p = 0; \nabla \cdot \underline{u} = 0 :$$

a) Let $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega) \oplus H^1(\Omega)$, and adopt the notation $\tilde{u} \equiv (\underline{u}, p)$ whenever $\tilde{u} \in D(\Omega)$.

b) Define the vector valued differential operator $\underline{\mathcal{L}}$ by $\underline{\mathcal{L}} \cdot \tilde{u} \equiv (-\Delta \underline{u} + \nabla p, -\nabla \cdot \underline{u})$.

c) Then $\underline{\mathcal{L}}$ is self-adjoint and, writing $\tilde{w} \equiv (\underline{w}, q)$, one has $\tilde{w} \cdot \underline{\mathcal{L}} \cdot \tilde{u} - \tilde{u} \cdot \underline{\mathcal{L}} \cdot \tilde{w} \equiv \nabla \cdot (\underline{u} \cdot \nabla \underline{w} - \underline{w} \cdot \nabla \underline{u} + p\underline{w} - q\underline{u})$.

d) Thus: $\underline{\mathcal{D}}(\tilde{u}, \tilde{w}) \equiv \underline{u} \cdot (\nabla \underline{w} - q\underline{I}) - \underline{w} \cdot (\nabla \underline{u} - p\underline{I})$

e) $\mathcal{B}(\tilde{u}, \tilde{w}) \equiv \underline{u} \cdot \left(\dfrac{\partial \underline{w}}{\partial n} - qn\right)$

f) $\mathcal{C}(\tilde{w}, \tilde{u}) \equiv \underline{w} \cdot \left(\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}\right)$

g) $\mathcal{J}(\tilde{u}, \tilde{w}) \equiv \dot{\underline{w}} \cdot \left[\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}|\right] - [\underline{u}] \cdot \overline{\dfrac{\partial \underline{w}}{\partial n} - q\underline{n}}$

h) $\mathcal{K}(\tilde{w}, \tilde{u}) \equiv \dot{\underline{u}} \cdot \left[\frac{\partial \underline{w}}{\partial n} - q\underline{n}|\right] - [\underline{w}] \cdot \left(\overline{\frac{\partial \underline{u}}{\partial n} - p\underline{n}}\right)$.

i) Data on the external boundary: $\underline{u}$

j) Data on the internal boundary: $[\underline{u}]$ and $\left[\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}|\right]$

k) Sought information at the external boundary: $\dfrac{\partial \underline{u}}{\partial n} - p\underline{n}$

l) Sought information at the internal boundary: $\dot{\underline{u}}$ and $\overline{\frac{\partial \underline{u}}{\partial n} - p\underline{n}}$.

D. Equations of Elasticity

Let $D_1(\Omega) \equiv D_2(\Omega) \equiv D(\Omega) \equiv H^2(\Omega) \oplus H^2(\Omega) \oplus H^2(\Omega)$, and define for every $\underline{u} \equiv (u_1, u_2, u_3) \in D(\Omega)$: $t_{ij}(\overline{u}) \equiv C_{ijpq}\frac{\partial u_p}{\partial x_q}$, where as usual it is assumed that the elastic tensor possesses the following symmetries: $C_{ijpq} = C_{jipq} = C_{ijqp}$.

a) Define the vector valued differential operator $\underline{\mathcal{L}}$ by $\underline{\mathcal{L}} \cdot \underline{u} \equiv -\nabla \cdot \underline{t}(\underline{u})$, whose adjoint is $\underline{\mathcal{L}}^* \cdot \underline{w} \equiv -\nabla \cdot \underline{t}(\underline{w})$

b) $\underline{\mathcal{D}}(\underline{u}, \underline{w}) \equiv \underline{u} \cdot \underline{t}(\underline{w}) - \underline{w} \cdot \underline{t}(\underline{u})$

c) $\mathcal{B}(\underline{u}, \underline{w}) \equiv \underline{u} \cdot \underline{t}(\underline{w}) \cdot \underline{n}$

d) $\mathcal{C}(\underline{u}, \underline{w}) \equiv \underline{w} \cdot \underline{t}(\underline{u}) \cdot \underline{n}$

e) $\mathcal{J}(\underline{u}, \underline{w}) \equiv \dot{\underline{w}} \cdot [\underline{t}(\underline{u})] \cdot \underline{n} - [\underline{u}] \cdot \dot{\overline{\underline{t}(\underline{w})}} \cdot \underline{n}$

f) $\mathcal{K}(\underline{w}, \underline{u}) \equiv \dot{\underline{u}} \cdot [\underline{t}(\underline{w})] \cdot \underline{n} - [\underline{w}] \cdot \dot{\overline{\underline{t}(\underline{u})}} \cdot \underline{n}$

m) Data on the external boundary: $\underline{u}$

n) Data on the internal boundary: $[\underline{u}]$ and $[\underline{t}(\underline{u})] \cdot \underline{n}$

o) Sought information at the external boundary: $\underline{t}(\underline{u}) \cdot \underline{n}$

p) Sought information at the internal boundary: $\dot{\underline{u}}$ and $\dot{\overline{\underline{t}(\underline{u})}} \cdot \underline{n}$.

VII. TREFFTZ METHODS

As mentioned in the Introduction, the method proposed originally by Trefftz, in 1926 [1], has been generalized, and to be precise the following definition is proposed.

Definition 7.1. *Let $\Pi = \{\Omega_1, \ldots, \Omega_E\}$ be a partition and for every $i = 1, \ldots, E$, let $\mathcal{H}_i$ be defined by the condition that $u_H^i \in \mathcal{H}_i$ if and only if $u_H^i \in \hat{D}(\Omega_i)$ and $\mathcal{L}u_H^i = 0$, in Ω_i. In addition, let $\mathcal{H} \equiv \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_E$. Then the problem of finding $u_H^i \in \mathcal{H}_i, i = 1, \ldots, E$, such that*

$$u = \sum_{i=1}^{E} u_\Omega^i + \sum_{i=1}^{E} u_H^i = u_\Omega + u_H \tag{7.1}$$

is the solution of the Boundary Value Problem with Prescribed Jumps, is referred as the Trefftz Problem.

Observe that the solution of Trefftz problem, $u_H \equiv \sum_{i=1}^{E} u_H^i$, is unique necessarily, because $u_H = u - u_\Omega$ and, by assumption the solution $u \in \hat{D}(\Omega)$ is unique, while $u_\Omega \in \hat{D}(\Omega)$ is a datum. The notation $u_H \in \mathcal{H}$, is reserved for it. Notice, however, that the definition of $u_H \in \mathcal{H}$, changes if the function u_Ω, used to specify the right-hand side of the differential equation, is modified.

Two approaches for constructing the solution of the Trefftz problem are considered; methods derived from one or the other are referred as *direct* (*Trefftz–Jirousek*) *and indirect methods* (*Trefftz–Herrera*), respectively. In the direct approach, the local solutions are put together in such a way that the boundary conditions and prescribed jumps on Σ are fulfilled, and the search for u_H is guided by such requirements. In the Trefftz–Herrera method, on the other hand, special test or weighting functions are applied to obtain enough information on the internal boundary Σ to define well-posed problems in each one of the subregions $\Omega_i, i = 1, \ldots, E$. This condition assures that the solution can be reconstructed locally from the information available.

A second point of view for classifying Trefftz methods, which is independent of the first one, yields two other wide groups: overlapping and nonoverlapping methods; i.e., the same classes that are considered when studying domain decomposition methods [24–31]. Because these two points of view are independent of each other, they may be combined to give four types of methods: direct nonoverlapping, direct overlapping, indirect nonoverlapping, and indirect overlapping.

For numerical applications, it is relevant to observe that the number of degrees of freedom is minimal when superfluous information is eliminated; i.e., when only information that is essential

for defining local well-posed problems is retained. Generally, to eliminate superfluous information and handle essential information only in both Trefftz–Jirousek and Trefftz–Herrera methods, it is necessary to resort to overlapping methods, as shown in the following sections.

VIII. VARIATIONAL FORMULATIONS OF TREFFTZ METHODS

In what follows, $\hat{u}_H$ stands for any function belonging to $\mathcal{H} \equiv N_P \subset \hat{D}_1(\Omega)$. For direct methods, a basic variational formulation derived from Eq. (5.9a) is that a function $\hat{u}_H \in N_P$ is a solution of the Trefftz problem if and only if

$$-\langle (B+J)\hat{u}_H, w\rangle = \langle (B+J)u_\Omega, w\rangle - \langle g+j; w\rangle; \cdots \forall w \in D_2. \tag{8.1}$$

The condition $\forall w \in \hat{D}_2(\Omega)$ may be relaxed. Indeed, generally it is enough to require that Eq. (8.1) be satisfied for $\forall w \in N_Q \subset \hat{D}_2(\Omega)$.

For the symmetric case, discussed in Section V, one can define the functional

$$Y(\hat{u}_H) \equiv -\tfrac{1}{2}\langle (B+J)\hat{u}_H, \hat{u}_H\rangle + \langle g+j-(B+J)u_\Omega, \hat{u}_H\rangle, \tag{8.2}$$

where $\hat{u}_H$ is any function belonging to $N_P \equiv N_Q \subset \hat{D}(\Omega)$. Then, $\hat{u}_H \equiv \sum_{i=1}^{E} \hat{u}_H^i$ is a solution of the Trefftz problem if and only if $Y'(\hat{u}_H) = 0$. When the bilinear functional $-(B+J)$ is positive definite in $N_P \equiv N_Q \subset \hat{D}(\Omega)$, the functional $Y(\hat{u}_H)$ yields a minimum principle. Observe that a sufficient condition for $-(B+J)$ to be positive definite in $N_P \equiv N_Q \subset \hat{D}(\Omega)$ is that P-B-J be positive definite in $\hat{D}(\Omega)$. More generally, when $N \subset N_P \equiv N_Q \subset \hat{D}(\Omega)$ is a subspace in which $-(B+J)$ is positive definite and $u_H \in N$, then $Y(\hat{u}_H)$ attains a minimum at $\hat{u}_H \in N$ if and only if $\hat{u}_H = u_H$.

IX. TREFFTZ–JIROUSEK METHODS

The application of direct methods to one-dimensional problems is relatively straight-forward [59]. However, their application in several dimensions is considerably more complicated. The search for the solution of Trefftz problem, $u_H \in \mathcal{H}$, can be done in several ways. In his pioneering work, Jirousek [2, 3] applied variational principles that were specific for the differential equations considered; they are particular cases of the general variational principles of Section VIII. However, other procedures can be, and have been used, for example, collocation in the internal boundaries [50]. The application of least-squares also possesses great generality and has the additional advantage of yielding symmetric and positive definite matrices [48, 49].

In the case of direct overlapping methods, it is possible to apply two different approaches; one that is more direct and the other one that is less direct. In the latter one, the base functions are used to impose a *compatibility condition* from which the global system of equations is derived [59]. In addition, in this manner information about the sought solution is obtained, which is enough to formulate well-posed local problems. This procedure handles only essential information, so that the number of degrees of freedom is minimal.

In the first and more direct of the overlapping methods, the reduction in the number of degrees of freedom is achieved using base functions, which fulfill some of the jump conditions, such as continuity conditions, from the start. These kinds of weighting functions are easy to construct, if numerical methods are used to build them. But this is not feasible, in most cases, when systems of

analytical solutions are applied. The construction of TH-complete systems of weighting functions is discussed and illustrated in Section XI.

Consider, as an example, the BVPJ for the general elliptic equation of second order, defined by Eqs. (4.3). When a direct method is applied, one can use the variational principle in terms of the data of the problem of Eq. (5.4a), with the help of Eqs. (4.4). Another possibility is to apply least squares to the quantities $[\hat{u} - u_\partial]$, on $\partial\Omega$, together with $[\hat{u} - u_\Sigma]$ and $[\underline{\mathbf{a}}_n \cdot \nabla\hat{u} - \underline{\mathbf{a}}_n \cdot \nabla u_\Sigma]$, on Σ, where $\hat{u} \in D$ is any trial function. When the coefficients of the differential operator are continuous, it is simpler to replace this latter quantity by $[\partial\hat{u}/\partial n - \partial u_\Sigma/\partial n]$. In addition, the following observation must be made: when the numerical method that is applied to solve the local problems is collocation [60], the boundary condition $\hat{u} = u_\partial$, on $\partial\Omega$, can be fulfilled by the trial functions from the start, so that the least squares on $[\hat{u} - u_\partial]$ need not be applied. Also, when overlapping methods are used, it is easy to construct trial functions that fulfill the condition $[\hat{u} - u_\Sigma]$, on Σ (see Section XII), and this reduces the number of degrees of freedom of the matrices of the global system of equations. As has already been mentioned, this is not possible when analytical solutions are applied.

To illustrate the alternative overlapping procedure [59], which in some sense is only *semi-direct*, consider the equation $\mathcal{L}u = 0$ in an interval of the real line, where $\mathcal{L}$ is a second-order differential operator. Let $x_i \in (x_{i-1}, x_{i+1})$, then $u(x_i)$ depends linearly on $u(x_{i-1})$ and $u(x_{i+1})$. Indeed, $u(x_i) = \varphi_i^-(x_i)u(x_{i-1}) + \varphi_i^+(x_i)u(x_{i+1})$, and this equation constitutes a three-diagonal system of equations, whose coefficients can be obtained solving locally, by collocation, a pair of boundary value problems in the interval (x_{i-1}, x_{i+1}): $\mathcal{L}\varphi_i^- = \mathcal{L}\varphi_i^+ = 0$, subjected to $\varphi_i^-(x_{i-1}) = \varphi_i^+(x_{i+1}) = 1$ and $\varphi_i^-(x_{i+1}) = \varphi_i^+(x_{i-1}) = 0$.

The generalization of this method to more complicated problems and to several dimensions is presented in [59]. In particular, it is shown that this is the basic procedure behind the well-known Schwarz alternating method [61].

X. TREFFTZ–HERRERA METHODS

The indirect Trefftz methods have been introduced and developed by Herrera and his collaborators [4–23]. They stem from the following observation [13]: when the method of weighted residuals is applied–-and this includes the Finite Element Method (FEM)–-the information about the sought solution contained in an approximate one is determined by the system of weighting functions that are applied, and it is independent of the base functions that are used. A convenient strategy is to apply test functions of a special kind, *specialized test functions*, with the property of yielding information in the boundaries $\partial\Omega$ and Σ, exclusively. To solve the Trefftz problem, i.e., to recover $u_H^i, i = 1, \ldots, E$, it is necessary to have enough information on Σ for defining well-posed problems in each one of the subregions Ω_i $(i = 1, \ldots, E)$, because this determines the functions u_H^i. In addition, Herrera's algebraic theory of boundary value problems supplies a very effective framework for guiding the construction of such test functions [19].

The point of view just mentioned yields the following interpretation of FEM formulations: the system of test functions that are applied determine the information about the sought solution contained in an approximate one, while the base functions interpolate (or extrapolate) such information. A strategy, which in some sense is optimal [62], is to obtain enough information to define well-posed problems locally and then use the solutions of these local problems, instead of base functions, to extending the information that is available, because this is the most efficient way of performing this function. Sometimes the specialized test functions have been referred as

Optimal Test Functions [5], and the extension of the information by means of the solution of the local boundary value problems, as Optimal Interpolation [62].

By inspection of Eqs. (5.5), it can be recognized that the information about the solution $u \in D$ is given by Q^*u, in the interior of the subregions Ω_i $(i = 1, \ldots, E)$; it is given by C^*u, in the outer boundary $\partial\Omega$; and it is given by K^*u, in the internal boundary Σ. Jirousek [48] refers to $\Sigma \cup \partial\Omega$ as the "generalized boundary." A first step to derive Trefftz–Herrera procedures is to manipulate the variational formulation in terms of the sought information of Eq. (5.9b) in such a way as to leave information in the generalized boundary, exclusively. This requires eliminating Q^*u in that equation, and can be achieved by taking special weighting functions such that $Qw = 0$. This yields

$$-\langle (C + K)^*u, w\rangle = \langle f - g - j; w\rangle; \cdots \forall w \in N_Q \subset \hat{D}_2(\Omega). \tag{10.1}$$

Generally, one is interested only in part of the information contained in $(C + K)^*u$; so it is useful to introduce a decomposition of the bilinear functional $C + K$ and write

$$C + K \equiv S + R, \tag{10.2}$$

where S is chosen so that S^*u is precisely "*the sought information*."

Definition 10.1. *Given R and S, which fulfill Eq.* (10.2), *let $\tilde{u} \in \hat{D}_1(\Omega)$ be such that there exists a solution, $u \in \hat{D}_1(\Omega)$, of the BVPJ with the property that $S^*\tilde{u}$ is the sought information*; *i.e.*,

$$S^*\tilde{u} = S^*u. \tag{10.3}$$

Then $\tilde{u} \in \hat{D}_1(\Omega)$ is said to contain "the sought information."

In what follows, the symbol $\tilde{u} \in D$ is reserved for functions that contain the sought information. Let $N_Q \subset \hat{D}_2(\Omega)$ and $N_R \subset \hat{D}_2(\Omega)$ be the null subspaces of Q and R, respectively. To formulate a necessary and sufficient condition for a function for $\hat{u} \in D_1$ to contain the sought information, it is necessary to define a concept of completeness, similar to that introduced by the author in 1980 [12] and which has been very effective in the study of complete families [63].

Definition 10.2. *A subset of weighting functions, $\mathcal{E} \subset N_Q \cap N_R$, is said to be TH-complete for S^* when, for any $\hat{u} \in \hat{D}_1(\Omega)$, one has*

$$\langle S^*\hat{u}, w\rangle = 0, \forall w \in \mathcal{E} \Rightarrow S^*\hat{u} = 0. \tag{10.4}$$

Clearly, a necessary and sufficient condition for the existence of TH-complete systems is that $N_Q \cap N_R$ be itself a TH-complete system.

Theorem 10.1. *Let $\mathcal{E} \subset N_Q \cap N_R$ be a system of weighting functions, TH-complete for S^*, and assume that there exists $u \in \hat{D}_1(\Omega)$, a solution of the BVPJ. Then a necessary and sufficient condition for $\hat{u} \in D_1$ to contain the sought information is that*

$$-\langle S^*\hat{u}, w\rangle = \langle f - g - j, w\rangle, \cdots \forall w \in \mathcal{E}. \tag{10.5}$$

Proof. The necessity of this condition can be derived using Eqs. (10.1) and (10.2). To prove the sufficiency, observe that the necessary condition just mentioned implies that for the solution $u \in \hat{D}_1(\Omega)$, whose existence is assumed, one has

$$-\langle S^*u, w\rangle = \langle f - g - j, w\rangle, \cdots \forall w \in \mathcal{E}. \tag{10.6}$$

Therefore, if $\hat{u} \in D_1$ fulfills Eq. (10.5), then Eqs. (10.6) and (10.5) together, imply

$$\langle S^* \hat{u}, w \rangle = \langle S^* u, w \rangle, \cdots \forall w \in \mathcal{E}. \tag{10.7}$$

Hence, Eq. (10.3) is TH-complete, because $\mathcal{E} \subset N_Q \cap N_R$.

In numerical applications of indirect methods, Theorem 10.1 yields the basic system of equations whose solution is sought. To obtain a formulation that is suitable for both elliptic and time-dependent problems, it is necessary, in addition, to introduce decompositions of the bilinear functionals C and K. These are

$$C = C^S + C^C \text{ and } K \equiv K^S + K^C. \tag{10.8}$$

When time-dependent problems are considered, Ω is a space-time region and the final state of the system that is modeled by the partial differential equation lies in the outer boundary, $\partial\Omega$. Thus, a suitable choice of C^S permits handling this situation. In applications to elliptic problems, on the other hand, it is frequently convenient to define $S \equiv K^S$, so that $R \equiv C + K^C$. In this case, the information on the external boundary is eliminated and the sought information $S^* u \equiv K^{S^*} u$ contains information in the internal boundary, exclusively. The choice $K^C = 0$ leads to nonoverlapping indirect methods, while $K^C \neq 0$ corresponds to overlapping indirect methods.

A corollary of Theorem 10.1 is that, when $u_P \in \hat{D}_1(\Omega)$ is such that $Pu_P = f$ and $Bu_P = g$, then Eq. (9.5) can be replaced by

$$-\langle K^{S^*} \hat{u}, w \rangle = -\langle K^{S^*} u_P, w \rangle + \langle J(u_P - u_\Sigma), w \rangle; \forall w \in \mathcal{E} \tag{10.9}$$

in that theorem. In applications, this result may be used to replace an expression involving integrals in the interior of the subregions $\Omega_i, (i = 1, \ldots, E)$, by one that involves integrals over the internal boundary only. In our discussions, it has been assumed that $u_P \in \hat{D}_1(\Omega)$ is a datum. Generally, when this is not available from the start, its construction requires solving local boundary value problems in each one of the subregions $\Omega_i, (i = 1, \ldots, E)$, exclusively.

When $-K^{S^*}$ is symmetric in $N \equiv N_Q \cap N_R$, the variational principles of Eqs. (10.5) and (10.9) can be derived from the potentials

$$Z(\hat{u}) = -\frac{1}{2}\langle K^S \hat{u}, \hat{u} \rangle - \langle f - g - j, \hat{u}_H \rangle \tag{10.10}$$

and

$$\tilde{Z}(\hat{u}) \equiv -\frac{1}{2}\langle K^S \hat{u}, \hat{u} \rangle + \langle K^{S^*} u_P, w \rangle - \langle J(u_P - u_\Sigma), w \rangle, \tag{10.11}$$

respectively. When it is positive definite on $N \equiv N_Q \cap N_R$, then a minimum principle holds, in addition.

As in Section V, let us illustrate the TH-method by applying it to the elliptic BVPJ of second-order of Eqs. (4.3). Since in the case of elliptic problems a convenient strategy is to concentrate all the sought information on Σ, a first possibility is to set $S \equiv K$; i.e., $K^C \equiv 0$ and $R \equiv C$, so that the test functions are required to fulfill $\mathcal{L}^* w = 0$, in each one of the subregions separately, together with $w = 0$, on $\partial\Omega$. Observe that no matching condition between the subregions is imposed. Thus, in this case the method is nonoverlapping. The sought information is $\dot{u}$ and $\overline{\partial u / \partial n}$ on Σ. This information is excessive, in the sense that when it is used to define local boundary value problems they turn out to be over-determined.

Indeed, it would be enough, for example, to prescribe $\dot{u}$ on Σ to have a well-posed problem, if that information is complemented with the data on $\partial\Omega$ (see Fig. 1). Thus, one strategy that

permits handling essential information only is to concentrate all the information in $\dot{u}$ on Σ. This is achieved, if one sets

$$\langle K^C w, u\rangle = -\int_\Sigma [w](\overline{\underline{\mathbf{a}}_n \cdot \nabla \dot{u} - \underline{b}_n u})\, dx \tag{10.12}$$

in Eq. (10.8), together with $C^S \equiv 0$ (and $C^C \equiv C$). In this case, the requirement $w \in N_R$ implies the condition $[w] = 0$ on Σ, in addition to the previous conditions. Thus, such functions must be continuous across Σ. The construction, by collocation, of test functions fulfilling these conditions is not difficult, but requires putting together several subregions. Thus, diminishing the information that has to be handled, and so the degrees of freedom, leads to an overlapping method. In Sections XI and XII TH-complete systems of functions and procedures for their construction are presented.

XI. TH-COMPLETE SYSTEMS

The application of Trefftz methods requires having available systems of functions that are complete for the space $\mathcal{H} \equiv \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_E$. A criterion of completeness that has permitted applying the function theoretic approach as an effective means to solving boundary value problems [63] is due to Herrera [12] and an extension of that concept was given in Section X: it is referred to as TH-completeness (Trefftz–Herrera completeness; it has also been referred as C-completeness or T-completeness). This section is devoted to discuss briefly the methods available for developing such systems of functions, which can be grouped into two broad categories: analytical and numerical.

The classical approach is based on analytical methods and a thorough account may be found in a book by Begehr and Gilbert [63]. The function theoretic method was pioneered by Bergman [64] and Vekua [65], and further developed by Colton [66–68], Gilbert [69–70], Kracht–Kreyszig [71], Lanckau [72], and others. The author has supplied such systems for Stokes problem [73], Helmholtz equation (in [23] it is shown that a system of plane waves possess that property) and the biharmonic equation [74]. Other means of constructing them use fundamental solutions and spectral methods, among others (see [63]).

The most general procedures for constructing TH-complete systems are, by far, numerical methods. Any such method can be applied, but collocation is quite suitable [60]. One has to construct families of solutions that span suitable spaces of boundary conditions, as illustrated in the next section, in the case of the general elliptic equation of second order.

XII. CONSTRUCTION OF TH-COMPLETE SYSTEMS BY COLLOCATION

Consider again the BVPJ for the general elliptic equation of second order. For simplicity, a rectangular region is considered and the subregions of the partition are rectangles [Fig. 2(a)].

For a system of functions to be TH-complete, for each subregion Ω_i, the traces of its members must span $H^0(\partial\Omega_i)$. When collocation methods are used in the construction of TH-complete systems, one may choose a system of functions that spans $H^0(\partial\Omega_i)$ and then solve a family of boundary value problems, taking as boundary conditions each one of the members of this system. A convenient choice for the system of functions that spans $H^0(\partial\Omega_i)$ is a system of piecewise polynomials. A linear basis of such a system of polynomials may be obtained by taking the four bilinear polynomials that have the property of assuming the value 1 at one corner of each

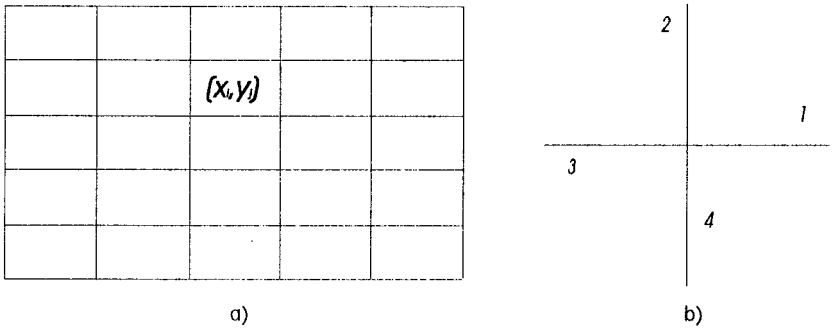

FIG. 2. (a) Rectangular domain decomposition of Ω; (b) numbering of internal boundaries.

given quadrilateral and vanishing at all the other three corners, together with all the piecewise polynomials defined on $\partial\Omega_i$, which vanish identically at three sides of the quadrilateral.

For constructing a TH-complete system, fulfilling a continuity condition, collocation methods are also quite suitable. With each internal node (x_i, y_j) a region Ω_{ij}, which is the union of the four rectangles of the original partition that surround that node, is associated. Then, the system of subregions $\{\Omega_{ij}\}$ is overlapping. The boundary of Ω_{ij} is $\partial\Omega_{ij}$, while that part of Σ laying in the interior of Ω_{ij} is denoted by $\sum_{ij}$ [Fig. 2(b)]; it is constituted by four segments, which are numbered as indicated in Fig. 2(b) and form a cross. Given any subregion Ω_{ij}, a system of functions that fulfill $\mathcal{L}^* w = 0$ in its interior and vanish on $\partial\Omega_{ij}$ is developed. Using the numbering already introduced with each interior node (x_i, y_j), five groups of weighting functions are constructed, which are identified by the conditions satisfied on $\sum_{ij}$:

Group 0—This group is made of only one function, which is linear in each one of the four segments of $\sum_{ij}$ and $w_{ij}(x_i, y_j) = 1$.

For $N = 1, \ldots, 4$, they are defined by:

Group N—The restriction to interval "N" of Fig. 1(b) is a polynomial in x, which vanishes at the end points of interval "N." For each degree ≥ 2, there is only one, linearly independent, such polynomial.

The support of the test function of Group 0, is the whole square, while those weighting functions associated with Groups 1–4 have as support rectangles that can be obtained from each other by rotation, as shown in Fig. 3.

Of course, when developing numerical algorithms for the solution of boundary value problems, only a few terms of these TH-complete systems are taken; it could be only one (see [4]). Generally, the order of precision of the resulting scheme depends on the number of terms taken.

XIII. CONCLUSIONS

A large class of numerical methods has been formulated, whose research thus far has been quite incomplete. The conclusion is drawn that a lot of work should be done on them, because they have great potential in the theory and practice of numerical methods for partial differential equations. The framework presented here would be valuable for this purpose. In particular, collocation methods could be greatly improved along these lines [32].

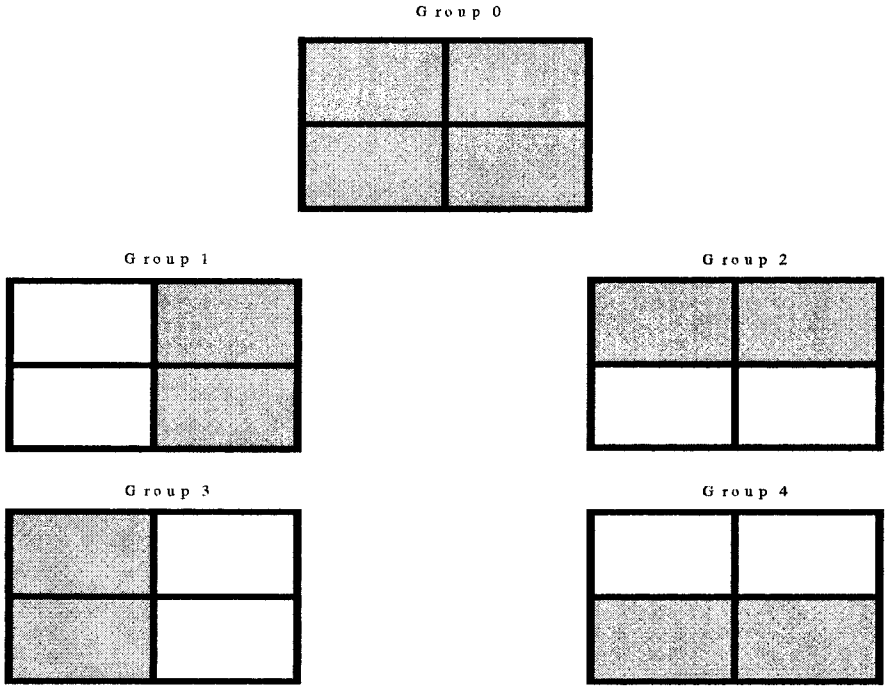

FIG. 3. The five groups of weighting functions, according to their supports.

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