General Theory of Domain Decomposition: Beyond Schwarz Methods

Ismael Herrera, Robert Yates
Instituto de Geofísica
Universidad Nacional Autonoma de Mexico (UNAM)
Apartado Postal 22-582
14000, Mexico, D.F.

Received 3 September 2000; accepted 6 February 2001

Recently, Herrera presented a general theory of domain decomposition methods (DDM). This article is part of a line of research devoted to its further development and applications. According to it, DDM are classified into direct and indirect, which in turn can be subdivided into overlapping and nonoverlapping. Some articles dealing with general aspects of the theory and with indirect (Trefftz–Herrera) methods have been published. In the present article, a very general direct-overlapping method, which subsumes Schwarz methods, is introduced. Also, this direct-overlapping method is quite suitable for parallel implementation.

Keywords: domain decomposition; collocation; Trefftz-Herrera; parallel computing; LAM

I. INTRODUCTION

Domain decomposition methods have received much attention in recent years [1], mainly because they supply very effective means for parallelizing computational models of continuous systems. In addition, it is useful to analyze numerical methods for partial differential equations from a domain-decomposition perspective, since the ideas related to domain decomposition are quite basic for them. According to J. L. Lions, one must distinguish between *a priori* domain decomposition and *a posteriori* domain decomposition, depending on whether the domain decomposition is applied before or after discretization, respectively.

Recently, Herrera has developed ''an *a priori* general theory of domain decomposition'' (a preliminary presentation of its ideas may be found in [2]), in which the general strategy consists in obtaining information about the sought solution in the internal boundary (Σ), which separates the subdomains from each other, sufficient for defining well-posed problems in each one of the

Correspondence to: Prof. Ismael Herrera, P.O. Box 22-582, 14000 Mexico D.F., Mexico (e-mail: iherrera@servidor.unam.mx)
Contract grant sponsor: DGAPA (UNAM)
Contract grant number: 1N105998

subdomains (to be referred as ''local problems''). In this manner, the solution can be reconstructed by solving such kind of problems exclusively. There are two general procedures that can be followed for gathering the information on Σ; they are referred as *direct* and *indirect* methods. The distinguishing feature of indirect methods, as developed by Herrera and coworkers [2–10], is the use of specialized test functions to obtain such information. On the other hand, methods such as that first proposed by Trefftz in 1926 [11], and more recently developed by Jirousek and coworkers [12, 13], consists in piecing together, just as *bricks*, the local solutions of the differential equations to build the global solution. These procedures, when seen as a manner of gathering information in the internal boundary, lead to the formulation of compatibility conditions that have to be satisfied by the sought solution and from which the required information on Σ can be derived. This is the strategy followed in the direct methods of Herrera's theory.

When the procedures described above are put to work, the main difference between direct and indirect methods is that, when the former are applied, the information on Σ is derived using local solutions of differential equations formulated in terms of the original differential operators, i.e., those occurring in the original formulation of the boundary value problem, while specialized test functions that fulfill equations formulated in terms of the adjoint differential operators are used in the latter. Each one of these procedures, in turn, can be applied in overlapping or nonoverlapping domain decompositions. Thus, four categories of methods are considered in the theory, which are derived by combining these two classes of domain decompositions with the two procedures for gathering the information on Σ, which have been described.

The scope of the theory is quite wide, since it is applicable to any partial (or ordinary) differential equation or system of such equations, which is linear, and possibly with discontinuous coefficients. It can also be applied to nonlinear problems via Newton's method, or alike, in the usual manner. In addition, the generality of Herrera's theory is considerably enhanced, because it is formulated in special classes of Sobolev spaces, developed specifically for this kind of application, in which trial and test functions are fully discontinuous. Even more, the methodologies apply to a general boundary value problem formulated in this setting, in which jumps of the functions and their derivatives are prescribed, on the internal boundaries of the domain decompositions.

A line of research is being carried out devoted to develop further the basic theory and the methods derived from it. The general theory is presented in [2] as well as in a forthcoming publication [13], and indirect methods are discussed in [7–9]. Research on direct methods is initiated with the present article, in which the basic ideas of a very general direct overlapping method, which subsumes Schwarz methods (including multiplicative and additive methods [14–16]), are introduced. In agreement with the general strategy of Herrera's theory, one of its essential features is the use that is made of solutions of the original differential equation—to be contrasted with those of the adjoint differential equation, in the case of indirect methods—to derive compatibility conditions from which the sought information is obtained. Because of this characteristic and because the domain decomposition is overlapping, it probably would be appropriate to call it a *semidirect overlapping method*. Although, as mentioned, the method so obtained contains Schwarz methods, as particular cases (see Section V), its formulation is more direct and general and turns out to be quite suitable for parallel processing.

This article is organized as follows: Section II is devoted to explain, through a simple example, the basic idea that motivates the semidirect overlapping method of this article. The notation that is used is presented in Section III. The problem formulation with prescribed jumps of Herrera's general theory is given in Section IV, while the version corresponding to the most general elliptic equation of second order in an arbitrary number of independent variables is explained in Section V. In this latter section, the Schwarz alternating method is derived from the equations of our method as a possible solution procedure that can only be applied when suitable conditions are

satisfied by the differential operators. As an illustration of the general semidirect overlapping method introduced here, in Section VI, it is applied to a one-dimensional version of the boundary value problem with prescribed jumps of the function and its derivative. Then, in Section VII, the orthogonal collocation procedure is used to construct the local solutions needed in several steps of the process of solution. The corresponding error analysis is carried out in Section VIII and the results of the numerical experiments that were performed are presented in Section IX. Finally, the conclusions are discussed in Section X.

II. BASIC IDEA

Consider the problem of solving the ordinary differential equation of second-order

$$\mathcal{L}u \equiv -\frac{d}{dx}\left(a\frac{du}{dx}\right) + \frac{d}{dx}(bu) + cu = 0 \tag{2.1}$$

in the interval $(0, \ell)$ of the real line, subjected to Dirichlet boundary conditions. Let, in addition, $\{x_0 = 0, x_1, \ldots, x_E = \ell\}$ be a partition of $(0, \ell)$. Let $x_i \in (x_{i-1}, x_{i+1})$, then $u(x_i)$ depends linearly on $u(x_{i-1})$ and $u(x_{i+1})$. Indeed, it can be seen that

$$u(x_i) = \varphi_i^-(x_i)u(x_{i-1}) + \varphi_i^+(x_i)u(x_{i+1}), \tag{2.2}$$

where $\varphi_i^-(x)$ and $\varphi_i^+(x)$ are solutions of two "local boundary value problems." More specifically, they satisfy the differential Eq. (2.1), subjected to the boundary conditions:

$$\varphi_i^-(x_{i-1}) = \varphi_i^+(x_{i+1}) = 1 \quad \text{and} \quad \varphi_i^-(x_{i+1}) = \varphi_i^+(x_{i-1}) = 0. \tag{2.3}$$

When i runs from 1 to E-1, Eq. (2.2) constitutes a tridiagonal system of equations from which the values at the internal nodes $u(x_i)$, $i = 1, \ldots, E-1$ can be obtained. Observe that a slight modification of this procedure permits the treatment of any other kind of boundary conditions.

The system of Eqs. (2.2) is exact, but its actual application requires knowing the coefficients $\varphi_i^-(x_i)$ and $\varphi_i^+(x_i)$; these in turn can be obtained by solving, in the interval $(x_{i-1},\ x_{i+1})$, the boundary value problem defined above. Except for very simple differential equations, this requires resorting to approximate numerical methods. When this approach is used as a discretization method, the numerical procedures to be applied for accomplishing this task are rather simple, since the interval of definition of such problem is small and generally only a few degrees of freedom need to be handled. In particular, a suitable method is collocation and, in this manner, a *nonstandard version of the method of collocation* is obtained for the global problem. Of course, the numerical method used to solve the local problems does not need to be collocation, and different numerical procedures can be derived in this manner, depending on this choice.

III. NOTATIONS

In what follows, unless otherwise explicitly stated, Ω is an open, bounded region. The closure of any set Ω is denoted by $\bar{\Omega}$. The (outer) boundary of Ω is denoted by $\partial\Omega$.

As usual, a collection $\Pi = \{\Omega_1, \ldots, \Omega_E\}$ of open subregions Ω_i $(i = 1, \ldots, E)$ of Ω, is said to be a *partition* of Ω, *iff*

$$\text{i.} \quad \Omega_i \cap \Omega_j = \phi, \text{ for every } i \neq j, \text{ and}$$
$$\text{ii.} \quad \bar{\Omega} = \bigcup_{i=1}^{i=E} \bar{\Omega}_i.$$

In addition, the partitions considered throughout this article are assumed to be such that the subregions Ω_i are *manifolds with corners*, in the sense of Loomis and Sternberg [17] (see also [18]). The manifold $\cup_{i=1}^{E} \partial\Omega_i$ is referred to as the "*generalized boundary*," while the "*internal boundary*" of Ω—to be denoted by Σ—is defined as the closed complement of $\partial\Omega$, considered as a subset of the generalized boundary. Observe that the internal boundary, and the generalized boundary as well, are concepts whose definition is relative to both the region Ω and the partition Π. Thus, when deemed necessary, the notation $\Sigma(\Omega, \Pi)$, which is more precise, is used.

A partition $\Pi' = \{\Omega'_1, \ldots, \Omega'_{E'}\}$ of Ω, is said to be a *subpartition* of Π, when for each given any $i = 1, \ldots, E'$, there is a subset of natural numbers $\mathcal{N}(i) \subset \{1, \ldots, E\}$, such that

$$\bar{\Omega}'_i = \bigcup_{j \in \mathcal{N}(i)} \bar{\Omega}_j. \tag{3.1}$$

Given a subpartition $\Pi' = \{\Omega'_1, \ldots, \Omega'_{E'}\}$ of Π, the function $\mu' : \{1, \ldots, E\} \to \{1, \ldots, E'\}$ is defined, for every $j = 1, \ldots, E$, by the equation $\mu'(j) = i$, whenever $j \in \mathcal{N}(i)$. Two partitions: $\Pi' = \{\Omega'_1, \ldots, \Omega'_{E'}\}$ and $\Pi'' = \{\Omega''_1, \ldots, \Omega''_{E''}\}$, respectively, are said to be *conjugate* with respect to a partition Π, when:

i. They are both subpartitions of Π;
ii. In the measure of the generalized boundary, the sets

$$\Sigma' - \left\{ \Sigma' \cap \left(\bigcup_{i=1}^{i=E'} \Omega''_i \right) \right\} \quad \text{and} \quad \Sigma'' - \left\{ \Sigma'' \cap \left(\bigcup_{i=1}^{i=E''} \Omega'_i \right) \right\} \tag{3.2}$$

have measure zero;
iii. And

$$\Sigma' \cup \Sigma'' = \Sigma. \tag{3.3}$$

Here, $\Sigma' = \Sigma(\Omega, \Pi')$ and $\Sigma'' = \Sigma(\Omega, \Pi'')$.

When $\Pi' = \{\Omega'_1, \ldots, \Omega'_E\}$ and $\Pi'' = \{\Omega''_1, \ldots, \Omega''_{E''}\}$ are conjugate partitions, in addition to the mapping μ' introduced above, it is necessary to consider a second mapping μ'', associated with Π'', which is defined correspondingly.

The formulation and treatment of boundary problems with prescribed jumps requires the introduction of a special class of Sobolev spaces in which some of their functions are fully discontinuous. In general, the definition of such spaces depends on the differential operator considered; in the case of the elliptic problems to be discussed in Section V, such spaces are defined by $\hat{H}^s(\Omega, \Pi) \equiv H^s(\Omega_1) \oplus \cdots \oplus H^s(\Omega_E)$, with $s \geq 0$. Elements $\hat{u} \equiv \{u^1, \ldots, u^E\} \in \hat{H}^s(\Omega, \Pi)$ are sequences of functions such that $u^i \in H^s(\Omega_i), i = 1, \ldots, E$. For $s \geq 0, u^i \in H^0(\Omega_i), i = 1, \ldots, E, H^s(\Omega_i) \subset H^0(\Omega_i)$ and the sequence $\{u^1, \ldots, u^E\}$ defines a unique function $u \in H^0(\Omega)$ with the property that $u|_{\Omega_i} = u^i, i = 1, \ldots, E$. The mapping $\hat{H}^s(\Omega, \Pi) \to H^0(\Omega)$, so defined, is referred as the *natural immersion* of $\hat{H}^s(\Omega, \Pi)$ into $H^0(\Omega)$.

When $v^i \in H^s(\Omega_i)$ and $s > \frac{1}{2}$, then the trace of v^i belongs to $H^{s-1/2}(\partial\Omega_i)$ [18]. Let be $\Sigma_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j$, then the traces on Σ_{ij} of each pair v^i and v^j are well defined, and using this pair of traces, the following notation is introduced:

$$v_+ \equiv \text{ Trace of } (v^i), \tag{3.4a}$$

when Ω_i lies on the positive side of Σ_{ij} and

$$v_- \equiv \text{Trace of } (v^i) \tag{3.4b}$$

otherwise. The *jump* of u across Σ_{ij} is defined by

$$[v] \equiv v_+ - v_-, \tag{3.5a}$$

and the *average* by

$$\dot{v} \equiv \frac{1}{2}\,(v_+ + v_-). \tag{3.5b}$$

Observe that under the natural immersion $H^0(\Omega) \supset \hat{H}^s(\Omega, \Pi) \supset H^s(\Omega)$, and it can be shown (see [10]) that, when s is an integer, an element $\hat{u} \in \hat{H}^s(\Omega, \Pi)$ belongs to $H^s(\Omega)$ if and only if the jump of the normal derivatives up to order $s-1$ vanishes on Σ.

IV. GENERAL PROBLEM WITH PRESCRIBED JUMPS

The idea explained in Section II can be generalized to obtain a procedure of general applicability capable of solving a very general class of boundary value problems for which jumps are prescribed in the internal boundaries. Given Ω, the region of definition of the problem, and a partition of Ω (or domain-decomposition) $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$, let $\Sigma \equiv \Sigma(\Omega, \Pi)$ be the internal boundary. Then, using a notation similar to that presented in [19], the general form of such a boundary value problem with prescribed jumps (BVPJ) is

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega; \quad \text{in } \Omega_i, \quad i = 1, \ldots, E \tag{4.1a}$$

$$B_j u = B_j u_\partial \equiv g_j; \quad \text{in } \partial\Omega \tag{4.1b}$$

and

$$[J_k u] = [J_k u_\Sigma] \equiv j_k; \quad \text{in } \Sigma, \tag{4.1c}$$

where the B_j's and J_k's are certain differential operators (the j's and k's run over suitable finite ranges of natural numbers) and $u_\Omega \equiv (u_\Omega^1, \ldots, u_\Omega^E)$, together with u_∂ and u_Σ are given functions of the space of trial functions. In addition, f_Ω, g_j and j_k may be defined by Eq. (4.1).

It must be emphasized that the scope of the methodology presented in this and the following articles of this series is quite wide, since in principle it is applicable to any partial differential equation or system of such equations that is linear, independently of its type. Although every kind of equation has its own peculiarities, which require special developments that have to be treated separately, among the equations that can be included, we would like to mention explicitly the following:

Single Equation

1. Elliptic

i) Second-Order
ii) Higher-Order
- Biharmonic.

2. Parabolic

i) Heat Equation
ii) Diffusive Transport (ELLAM).

3. Hyperbolic.

Systems of Equations

i) Stokes Problems
ii) Mixed Methods (Raviart-Thomas)
iii) Elasticity.

V. ELLIPTIC EQUATION OF SECOND ORDER

In this section, we describe the overlapping direct method under investigation, for the second-order differential equation of elliptic type, when the problem is defined in a space of arbitrary dimension. For definiteness, only boundary conditions of Dirichlet type are presented, but the procedure is applicable to any kind of boundary conditions for which the problem is well posed, as was done in [7]. With the notation introduced in Section III, a region Ω and a partition $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$ of Ω, are considered. The solution to the boundary value problem with prescribed jumps in this case, is sought in a Sobolev space of the kind introduced in that section. More precisely, a function $u \in \hat{H}^2(\Omega) \equiv H^2(\Omega_1) \oplus \cdots \oplus H^2(\Omega_E)$ is sought, such that

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b} u) + cu = f_\Omega; \quad \text{in } \Omega_{i,i=1,\ldots,E}, \tag{5.1}$$

subjected to the boundary conditions

$$u = u_\partial; \quad \text{in } \partial\Omega \tag{5.2}$$

and jump conditions

$$[u] = j^0 = [u_\Sigma]; \quad \text{on } \Sigma \tag{5.3}$$

$$[\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n} = j^1 = [\underline{\underline{a}} \cdot \nabla u_\Sigma] \cdot \underline{n}; \quad \text{on } \Sigma. \tag{5.4}$$

The above formulation and the methodology that follows applies even if the coefficients of the differential operator are discontinuous. In the particular case when the coefficients are continuous, the jump condition of Eq. (5.4), in the presence of (5.3), is equivalent to

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right]; \quad \text{on } \Sigma. \tag{5.5}$$

In what follows, it is assumed that this problem possesses one and only one solution. Conditions under which this assumption is fulfilled are discussed elsewhere.

In the Theorem that follows, two conjugate partitions $\Pi' = \{\Omega'_1, \ldots, \Omega'_{E'}\}$ and $\Pi'' = \{\Omega''_1, \ldots, \Omega''_{E''}\}$, as well as the mappings μ' and μ'' associated to them in the manner explained in Section III are considered. Also, the notations $\Sigma' \equiv \Sigma(\Omega, \Pi')$ and $\Sigma'' \equiv \Sigma(\Omega, \Pi'')$ are adopted.

Theorem. *Let $\Pi' = \{\Omega'_1, \ldots, \Omega'_{E'}\}$ and $\Pi'' = \{\Omega''_1, \ldots, \Omega''_{E''}\}$ be two partitions of Ω that are conjugate with respect to Π, and let $\{\hat{u}^1, \ldots, \hat{u}^{E'}\}$ and $\{\breve{u}^1, \ldots, \breve{u}^{E''}\}$ be two families of functions, such that*

1) *For every $i = 1, \ldots, E'$, the function $\hat{u}^i \in \hat{H}^2(\Omega'_i, \Pi')$ fulfills Eqs.* (5.1)–(5.3) *and satisfies Eq.* (5.4) *in Σ';*
2) *For every $j = 1, \ldots, E''$, the function $\breve{u}^j \in \hat{H}^2(\Omega''_j, \Pi'')$ fulfills Eqs.* (5.1)–(5.3) *and satisfies Eq.* (5.4) *in Σ''.*

Then, define $u' = (u'^1, \ldots, u'^E) \in \hat{H}^2(\Omega, \Pi)$ and $u'' = (u''^1, \ldots, u''^E) \in \hat{H}^2(\Omega, \Pi)$, by

$$u'^i = \hat{u}^{\mu'(i)}|_{\Omega_i}; \quad i = 1, \ldots, E \tag{5.6a}$$

and

$$u''^j = \hat{u}^{\mu''(j)}|_{\Omega_j}; \quad i = 1, \ldots, E. \tag{5.6b}$$

Under these assumptions the following statements are equivalent:

i. *u' and u'' are solutions of the BVPJ in Ω;*

ii.

$$u' \equiv u''; \tag{5.7}$$

iii.

$$\dot{u}'(\underline{x}) = \dot{u}''(\underline{x}), \quad \textit{a.e. on } \Sigma = \Sigma' \cup \Sigma''. \tag{5.8}$$

Proof. That (i) implies (ii) is immediate, because of the assumption of uniqueness of solution for the BVPJ. That (ii) implies (iii) follows from the jump condition of Eq. (5.3) and the definition of the average across Σ. Equation (5.8) in the presence of Eq. (5.3) in turn imply

$$u'(\underline{x}+) = \dot{u}'(\underline{x}) + \frac{1}{2}[u'] = \dot{u}'(\underline{x}) + \frac{1}{2}j^0 = \dot{u}''(\underline{x}) + \frac{1}{2}j^0 = \dot{u}''(\underline{x}) + \frac{1}{2}\,[u''] = u''(\underline{x}+). \tag{5.9}$$

Recalling that $\Sigma = \Sigma' \cup \Sigma''$ and that $\Sigma \cup \partial\Omega = \cup_{i=1}^{E} \partial\Omega_i$, it is seen that the boundary values of u' and u'' coincide on each side of Σ. This, together with the assumed uniqueness of solution of the boundary value problem at each one of the subregions of the partition, imply $u' \equiv u''$.

It is timely to point out the connections between the method discussed in this article and the Schwarz alternating methods. Indeed, this latter approach can be derived from Eqs. (5.1)–(5.3) and (5.8), when an iterative procedure is adopted for fulfilling Eq. (5.8). To show this, let u^{2n} $(n = 0, 1, \cdots)$ and u^{2n+1} $(n = 0, 1, \cdots)$ satisfy Eqs. (5.1)–(5.3), together with

$$\overbrace{u^{2n+1}}^{\bullet} = \overbrace{u^{2n}}^{\bullet}, \quad \text{on } \Sigma', (n = 0, 1, \cdots) \tag{5.10a}$$

and

$$\overbrace{u^{2n+2}}^{\bullet} = \overbrace{u^{2n+1}}^{\bullet}, \quad \text{on } \Sigma'', (n = 0, 1, \cdots). \tag{5.10b}$$

Then, if the sequence u^{2n} $(n = 0, 1, \cdots)$ converges to $\hat{u}$, while the sequence u^{2n+1} $(n = 0, 1, \cdots)$ converges to $\breve{u}$, one has $\hat{u} = \breve{u} = u$, and this function fulfills Eqs. (5.1)–(5.3), together with Eq. (5.8). In the cases when a variational principle can be applied, the projection interpretation is possible and the Schwarz alternating procedure can be derived (see, for example, [14–16]). ∎

VI. ONE-DIMENSIONAL PROBLEM

The one dimensional version of the problem described in Section V corresponds to the two-point boundary value problem of the general differential equation of second order. Let be $\Omega \equiv (0, l)$ and $\Pi \equiv \{(0, x_1), (x_1, x_2), \ldots, (x_{E-1}, x_E = l)\}$. Then

$$\mathcal{L}u \equiv -\frac{d}{dx}\left(a\frac{du}{dx}\right) + \frac{d}{dx}(bu) + cu = f_\Omega, \text{ in } (x_{i-1}, x_i), i = 1, \ldots, E. \tag{6.1}$$

Assume that the boundary and jump conditions are:

$$u(0) = g_{\partial 0}, \quad u(\ell) = g_{\partial \ell} \tag{6.2a}$$

and

$$[u] = j_i^0 \equiv [u_\Sigma] \text{ and } \left[\frac{du}{dx}\right] = j_i^1 \equiv \left[\frac{du_\Sigma}{dx}\right]; \quad i = 1, \ldots, E-1, \tag{6.2b}$$

respectively. In addition, it is assumed that the Dirichlet problem is well posed in each one of the subintervals and that $u(x) \in H^2(\Omega)$ is the unique solution of this BVPJ, in Ω.

In every subinterval $(x_{i-1}, x_{i+1}), i = 1, \ldots, E-1$, define the function $u^i(x)$ to be the restriction of $u(x)$ to Ω_i. Then, for every $i = 1, \ldots, E-1, u^i(x)$, is the unique solution of a boundary value problem with prescribed jumps defined in the subinterval (x_{i-1}, x_{i+1}), which is derived from the following conditions:

$$\mathcal{L}u^i = f_\Omega, \text{ in } (x_{i-1}, x_{i+1}); \quad i = 1, \ldots, E-1, \tag{6.3a}$$

$$[u^i]_i = j_i^0; \quad \left[\frac{du^i}{dx}\right]_i = j_i^1; \quad i = 1, \ldots, E-1, \tag{6.3b}$$

$$u^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0; \quad i = 2, \ldots, E-1, \tag{6.3c}$$

$$u^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0; \quad i = 1, \ldots, E-2, \tag{6.3d}$$

$$u^1(0) = u(0) = g_{\partial 0}, \tag{6.3e}$$

and

$$u^{E-1}(\ell) = u(\ell) = g_{\partial \ell}. \tag{6.3f}$$

Let the functions $u_H^i(x)$ and $u_P^i(x)$ be defined in (x_{i-1}, x_{i+1}) by the following conditions:

$$\mathcal{L}u_H^i = 0, \text{ in } (x_{i-1}, x_{i+1}); \quad i = 1, \ldots, E \tag{6.4a}$$

$$[u_H^i]_i = \left[\frac{du_H^i}{dx}\right]_i = 0; \quad i = 1, \ldots, E-1 \tag{6.4b}$$

$$u_H^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0; \quad i = 2, \ldots, E-1 \tag{6.4c}$$

$$u_H^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0; \quad i = 1, \ldots, E-2 \tag{6.4d}$$

$$u_H^1(x_0) = u(0) = g_{\partial 0}; \tag{6.4e}$$

and

$$u_H^{E-1}(x_E) = u(\ell) = g_{\partial \ell}; \tag{6.4f}$$

together with

$$\mathcal{L}u_P^i = f_\Omega, \text{ in } (x_{i-1}, x_i) \quad \text{and} \quad (x_i, x_{i+1}), \quad \text{separately,} \quad \text{for } i = 1, \ldots, E-1 \tag{6.5a}$$

$$u_P^i(x_{i-1}+) = u_P^i(x_{i+1}-) = 0, \quad \text{for } i = 1, \ldots, E-1 \tag{6.5b}$$

$$[u_P^i]_i = j_i^0 \quad \text{and} \quad \left[\frac{du_P^i}{dx}\right]_i = j_i^1; \quad i = 1, \ldots, E-1. \tag{6.5c}$$

Then, it can be verified that

$$u^i(x) = u_H^i(x) + u_P^i(x); \quad i = 1, \ldots, E-1. \tag{6.6}$$

Even more:

$$u_H^i(x) = u_H^i(x_{i-1}-)\phi_-^i(x) + u_H^i(x_{i+1}+)\phi_+^i(x), \tag{6.7}$$

when $\phi_-^i(x)$ and $\phi_+^i(x)$ are defined by the conditions:

$$\mathcal{L}\phi_+^i = 0; \quad \phi_+^i(x_{i-1}) = 0, \quad \phi_+^i(x_{i+1}) = 1 \tag{6.8a}$$

$$\mathcal{L}\phi_-^i = 0; \quad \phi_-^i(x_{i-1}) = 1, \quad \phi_-^i(x_{i+1}) = 0 \tag{6.8b}$$

together with

$$[\phi_+^i]_i = [\phi_-^i]_i = \left[\frac{d\phi_+^i}{dx}\right]_i = \left[\frac{d\phi_-^i}{dx}\right]_i = 0. \tag{6.8c}$$

From Eqs. (6.6), (6.7), and (6.4c,d), it follows that

$$\dot{u}(x_i) - \dot{u}_P^i(x_i) = \dot{u}_H^i(x_i) = \{\dot{u}(x_{i-1}) + \frac{1}{2}j_{i-1}^0\}\phi_-^i(x_i) + \{\dot{u}(x_{i+1}) - \frac{1}{2}j_{i+1}^0\}\phi_+^i(x_i). \tag{6.9}$$

Hence,

$$-\rho_-^i\dot{u}_{i-1} + \dot{u}_i - \rho_+^i\dot{u}_{i+1} = \mu_i; \quad i = 2, \ldots, E-2 \tag{6.10a}$$

$$\dot{u}_i - \rho_+^i\dot{u}_{i+1} = \mu_i; \quad i = 1 \tag{6.10b}$$

and

$$-\rho_-^i \dot{u}_{i-1} + \dot{u}_i = \mu_i; \quad i = E-1, \tag{6.10c}$$

where

$$\rho_-^i = \phi_-^i(x_i), \rho_+^i = \phi_+^i(x_i); \quad i = 1, \ldots, E-1 \tag{6.11a}$$

$$\mu_i = \frac{\rho_-^i}{2} j_{i-1}^0 + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2} j_{i+1}^0; \quad i = 2, \ldots, E-2 \tag{6.11b}$$

$$\mu_i = \rho_-^i g_{\partial 0} + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2} j_{i+1}^0, \quad i = 1 \tag{6.11c}$$

and

$$\mu_i = \frac{\rho_-^i}{2} j_{i-1}^0 + \dot{u}_P^i(x_i) + \rho_+^i g_{\partial \ell}; \quad i = E-1. \tag{6.11d}$$

Equations (6.10) constitute an $E-1$ tridiagonal system of equations, which can be solved for $\dot{u}_i$ $(i = 1, \ldots, E-1)$.

Once the averages $\dot{u}_i$ $(i = 1, \ldots, E-1)$ are known, it is possible to derive the information that is required to define well-posed problems in each one of the subintervals of the partition. Indeed, all that is required is to apply the identities

$$u(x_i+) \equiv \dot{u}_i + \frac{1}{2}[u]_i = \dot{u}_i + \frac{1}{2} j_i^0 \quad \text{and} \quad u(x_i-) \equiv \dot{u}_i - \frac{1}{2}[u]_i = \dot{u}_i - \frac{1}{2} j_i^0. \tag{6.12}$$

When these values are complemented with the prescribed boundary values of Eq. (6.2a), well-posed boundary value problems in each one of the subintervals of the partition can be defined. In this manner, all that is required to reconstruct the exact solution of the BVPJ is to solve such "local problems" in each one of the subintervals. Using the previous developments, one can apply Eqs. (6.6) and (6.7), to obtain $u(x)$ in the interior of the subintervals of the partition.

Up to now, all the developments have been exact. However, application of the system of Eqs. (6.10), as well as that of Eqs. (6.6) and (6.7), requires having available the functions ϕ_-^i, ϕ_+^i, and u_P^i, $(i = 1, \ldots, E-1)$. In general applications it is necessary to resort to numerical approximations for the construction of such functions and the system of equations so obtained is not exact any longer. Instead, its precision depends on the error introduced by the numerical procedure that is applied for solving the problems defined by Eqs. (6.4) and (6.5). A similar comment can be made with respect to the construction of the solution of the local boundary value problem whose solution is given by Eqs. (6.6) and (6.7).

VII. COLLOCATION PROCEDURE

The numerical procedure chosen in this article for the construction of the functions ϕ_-^i, ϕ_+^i, and u_P^i $(i = 1, \ldots, E-1)$, is orthogonal collocation, because of its precision and easiness to apply. First, in Subsection VIIA, it is used with two collocation points, using cubic polynomials, and later, in Subsection VIIB, the procedure is extended to an arbitrary number "n" of collocation points, in each subinterval of the partition, with polynomials of degree $G \equiv n+1$.

A. Method with Two Collocation Points in Each Subinterval

Define

$$\underline{H}_i = \{H^1_{i-1}, H^0_i, H^1_i, H^1_{i+1}\} \tag{7.1a}$$

$$\mathcal{L}\underline{H}_i = \{\mathcal{L}H^1_{i-1}, \mathcal{L}H^0_i, \mathcal{L}H^1_i, \mathcal{L}H^1_{i+1}\}, \tag{7.1b}$$

where H^0_i and $H^1_i, i = 0, \ldots, E$, are the cubic Hermite polynomials for which the function or its derivative takes the value one at x_i, respectively. In addition,

$$\underline{X}^i_- \equiv \{\phi^{i1}_-(x_{i-1})\phi^i_-(x_i), \phi^{i1}_-(x_i), \phi^{i1}_-(x_{i+1})\} \tag{7.2a}$$

and

$$\underline{X}^i_+ \equiv \{\phi^{i1}_+(x_{i-1}), \phi^i_+(x_i), \phi^{i1}_+(x_i), \phi^{i1}_+(x_{i+1})\}. \tag{7.2b}$$

Then, in view of the boundary conditions for ϕ^i_- and ϕ^i_+, these functions can be written as

$$\phi^i_- \equiv H^0_{i-1} + \underline{H}_i * \underline{X}^i_- \tag{7.3a}$$

and

$$\phi^i_+ \equiv H^0_{i+1} + \underline{H}_i * \underline{X}^i_+, \tag{7.3b}$$

or more explicitly

$$\begin{aligned}\phi^i_-(x) = H^0_{i-1}(x) + \phi^{i1}_-(x_{i-1})H^1_{i-1}(x) + \phi^i_-(x_i)H^0_i(x)\\ + \phi^{i1}_-(x_i)H^1_i(x) + \phi^{i1}_-(x_{i+1})H^1_{i+1}(x); \quad x_{i-1} < x < x_{i+1}\end{aligned} \tag{7.4a}$$

and

$$\begin{aligned}\phi^i_+(x) = H^0_{i+1}(x) + \phi^{i1}_+(x_{i-1})H^1_{i-1}(x) + \phi^i_+(x_i)H^0_i(x) + \phi^{i1}_+(x_i)H^1_i(x)\\ + \phi^{i1}_+(x_{i+1})H^1_{i+1}(x); \quad x_{i-1} < x < x_{i+1}.\end{aligned} \tag{7.4b}$$

The collocation equations are

$$\mathcal{L}\phi^i_-(x) = 0; \quad x = x^*_i, x^{**}_i, x^*_{i+1}x^{**}_{i+1} \tag{7.5a}$$

and

$$\mathcal{L}\phi^i_+(x) = 0; \quad x = x^*_i, x^{**}_i, x^*_{i+1}, x^{**}_{i+1}, \tag{7.5b}$$

where x^*_i and x^{**}_i, for $i = 1, \ldots, E$, are the Gaussian points of the subinterval (x_{i-1}, x_i). These systems of equations can be written as

$$\mathcal{L}\underline{H}_i(x) * \underline{X}^i_- = -\mathcal{L}H^0_{i-1}(x); \quad x = x^*_i, x^{**}_i, x^*_{i+1}, x^{**}_{i+1} \tag{7.6a}$$

and

$$\mathcal{L}\underline{H}_i(x) * \underline{X}^i_+ = -\mathcal{L}H^0_{i+1}(x); \quad x = x^*_i, x^{**}_i, x^*_{i+1}, x^{**}_{i+1}, \tag{7.6b}$$

respectively. Observe that

$$\begin{bmatrix} \mathcal{L}H^1_{i-1}(x^*_i) & \mathcal{L}H^0_i(x^*_i) & \mathcal{L}H^1_i(x^*_i) & 0 \\ \mathcal{L}H^1_{i-1}(x^{**}_i) & \mathcal{L}H^0_i(x^{**}_i) & \mathcal{L}H^1_i(x^{**}_i) & 0 \\ 0 & \mathcal{L}H^0_i(x^*_{i+1}) & \mathcal{L}H^1_i(x^*_{i+1}) & \mathcal{L}H^1_{i+1}(x^*_{i+1}) \\ 0 & \mathcal{L}H^0_i(x^{**}_{i+1}) & \mathcal{L}H^1_i(x^{**}_{i+1}) & \mathcal{L}H^1_{i+1}(x^{**}_{i+1}) \end{bmatrix} \tag{7.7}$$

is the matrix for both systems of equations.

For constructing the functions u_P^i define $\underline{Y}^i \equiv (Y_1^i, Y_2^i, Y_3^i, Y_4^i)$ and $\underline{j}_i \equiv (0, j_i^0, j_i^1, 0)$ with

$$Y_1^i \equiv u_P^{i1}(x_{i-1}), Y_2^i \equiv \dot{u}_P^i(x_i), Y_3^i \equiv \dot{u}_P^{i1}(x_i) \quad \text{and} \quad Y_4^j \equiv u_P^{i1}(x_{i+1}). \tag{7.8}$$

Then

$$u_P^i(x) = \underline{Y}^i * \underline{H}_i(x) - \frac{1}{2}\underline{j}_i * \underline{H}_i(x), \quad x_{i-1} \le x < x_i \tag{7.9a}$$

and

$$u_P^i(x) = \underline{Y}^i * \underline{H}_i(x) + \frac{1}{2}\underline{j}_i * \underline{H}_i(x), \quad x_i < x \le x_{i+1}. \tag{7.9b}$$

The collocation equations are

$$\mathcal{L}u_P^i(x) = f_\Omega(x), \text{ at } x = x_i^*, x_i^{**}, x_{i+1}^*, x_{i+1}^{**}, \tag{7.10}$$

and can be written as

$$\underline{Y}^i * \mathcal{L}\underline{H}_i(x) = f_\Omega(x) + \frac{1}{2}\underline{j}_i * \underline{\mathcal{L}H}_i(x), \quad \text{at } x = x_i^*, x_i^{**} \tag{7.11a}$$

together with

$$\underline{Y}^i * \mathcal{L}\underline{H}_i(x) = f_\Omega(x) - \frac{1}{2}\underline{j}_i * \mathcal{L}\underline{H}_i(x), \quad \text{at } x = x_{i+1}^*, x_{i+1}^{**}. \tag{7.11b}$$

Therefore, the matrix of the system is again given by Eq. (7.7).

B. Extension to Arbitrary Order

The extension of the second-order procedure just presented into an arbitrary order method is straightforward. Indeed, it is only required to modify Eqs. (5.3) to be

$$\phi_-^i \equiv H_{i-1}^0 + \underline{H}_i * \underline{X}_-^i + Q_-^i \tag{7.12a}$$

and

$$\phi_+^i \equiv H_{i+1}^0 + \underline{H}_i * \underline{X}_+^i + Q_+^i, \tag{7.12b}$$

where Q_-^i and Q_+^i are piecewise polynomials of degree G. Let

$$p^i(x) \equiv (x - x_{i-1})^2(x - x_i)^2, \tag{7.13}$$

then $Q_-^i(x) \equiv p^i(x)q_-^i(x)$, *in* (x_{i-1}, x_i) and $Q_-^i(x) \equiv p^{i+1}(x)q_-^i(x)$, *in* $(x_i, x_i + 1)$, where q_-^i is a piecewise polynomial in (x_{i-1}, x_{i+1}), which in turn is a polynomial of degree G-4 in each one of the subintervals (x_{i-1}, x_i) and (x_i, x_{i+1}) separately. The function Q_+^i is defined correspondingly. The definition of u_P^i, given Eqs. (7.9), has to be modified similarly. This was done adding on the right-hand member of Eqs. (7.9), a piecewise polynomial Q_P^i of degree G, which vanishes, together with its first-order derivative, at x_{i-1}, x_i, and x_{i+1}. The expression for Q_P^i that was used is

$$Q_P^i(x) \equiv p^i(x)q_P^i(x), \text{ in } (x_{i-1}, x_i) \tag{7.14a}$$

and

$$Q_P^i(x) \equiv p^{i+1}(x)q_P^i(x), \text{ in } (x_i, x_{i+1}), \tag{7.14b}$$

where q_P^i is a polynomial of degree G-4, in each one of the subintervals (x_{i-1}, x_i) and (x_i, x_{i+1}), separately.

For $i = 1, \ldots, E$, let x_i^α $(\alpha = 1, \ldots, G-1)$ be the Gaussian points of the subinterval (x_{i-1}, x_i). Then the system of equations to be fulfilled for the construction of ϕ_-^i and ϕ_+^i are

$$\mathcal{L}\underline{H}_i(x) * \underline{X}_-^i + \mathcal{L}Q_-^i = -\mathcal{L}H_{i-1}^0(x); \quad x = x_i^\alpha, x_{i+1}^\alpha, \alpha = 1, \ldots, G-1 \tag{7.15a}$$

and

$$\mathcal{L}\underline{H}_i(x) * \underline{X}_+^i + \mathcal{L}Q_+^i = -\mathcal{L}H_{i+1}^0(x); \quad x = x_i^\alpha, x_{i+1}^\alpha, \alpha = 1, \ldots, G-1. \tag{7.15b}$$

In addition, the system of equations for u_P^i is

$$\underline{Y}^i * \mathcal{L}\underline{H}_i(x) + \mathcal{L}Q_P^i = f_\Omega(x) + \frac{1}{2}\underline{j}_i * \underline{\mathcal{L}H}_i(x), \quad \text{at } x = x_i^\alpha, \alpha = 1, \ldots, G-1 \tag{7.16a}$$

together with

$$\underline{Y}^i * \mathcal{L}\underline{H}_i(x) + \mathcal{L}Q_P^i = f_\Omega(x) - \frac{1}{2}\underline{j}_i * \underline{\mathcal{L}H}_i(x), \quad \text{at } x = x_{i+1}^\alpha, \alpha = 1, \ldots, G-1. \tag{7.16b}$$

C. Approximate Solution

Theorem 7.1. *Let $\hat{\phi}_+^i, \hat{\phi}_-^i$, and $\hat{u}_P^i$ be the approximations of the functions ϕ_+^i, ϕ_-^i, and u_P^i, respectively, obtained by the method of collocation of this section. Assume that $\dot{\hat{u}}_i$ $(i = 1, \ldots, E-1)$ are the solutions of the system of Eqs.* (6.10), *with the coefficients given by Eqs.* (6.11), *when the functions ϕ_+^i, ϕ_-^i, and u_P^i are replaced by $\hat{\phi}_+^i, \hat{\phi}_-^i$, and $\hat{u}_P^i$, respectively. Define in (x_{i-1}, x_{i+1}), for $(i = 1, \ldots, E-1)$, the functions*

$$\hat{u}^i(x) = \hat{u}_H^i(x) + \hat{u}_P^i(x) \tag{7.17}$$

with

$$\hat{u}_H^i(x) = (\dot{\hat{u}}_{i-1} + \frac{1}{2}j_{i-1}^0)\hat{\phi}_-^i(x) + (\dot{\hat{u}}_{i+1} - \frac{1}{2}j_{i+1}^0)\hat{\phi}_+^i(x) \tag{7.18}$$

and, in $(0, l)$, let be:

$$\hat{u}(x) = \hat{u}^i(x), \quad \textit{in } (x_{i-1}, x_{i+1}), \quad \textit{for } (i = 1, \ldots, E-1). \tag{7.19}$$

Then:

i) The definition of Eq. (7.19) *is consistent.*

ii) The jump conditions are fulfilled:

$$[\hat{u}]_i = j_i^0 \quad \textit{and} \quad \left[\frac{d\hat{u}}{dx}\right]_i = j_i^1; \textit{ for } i = 1, \ldots, E-1. \tag{7.20}$$

iii) The boundary conditions are fulfilled:

$$\hat{u}(0) = g_{\partial 0} \quad \textit{and} \quad \hat{u}(l) = g_{\partial l}. \tag{7.21}$$

iv) The collocation differential equation is satisfied at the $n(\equiv G-1)$ Gaussian points of each subinterval (x_{i-1}, x_i), for $(i = 1, \ldots, E)$:

$$\mathcal{L}\hat{u} \equiv -\frac{d}{dx}\left(a\frac{d\hat{u}}{dx}\right) + \frac{d}{dx}(b\hat{u}) + c\hat{u} = f_\Omega. \tag{7.22}$$

Proof. To prove condition (i), observe that for each $i = 1, \ldots, E-2$ the intersection of the subintervals (x_{i-1}, x_{i+1}) and (x_i, x_{i+2}), is the open subinterval (x_i, x_{i+1}). In addition,

$$\hat{u}^{i+1}(x_i+) = \dot{\hat{u}}_i + \frac{1}{2}j_i^0 = \dot{\hat{u}}^i(x_i) + \frac{1}{2}j_{i-1}^0 = u^i(x_i+), \tag{7.23a}$$

and similarly,

$$\hat{u}^i(x_{i+1}-) = \dot{\hat{u}}_{i+1} - \frac{1}{2}j_{i+1}^0 = \dot{\hat{u}}^{i+1}(x_{i+1}) - \frac{1}{2}j_{i+1}^0 = u^{i+1}(x_{i+1}-). \tag{7.23b}$$

Therefore, the functions $\hat{u}^i(x)$ and $\hat{u}^{i+1}(x)$ satisfy the same boundary conditions at the end points of the subinterval (x_i, x_{i+1}); they are polynomials of degree G and fulfill the differential Eq. (6.1) at the $n(\equiv G-1)$ Gaussian points of that subinterval; thus, they are identical. Having proved condition (i), conditions (ii)–(iv) become obvious, because, for each $i = 1, \ldots, E-2$, they are mere repetitions of the conditions imposed either on the function $\hat{u}^i(x)$ or $\hat{u}^{i+1}(x)$. ∎

VIII. ERROR ANALYSIS

Let the error be defined by $e(x) \equiv \hat{u}(x) - u(x)$, where $u(x)$ is the exact solution of the problem of Section VI, and $\hat{u}(x)$ its approximate solution, obtained by the collocation method of Section VII.

Using the results presented in Theorem 7.1, it is seen that $e(x)$ satisfies the following

i) Jump conditions:

$$[e]_i = 0 \quad \text{and} \quad \left[\frac{de}{dx}\right]_i = 0; \quad \text{for } i = 1, \ldots, E-1; \tag{8.1}$$

ii) Boundary conditions:

$$e(0) = 0 \quad \text{and} \quad e(l) = 0; \tag{8.2}$$

iii) Differential equation

$$\mathcal{L}e \equiv -\frac{d}{dx}\left(a\frac{de}{dx}\right) + \frac{d}{dx}(be) + ce = 0 \tag{8.3}$$

is satisfied at the $n(\equiv G-1)$. Gaussian points of each subinterval (x_{i-1}, x_i), for $(i = 1, \ldots, E)$. Therefore, the error estimates usual for orthogonal collocation are applicable (see [20]). In particular, if the coefficients a, b, and c are sufficiently smooth (piecewise)—with discontinuities possibly occurring on Σ—and f_Ω is piecewise C^n, again with discontinuities possibly occurring on Σ, then

$$|e|_\infty \leq \beta h^{2n}, \tag{8.4}$$

where β is a constant independent of E.

IX. NUMERICAL EXPERIMENTS

The numerical experiments that were performed consist in solving Eq. (6.1) subject to Dirichlet boundary conditions. Four examples were chosen to compare p-convergence (higher-order polynomial degree convergence) with h-convergence (mesh size), as well as to verify the solution

TABLE I. Definitions of the examples treated.

Example	a	b	c	f_Ω
1	1	$2px/q$	$-\left\{\frac{4p(1+p)}{q^2}+\frac{2p^2}{q}+p^2\right\}$	0
2	-1	$-\alpha$	0	0
3	-1	0	-1	0
4	$-1; 0 \le x \le 1$	0	1	0
	$-4; 1 < x \le 2$			

$p = \sqrt{40}\pi; q = 1 + p(1 + x^2)$.

behavior in the cases of prescribed jumps and discontinuous coefficients. Table I gives the coefficients used for each of the examples for which the analytical solutions are known (Table II). In each case, a graph of the solution is given along with the graph of the error, measured using the $\| \quad \|_\infty$ norm.

The error analysis for the method studied in this article is presented in Section VIII, where a bound for the error $\|e\|_\infty$ is established, which behaves as $h^{2n} \equiv h^{2(G-1)}$. Here, $n \equiv G - 1$ is the number of collocation points at each subinterval of the partition, G is the degree of the polynomial approximations and $h \equiv \max |x_i - x_{i-1}|$ is the norm of the mesh size. For larger values of n, hyper-convergence is obtained, producing great savings in the required number of elements to achieve a desired accuracy. In the first two examples, with C^1 coefficients, the error is seen to follow the predicted theoretical behavior fairly well for smaller values of n, but differ slightly for larger ones. In all cases, the convergence improves rapidly when n is increased and the deviations from the predicted bound are not important. There is a definite trade-off between the use of higher-order polynomials and a reduced number of mesh points. While no specific computer-time results are given, it should be mentioned that the method entails first the establishment and then the solution of $2(E-1)$ matrix equations, each one $n \times n$. The global matrix equation derived in this manner is an $(E - 1)$ tridiagonal system. This gives a linear time (based upon the total number of numerical operations performed) in the parameter E and cubic in n, which is usually much smaller than E. Although the number of required computations is similar to the standard Hermite collocation method, and the convergence properties are also similar, several advantages are apparent from our approach. First, the method handles discontinuous

TABLE II. Solution for each one of the examples.

Example	Exact Solution
1	$\sin(px) + x\cos(px)$;
2	$\frac{e^{\alpha x}-e^{\alpha}}{1-e^{\alpha}}; \alpha = 20{,}100;$
3	$e^x; 0 \le x < \frac{1}{2}$
	$\frac{1}{4} + e^{1/2}; x = \frac{1}{2}$
	$\left(1 - \frac{1}{2}\frac{e^{-3/2}}{e-1}\right)e^x + \frac{1}{2}\frac{e^{3/2}}{e-1}e^{-x}; \frac{1}{2} < x \le 1;$
4	$A\sin x; 0 \le x \le 1$
	$C\sin\frac{x}{2} + D\cos\frac{x}{2}; 1 < x \le 2;$
	where A, C, and D are defined by
	$\begin{bmatrix} \sin(1) & -\sin(\frac{1}{2}) & -\cos(\frac{1}{2}) \\ \cos(1) & -2\cos(\frac{1}{2}) & 2\sin(\frac{1}{2}) \\ 0 & \sin(\frac{1}{2}) & \cos(1) \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

coefficients as well as prescribed jumps of the function and its derivative. Second, in our method the matrix of the global system of equations is always tridiagonal regardless of the number of collocation points used—i.e., independently of the precision of the algorithm that is being developed—while the bandwidth obtained in the standard Hermite collocation method is three times the number of collocation points $(3n \equiv 3(G-1))$. Third, the $n \times n$ systems of equations occurring in the local problems can be easily computed in parallel. Finally, the procedure is readily extended to higher dimensions, where the advantages mentioned above become increasingly more significant.

Although the numerical results here presented are only for one-dimensional problems, work is underway towards higher-dimensional implementations, which looks promising. It must be stressed that the local solutions required for the construction of the global matrix equation as well as the computation of the final solution are independent of each other and, therefore, their computation can be easily performed using parallel processing.

Example 1

In this case, the solution is an oscillating function $u(x) = \sin px + x \cos px$, where $p = (40\pi)^{1/2}$. For the usual Hermite cubic polynomials, convergence starts with $E = 10$ and follows a fourth-degree curve. However, for higher-degree polynomials, convergence starts to occur for as little as $E = 3$ and proceeds with a higher-degree slope. See Figs. 1 and 2.

Example 2

The differential equation yields as a solution a function giving a ''sharp front.'' Values for alpha of 20 and 100 are illustrated. A similar behavior is observed with convergence occurring at a lower

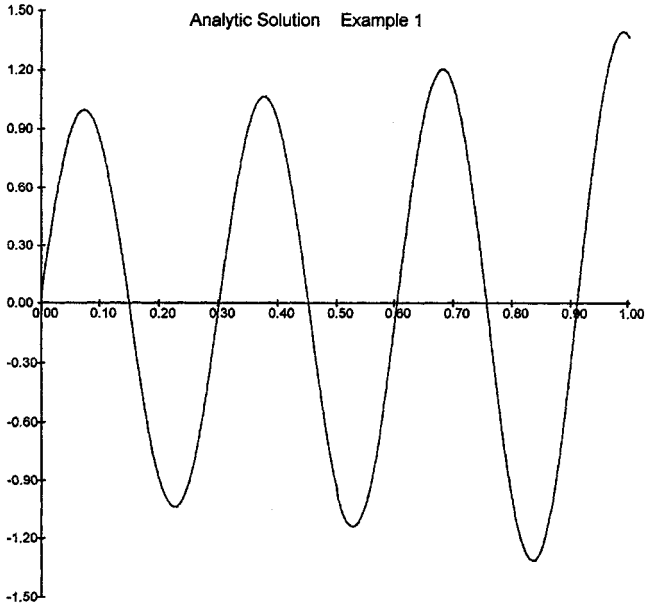

FIG. 1. The exact solution for Example 1.

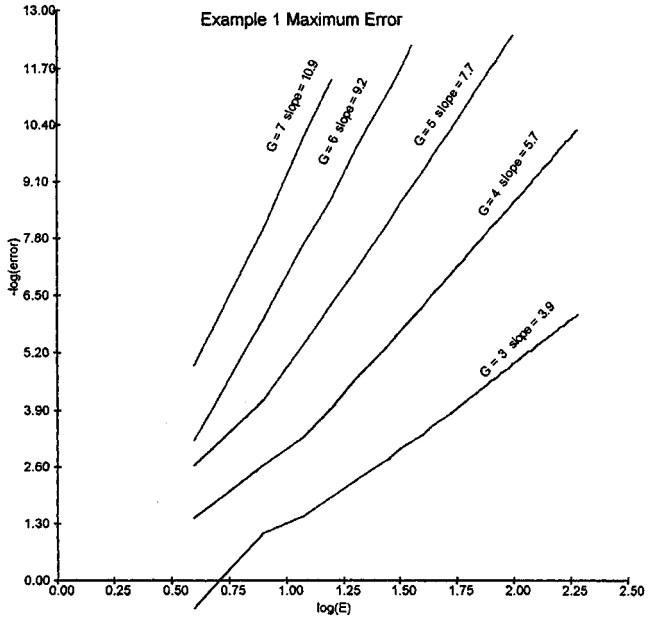

FIG. 2. Error behavior for Example 1.

number of node points and with a higher slope for higher-degree approximation polynomials. As is to be expected, the convergence is faster for the lower value of alpha (20), since the function is more easily approximated at the inflection point of the curve. See Figs. 3–6.

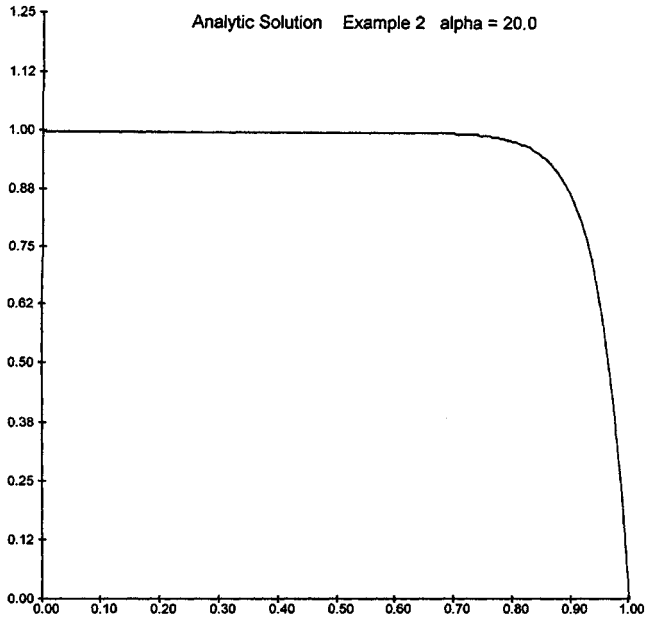

FIG. 3. Exact solution for Example 2 and $\alpha = 20$.

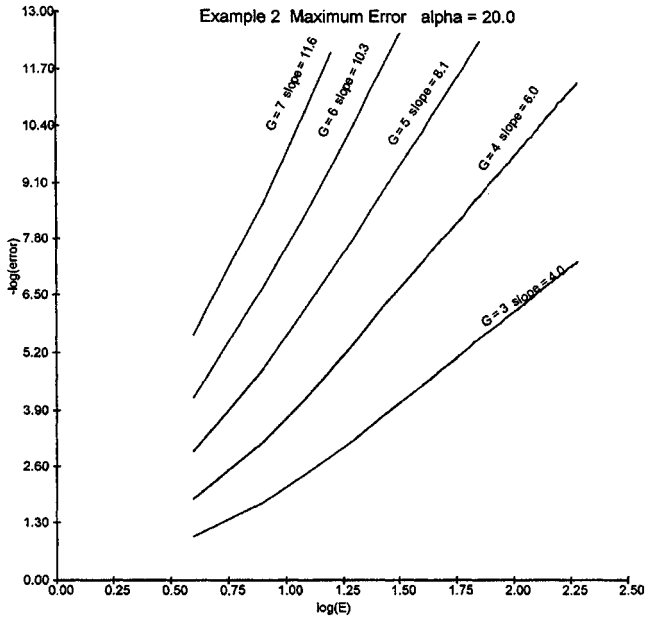

FIG. 4. Error behavior for Example 2 and $\alpha = 20$.

Example 3

In this case, prescribed jumps for the solution and its first derivative were specified at the value $x = 0.5$. Since the underlying numerical method essentially solves for the solution in each

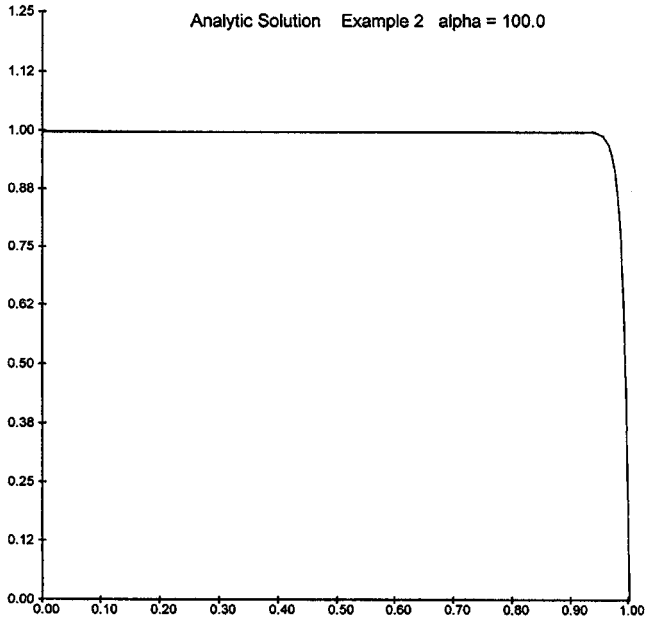

FIG. 5. Exact solution for Example 2 and $\alpha = 100$.

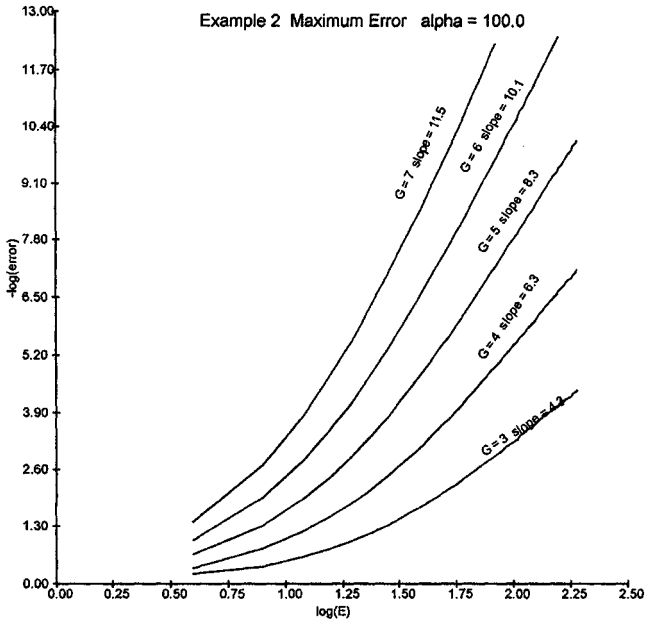

FIG. 6. Error behavior for Example 2 and $\alpha = 100$.

of the ''smooth'' interiors and then pieces these solutions together in the global tridiagonal linear equation, it is to be expected that similar convergence results would also occur. In this case, only two graphs were available due to numerical ''rounding'' errors after about 11 digits.

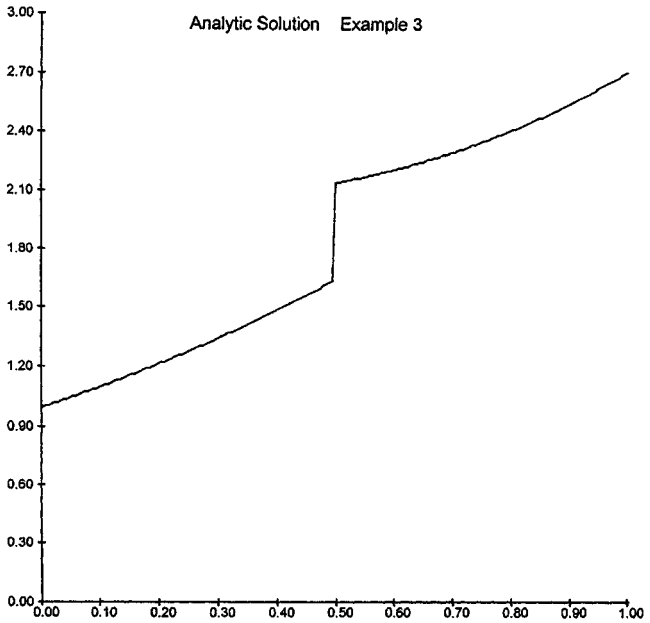

FIG. 7. Exact solution for a problem with prescribed jumps (Example 3).

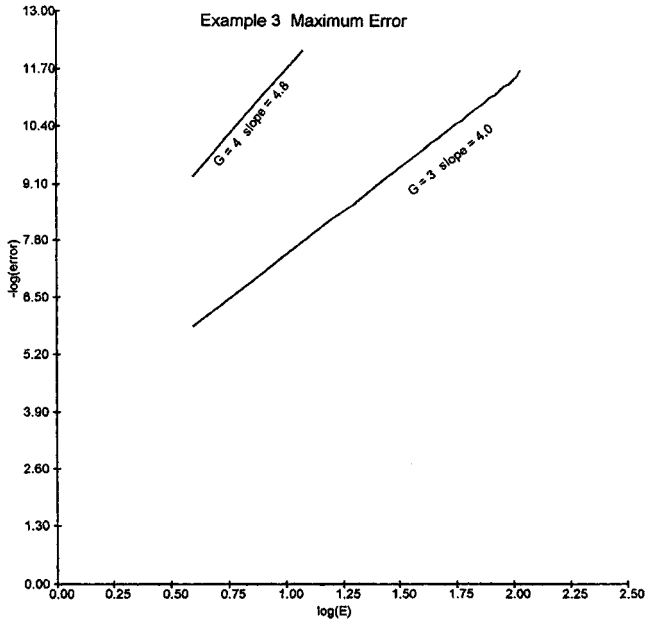

FIG. 8. Error behavior for Example 3.

Again, the initial slope of h^4 is seen for the standard Hermite polynomials, while it is difficult to ascertain from this example exactly what kind of slopes result from higher values of G. See Figs. 7 and 8.

Example 4

This example was included to illustrate how the method of this article can be used for the case of piecewise discontinuous coefficients. Here $a(x)$ was taken as the constant 1 for $0 \leq x \leq 1$ and as the constant 4 for $1 < x \leq 2$. The functions $u(x)$ and $a(x)u'(x)$ were assumed smooth across the point $x = 1$. Again the numerical solution obtained gave results in accordance with the expected values for G and h but again, the ''round-off'' problem occurred too quickly to obtain a clearer pattern for the convergence slopes. See Figs. 9 and 10.

X. CONCLUSIONS

This article is part of a line of research devoted to developing a general theory of domain decomposition and the methods derived from it. Such general theory divides domain decomposition methods into two broad classes: direct and indirect approaches. Indirect methods (Trefftz–Herrera methods), as well as the general theory, although not fully developed yet, have been the subject of several publications. In the present article, a general direct overlapping method called ''semidirect overlapping method,'' which subsumes Schwarz methods (including multiplicative and additive methods), has been introduced. This method, the semidirect overlapping method, is quite suitable for parallel implementation.

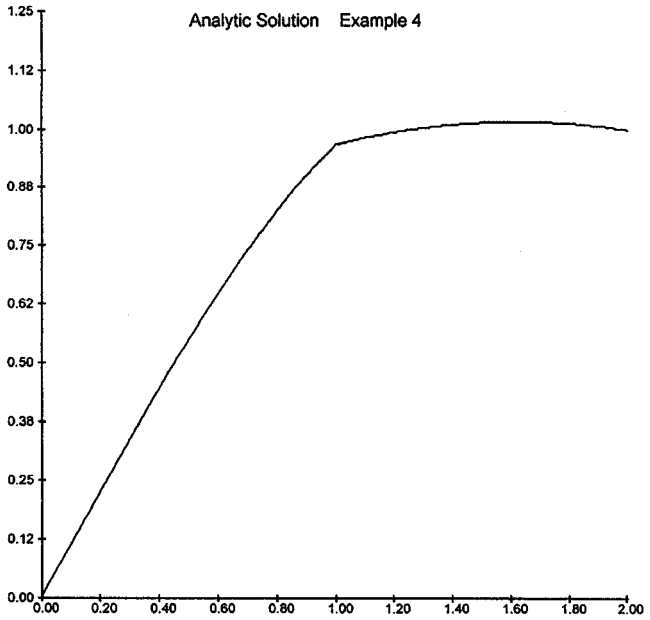

FIG. 9. Exact solution for a differential equation with discontinuous coefficients (Example 4).

In the examples here presented, to test the numerical performance of our approach, several additional advantages became apparent. These advantages can be expected to be increasingly more significant, since the procedure is readily extended to higher dimensions and its appli-

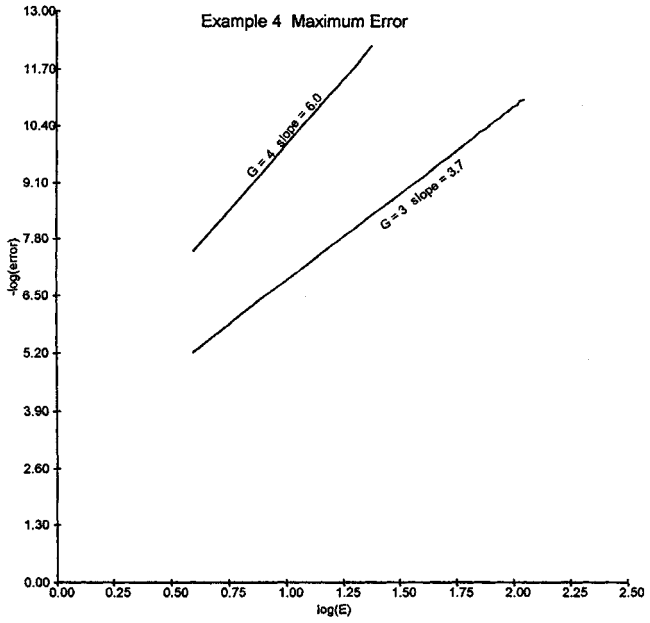

FIG. 10. Error behavior for Example 4.

cation to such problems is under investigation at present. Furthermore, the methodology proposed in this article possesses great generality, since it handles many kinds of discontinuities (discontinuous coefficients, discontinuous functions, and any boundary value problem with prescribed jumps in the internal boundaries of the domain decomposition) and can be applied to any differential equation or system of such equations. Using it one can treat, for example, elliptic, parabolic, and hyperbolic equations, as well as systems such as Stokes problems and elasticity.

The authors acknowledge the valuable help given by Martin Diaz in different aspects of the developments presented in this article and the financial support received from CONACYT and DGAPA (UNAM) through their Projects #32051-A and IN105998, respectively.

References

1. International Scientific Committee for Domain Decomposition, Proceedings of 13 conferences on Domain Decomposition Methods. www.ddm.org, 1988–2000.
2. I. Herrera, Trefftz method: a general theory, Numer Methods Partial Differential Eq 16 (2000), 561–580.
3. E. Trefftz, Ein Gegenstück zum Ritzschen Verfahren, Proc 2nd Int Congr Appl Mech (Zurich), 1926, pp. 131–137.
4. J. Jirousek and A. Wróblewski, T-elements: state of the art and future trends, Archives of Computational Methods in Engineering, State of the Art Reviews, 3 (1996), 323–434.
5. J. Jirousek and A. P. Zielinski, Survey of Trefftz-type element formulations, Comp Struct 63 (1997), 225–242.
6. I. Herrera, Unified approach to numerical methods. Part 1. Green's formulas for operators in discontinuous fields, Numer Methods Partial Differential Eq 1 (1985), 12–37.
7. I. Herrera, Unified approach to numerical methods. Part 3. Finite differences and ordinary differential equations, Numer Methods Partial Differential Eq 1 (1985), 241–258.
8. I. Herrera, The algebraic theory approach for ordinary differential equations: highly accurate finite differences, Numer Methods Partial Differential Eq 3 (1987), 199–218.
9. I. Herrera, Trefftz–Herrera domain decomposition, Special Volume on Trefftz Method: 70 Years Anniversary, Adv Eng Software 24 (1995), 43–56.
10. I. Herrera, Trefftz–Herrera method, Computer Assisted Mechanics and Engineering Sciences, Polish Acad Sci 4 (1997), 369–382.
11. I. Herrera and M. Díaz, Indirect methods of collocation: Trefftz–Herrera collocation, Numer Methods Partial Differential Eq 15 (1990), 709–738.
12. I. Herrera and M. Díaz, Indirect methods of domain decomposition, Numer Methods Partial Differential Eq, to appear.
13. I. Herrera, Domain decomposition methods: A general theory, to appear.
14. M. Dryja and O. B. Widlund, Towards a unified theory of domain decomposition algorithms for elliptic problems, Domain Decomposition Methods for PDE (DD3, 1989), SIAM, pp. 3–21.
15. M. Dryja and O. B. Widlund, Additive Schwarz methods for elliptic finite element problems in three dimensions, Domain Decomposition Methods for PDE (DD5, 1991), SIAM, pp. 3–21.
16. B. Bialecki and M. Dryja, Multilevel additive and multiplicative methods for orthogonal spline collocation problems, Numer Math 77 (1997), 35–58.
17. L. Loomis and S. Sternberg, Advanced calculus, Addison–Wesley, Reading, MA, 1968.

18. R. Berlanga and I. Herrera, The Gauss theorem for domain decompositions in Sobolev spaces, Applic Anal 76 (2000), 67–81.

19. J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. 1, Springer–Verlag, New York—Heidelberg—Berlin, 1972.

20. P. M. Prenter, Splines and variational methods, Wiley, New York, 1975.