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"A GENERAL THEORY OF DOMAIN DECOMPOSITON AND TREFFTZ METHODS"

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Abstract. *Herrera has developed a general theory of domain decomposition in which the unifying feature, for the different domain decomposition methods, is the search for information about the sought solution in the internal boundary (Σ) which separates the subdomains from each other. According to that theory, every domain decomposition method adopts a procedure for gathering information of that kind, sufficient for defining well-posed "local problems" in each one of the subdomains of the domain decomposition. Such procedures are classified into two very broad groups: 'direct' and 'indirect' methods. Indirect methods were introduced in numerical analysis by Herrera and coworkers, and are referred as Trefftz-Herrera Methods; its distinguishing feature is the use of specialized test functions which have the property of yielding any desired information on Σ. The guidelines for the construction of such weighting functions is supplied by a special kind of Green's formulas (Green-Herrera formulas), formulated in Sobolev spaces of discontinuous functions, which permit analyzing the information on Σ, contained in approximate solutions. Direct methods, when seen from the perspective of Herrera's general theory, lead to the formulation of compatibility conditions that have to be satisfied by the sought solution and from which the required information on Σ, can be derived. Also, a general description of direct methods and their foundations is given, existing procedures which fall in this category are identified and new methods suggested by the theory are discussed. In the presentation, special emphasis is given to a very general direct-overlapping method, to be called 'semidirect overlapping method', that subsumes Schwarz methods. This latter class of methods have received much attention in recent years, but this new 'semidirect overlapping method' is more direct and general, it is not restricted to positive definite operators, and it is quite suitable for parallel implementation. The generality of the methodologies so obtained is quite wide, being applicable to any linear differential equation, or system of such equations, independently of its type (elliptic, parabolic and hyperbolic equations are included), and to problems with prescribed jumps and with discontinuous coefficients.*

1. Introduction

Domain decomposition methods have received much attention in recent years [1], mainly because they supply very effective means for parallelizing computational models of continuous systems. In addition, it is useful to analyze numerical methods for partial differential equations from a domain-decomposition perspective, since the ideas related to domain decomposition are quite basic for them. Herrera has proposed a general theory of domain decomposition [2], based on the following unifying observation: "Every domain decomposition method (DDM) performs in some way the function of gathering information about the sought solution in the internal boundary (Σ), sufficient for defining well-posed *'local problems'* in each one of the subdomains". There are two main procedures for performing this function: *'direct'* and *'indirect* (or *Trefftz-Herrera)'* methods.

Trefftz-Herrera methods were introduced in numerical analysis by Herrera and coworkers [3-5], and its distinguishing feature is the use of specialized test functions which have the property of yielding any desired information on Σ. The guidelines for the construction of such weighting functions is supplied by a special kind of Green's formulas (Green-Herrera formulas), formulated in Sobolev spaces of fully discontinuous functions [6], which permit analyzing the information on Σ, contained in approximate solutions and allow deriving 'indirect methods' from a very general variational principle, to be presented in Section 4. Direct methods, on the other hand, derive the required information on Σ from compatibility conditions that the local solutions of the original problem must satisfy when they are assembled for constructing the global solution [7]. It must be mentioned that a direct (to be called 'semidirect') method derived in this manner subsumes Schwarz methods [8-10] (see [7]). When direct and indirect procedures are put to work, the main difference between them is that when the former are applied, the information on Σ is derived using local solutions of differential equations formulated in terms of the original differential operators –i.e., those occurring in the original formulation of the boundary value problem-, while the specialized test functions that are used in indirect methods fulfill equations that are formulated in terms of the adjoint differential operators. Each one of these methods, in turn, can be applied in overlapping or non-overlapping domain decompositions. The generality of the methodologies must be stressed, since they are applicable to any linear differential equation, or system of such equations, independently of its type, and to problems with prescribed jumps and with discontinuous coefficients.

In this talk an overview of this unifying theory, is presented. In Section 2, the Sobolev spaces with discontinuous functions in which later developments are carried out, are introduced [6]. When boundary value problems are formulated in such setting, one is lead to consider problems with prescribed jumps and a general formulation of them is presented in Section 3. A very general and abstract formulation of indirect methods is introduced in Section 4, while Section 5 is devoted to illustrate them in connection with the general elliptic equation of second order. Direct methods are briefly explained in Section 6. Complete systems of functions which are required to apply the methodologies of the general theory are discussed in Section 7. Some quite appealing results, that have already been obtained are explained in Section 8, devoted to conclusions.

2 Sobolev spaces with discontinuous functions

In what follows, unless otherwise explicitly stated, Ω will be an open, bounded region. The closure of any set Ω will be denoted by $\overline{\Omega}$. The (outer) boundary of Ω will be denoted by $\partial\Omega$. As usual, a collection $\Pi = \{\Omega_1,...,\Omega_E\}$ of open subregiones Ω_i (i=1,...,E) of Ω, is said to be a *partition* of Ω, *iff*

i. $\Omega_i \cap \Omega_j = \phi$, *for every* $i \neq j$, and

ii. $\overline{\Omega} = \bigcup_{i=1}^{i=E} \overline{\Omega}_i$

In addition, the partitions considered throughout this paper are assumed to be such that the subregiones Ω_i are *manifolds with corners*, in the sense of Loomis and Sternberg [11] (see also [6]). The manifold $\bigcup_{i=1}^{E} \partial\Omega_i$ will be referred to as the '*generalized boundary*', while the '*internal boundary*' of Ω -to be denoted by Σ- is defined as the closed complement of $\partial\Omega$, considered as a subset of the generalized boundary.

The formulation and treatment of boundary problems with prescribed jumps requires the introduction of a special class of Sobolev spaces in which some of their functions are fully discontinuous. In general, the definition of such spaces which is suitable for a given problem, depends on the differential operator considered; in the case of the elliptic problems to be discussed later in this paper, such spaces are defined by $\hat{H}^s(\Omega,\Pi) \equiv H^s(\Omega_1) \oplus ... \oplus H^s(\Omega_E)$, with $s \geq 0$. Elements $\hat{u} \equiv \{u^1,...,u^E\} \in \hat{H}^s(\Omega,\Pi)$ are sequences of functions such that $u^i \in H^s(\Omega_i)$, $i=1,...,E$. For $s \geq 0$, $u^i \in H^0(\Omega_i)$, $i=1,...,E$, $H^s(\Omega_i) \subset H^0(\Omega_i)$ and the sequence $\{u^1,...,u^E\}$ defines a unique function $u \in H^0(\Omega)$ with the property that $u|_{\Omega_i} = u^i$, $i=1,...,E$. The mapping $\hat{H}^s(\Omega,\Pi) \rightarrow H^0(\Omega)$, so defined, is referred as the *natural immersion* of $\hat{H}^s(\Omega,\Pi)$ into $H^0(\Omega)$.

When $v^i \in H^s(\Omega_i)$ and $s > 1/2$, then the trace of v^i belongs to $H^{s-1/2}(\partial\Omega_i)$ [18]. Let be $\Sigma_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j$, then the traces on Σ_{ij} of each pair v^i and v^j are well-defined, and using this pair of traces, the following notation is introduced:

$$v_+ \equiv \text{Trace of } (v^i) \tag{2.4a}$$

when Ω_i lies on the positive side of Σ_{ij} and

$$v_- \equiv \text{Trace of } (v^i) \tag{2.4b}$$

otherwise. The *jump* of u across Σ_{ij}, is defined by

$$[v] \equiv v_+ - v_-, \tag{2.5a}$$

and the *average* by

$$\dot{v} \equiv \frac{1}{2}(v_+ + v_-), \tag{2.5b}$$

Observe that under the natural immersion $H^0(\Omega) \supset \hat{H}^s(\Omega,\Pi) \supset H^s(\Omega)$, and it can be shown that, when s is an integer, an element $\hat{u} \in \hat{H}^s(\Omega,\Pi)$ belongs to $H^s(\Omega)$, if and only if the jump of the normal derivatives up to order $s-1$, vanish on Σ.

3. Problems with prescribed jumps and scope

Herrera's unified theory of domain decomposition possesses considerable generality, since it applies to a very general class of boundary value problems for which jumps are prescribed in the internal boundary Σ. Given Ω, the region of definition of the problem, and a partition of Ω (or domain-decomposition) $\Pi \equiv \{\Omega_1,...,\Omega_E\}$, let $\Sigma \equiv \Sigma(\Omega,\Pi)$ be the internal boundary. Then, using a notation similar to that presented in [12], the general form of such boundary value problem with prescribed jumps (BVPJ) is

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega; \quad in\ \Omega_i, \quad i = 1,...,E \tag{3.1a}$$

$$B_j u = B_j u_\partial \equiv g_j; \quad in\ \partial\Omega \tag{3.1b}$$

and

$$[J_k u] = [J_k u_\Sigma] \equiv j_k; \quad in\ \Sigma \tag{3.1c}$$

where the B_j's and J_k's are certain differential operators (the j's and k's run over suitable finite ranges of natural numbers) and $u_\Omega \equiv (u_\Omega^1,...,u_\Omega^E)$, together with u_∂ and u_Σ are given functions of the space of trial functions. In addition, f_Ω, g_j and j_k may be defined by Eq.(3.1).

It must be emphasized that the scope of the general theory presented in this paper is quite wide, since they are applicable to any partial differential equation or system of such equations which is linear, independently of its type, Although every kind of equation has its own peculiarities. In particular, we would like to mention explicitly the following:

A.- A single equation

1).- Elliptic

i) Second Order

ii) Higher-Order

- Biharmonic

2).- Parabolic

i) Heat Equation

3).- Hyperbolic

i) Wave Equation

B.- Systems of equations

i) Stokes Problems
ii) Mixed Methods (Raviart-Thomas)
iii) Elasticity

4. Trefftz-herrera methods

As mentioned in the Introduction, Trefftz-Herrera methods were introduced in numerical analysis by Herrera and coworkers, and its distinguishing feature is the use of specialized test functions for gathering the required information on Σ. Since the set of data that would define well-posed problems in each one of the subdomains is not unique, it is necessary to define which information will be 'sought' on Σ. Once the sought information is defined, a special kind of Green's formulas (Green-Herrera formulas), formulated in Sobolev spaces of discontinuous functions, allow identifying necessary and sufficient conditions that the test functions must fulfill in order to have the property of yielding precisely such information on Σ. Green-Herrera formulas are presented in the first part of this Section and then Trefftz-Herrera method is explained. These results, in the form here presented, have remarkable generality since they can applied to a single equation or to systems of equations, independently of their type, as well as to problems with prescribed jumps and with discontinuous coefficients.

To start, let $\mathcal{L}$ and $\mathcal{L}^*$ be a differential operator and its formal adjoint, then there exists a vector-valued bilinear function $\mathcal{Q}(u,w)$, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla \bullet \mathcal{Q}(u,w), \tag{4.1}$$

It will also be assumed that there are bilinear functions $\mathcal{B}(u,w)$, $\mathcal{C}(w,u)$, $\mathcal{J}(u,w)$ and $\mathcal{K}(w,u)$, the first two defined on $\partial\Omega$ and the last two on Σ, such that

$$\mathcal{Q}(u,w) \bullet \underline{n} = \mathcal{B}(u,w) - \mathcal{C}(w,u), \quad \text{on } \partial\Omega \tag{4.2a}$$

$$-[\mathcal{Q}(u,w)] \bullet \underline{n} = \mathcal{J}(u,w) - \mathcal{K}(w,u), \quad \text{on } \Sigma \tag{4.2b}$$

Applying the generalized divergence theorem [6], this implies the following Green-Herrera formula [3,13,14]:

$$\int_\Omega w\,\mathcal{L}\,u\,dx - \int_{\partial\Omega} \mathcal{B}(u,w)\,dx - \int_\Sigma \mathcal{J}(u,w)\,dx = \int_\Omega u\mathcal{L}^* w dx - \int_\Omega \mathcal{C}^*(u,w)dx - \int_\Sigma \mathcal{K}^*(u,w)dx \tag{4.3}$$

Introduce the following notation:

$$\langle Pu,w\rangle = \int_\Omega w\mathcal{L}u dx; \qquad \langle Q^* u,w\rangle = \int_\Omega u\mathcal{L}^* w dx \tag{4.4a}$$

$$\langle Bu,w\rangle = \int_\Omega \mathcal{B}(u,w)dx; \qquad \langle C^* u,w\rangle = \int_\Omega \mathcal{C}^*(u,w)dx \tag{4.4b}$$

$$\langle Ju,w\rangle = \int_{\Sigma} \mathcal{J}(u,w)dx; \qquad \langle K^{*}u,w\rangle = \int_{\Sigma} \mathcal{K}^{*}(u,w)dx \tag{4.4c}$$

With these definitions, each one of P, B, J, Q*, C* and K*, are real-valued bilinear functionals defined on $\hat{D}_1(\Omega)X\hat{D}_2(\Omega)$ and Eq. (4.3) can be written as

$$\left\langle (P-B-J)u,w\right\rangle \equiv \left\langle (Q^{*}-C^{*}-K^{*})u,w\right\rangle, \quad \forall (u,w)\in \hat{D}_1(\Omega)X\hat{D}_2(\Omega) \tag{4.5}$$

or more briefly

$$P-B-J \equiv Q^{*}-C^{*}-K^{*} \tag{4.6}$$

A weak formulation of the BVPJ is

$$\left\langle (P-B-J)u,w\right\rangle \equiv \left\langle Pu_{\Omega}-Bu_{\partial}-Ju_{\Sigma},w\right\rangle, \quad \forall w\in \hat{D}_2(\Omega) \tag{4.7}$$

When f, g and j, are defined by $f \equiv Pu_{\Omega};\cdots g \equiv Bu_{\partial};\cdots j \equiv Ju_{\Sigma}$, this equation can also be written as

$$(P-B-J)u = f-g-j; \tag{4.8}$$

or, in view of Eq.(4.6),

$$(Q^{*}-C^{*}-K^{*})u = f-g-j; \tag{4.9}$$

Eqs. (4.8) and (4.9) supply two alternative and equivalent variational formulations of the BVPJ. The first one is referred as the *'variational formulation in terms of the data of the problem'*, while the second one will be referred as the *'variational formulation in terms of the complementary information'*.

Notice that Eqs. (4.8) and (4.9), alternatively, may be written as

$$<(P-B-J)u,w>=<f-g-j;w>;\cdots \forall w\in D_2 \tag{4.10}$$

and

$$<(Q-C-K)^{*}u,w>=<f-g-j;w>;\cdots \forall w\in D_2 \tag{4.11}$$

respectively. These equations exhibit more clearly their variational character.

Generally, the definitions of B and J depend on the kind of boundary conditions and the *"smoothness criterion"* of the specific problem considered. For the case when the coefficients of the differential operators are continuous, Herrera [3,14,15] has given very general formulas for J and K; they are:

$$\mathcal{J}(u,w) \equiv -\mathcal{D}([u],w)\bullet \underline{n}, \text{ and } \mathcal{K}(w,u) \equiv \mathcal{D}(\dot{u},[w])\bullet \underline{n}; \tag{4.12}$$

By inspection of Eqs. (4.4), it can be recognized that the information about the solution $u\in D$ is given by $Q^{*}u$, in the interior of the subregions Ω_i $(i=1,...,E)$; it is given by $C^{*}u$, in the outer boundary $\partial\Omega$; and it is given by $K^{*}u$, in the internal boundary Σ. A first step to derive Trefftz-Herrera procedures, is to manipulate the variational formulation in terms of the sought information of Eq.(4.9), in such a way as to leave information in the internal boundary, exclusively. This requires eliminating $Q^{*}u$ and $K^{*}u$, in that equation. A weighting function possesses this ability if and only if $Cw=0$ and $Qw=0$. Indeed, by virtue of Eq.(4.11), it seen that

$$-<K^{*}u,w>=<f-g-j;w>;\cdots\forall w\in N_{Q}\cap N_{C}\subset\hat{D}_{2}(\Omega) \qquad (4.13)$$

The information contained in $K^{*}u$, will be referred as the "complemenray information on Σ". Generally, this information when it is complemented with the boundary data, is sufficient to define well-posed problems in the subdomains of the domain decomposition. However, it can be seen through specific examples that it is more than what is needed to achieve this goal and, to develop numerical methods of optimal efficiency, it is necessary to eliminate part of such information.

The general procedure for carrying out such elimination consists in introducing a "strong decomposition" $\{S,R\}$ of the bilinear functional K (for a definition of strong decomposition, see [16]). Then, S and R are bilinear functionals and fulfill

$$K\equiv S+R \qquad (4.14)$$

Then *"the sought information"* is defined to be $S^{*}u$, where $u\in D_{1}$ is the solution of the BVPJ. In particular, a function $\tilde{u}\in D_{1}$ is said to "contain the sought information" when $S^{*}\tilde{u}=S^{*}u$.

An auxiliary concept, quite useful for formulating Trefftz-Herrera domain decomposition procedure, is the following.

Definition 4.1.- *A subset of weighting functions, $\mathcal{E}\subset N_{Q}\cap N_{C}\cap N_{R}$, is said to be TH-complete for S*, when for any $\hat{u}\in\hat{D}_{1}(\Omega)$, one has:*

$$<S^{*}\hat{u},w>=0,\ \ \forall w\in\mathcal{E}\Rightarrow S^{*}\hat{u}=0; \qquad (4.15)$$

Clearly, a necessary and sufficient condition for the existence of TH-complete systems, is that $N_{Q}\cap N_{C}\cap N_{R}$ be, itself, a TH-complete system.

Theorem 4.1.- *Let $\mathcal{E}\subset N_{Q}\cap N_{C}\cap N_{R}$ be a system weighting functions, TH-complete for S*, and let $u\in\hat{D}_{1}(\Omega)$ be the solution of the BVPJ. Then, a necessary and sufficient condition for $\hat{u}\in D_{1}$ to contain the sought information, is that*

$$-<S^{*}\hat{u},w>=<f-g-j,w>,\cdots\forall w\in\mathcal{E}; \qquad (4.16)$$

Proof.- The proof of this result is straight-forward.

Theorem 4.1, supplies a very **General Formulation of Indirect Methods (or Trefftz-Herrera Methods) of Domain Decomposition** which can be applied to any linear equation or system of such equations. When $u_{P}\in\hat{D}_{1}(\Omega)$ is a function satisfying $Pu_{P}=f$ and $Bu_{P}=g$, then Eq.(4.16) can be replaced by

$$-<S^{*}\hat{u},w>=-<S^{*}u_{P},w>+<J(u_{P}-u_{\Sigma}),w>;\qquad\forall w\in\mathcal{E} \qquad (4.17)$$

5. The elliptic second-order equation

As an illustration, in this Section the methodology will be presented in detail in connection with the second-order differential equation of elliptic type, when the problem is defined in a space of an arbitrary number of dimensions. The procedures are applicable to any kind of boundary conditions for which the problem is well-posed, but here only boundary conditions of Dirichlet type will be considered.

Recall the notation introduced in Section 2; Ω is a region and $\Pi \equiv \{\Omega_1,...,\Omega_E\}$ a partition of Ω. A function $u \in \hat{H}^2(\Omega) \equiv H^2(\Omega_1) \oplus ... \oplus H^2(\Omega_E)$ is sought, such that

$$\mathcal{L}u \equiv -\nabla \bullet (\underline{\underline{a}} \bullet \nabla u) + \nabla \bullet (\underline{b} u) + cu = f_{\Omega}; \quad \text{in } \Omega_i \ , \ i = 1,...,E \tag{5.1}$$

subjected to the boundary conditions

$$u = u_{\partial}; \qquad in\ \partial\Omega \tag{5.2}$$

and jump conditions

$$[u] = j^0 = [u_{\Sigma}]; \qquad on\ \Sigma \tag{5.3}$$

$$[\underline{\underline{a}} \bullet \nabla u] \bullet \underline{n} = j^1 = [\underline{\underline{a}} \bullet \nabla u_{\Sigma}] \bullet \underline{n}; \qquad on\ \Sigma \tag{5.4}$$

Here, as it is usual when dealing with elliptic operators of second order, it will be assumed that $\underline{\underline{a}}$ is coercive. Also, it will be assumed that this problem possesses one and only one solution. The developments that follow apply even if the coefficients of the differential operator are discontinuous. In the particular case when the coefficients are continuous, the jump condition of Eq.(5.4), in the presence of (5.3), is equivalent to

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_{\Sigma}}{\partial n}\right]; \qquad on\ \Sigma, \tag{5.5}$$

Define the bilinear functions

$$\mathcal{B}(u,w) \equiv u(\underline{\mathbf{a}}_n \bullet \nabla w + \mathbf{b}_n w) \text{ and } \mathcal{J}(u,w) \equiv \dot{w}[\underline{\mathbf{a}}_n \bullet \nabla u] - [u](\overline{\dot{\underline{\mathbf{a}}}_n \bullet \nabla w + \dot{\mathbf{b}}_n w}), \tag{5.6a}$$

together with

$$\mathcal{C}(w,u) \equiv w(\underline{\mathbf{a}}_n \bullet \nabla u) \text{ and } \mathcal{K}(w,u) \equiv \dot{u}[\underline{\mathbf{a}}_n \bullet \nabla w] - [w](\overline{\dot{\underline{\mathbf{a}}}_n \bullet \nabla u - \dot{\mathbf{b}}_n u}), \tag{5.6b}$$

The linear functionals $g \in D_2^*$ and $j \in D_2^*$ are defined by the condition that $\forall w \in D_2$, one has $g(w) \equiv \mathcal{B}(u_{\partial}, w)$ together with $j(w) \equiv \mathcal{J}(u_{\Sigma}, w)$.

The "complementary information" in this case, at every point of Σ, is made of $\dot{u}$ and $-\overline{\dot{\underline{\mathbf{a}}}_n \bullet \nabla u + \dot{\mathbf{b}}_n u}$. This information, when it is complemented with the prescribed jumps on Σ, is more than what is required to define well-posed problems in the subdomains. Indeed, from the fact that

$$u_+ = \dot{u} + \frac{1}{2}[u],\ u_- = \dot{u} - \frac{1}{2}[u] \tag{5.7a}$$

and

$$\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_+ = \overline{\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u}^{\bullet} + \frac{1}{2}\left[\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right],$$

$$\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_- = \overline{\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u}^{\bullet} - \frac{1}{2}\left[\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right] \tag{5.7b}$$

it is seen that when both $\dot{u}$ and $\overline{\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u}^{\bullet}$ are available, then u_+, u_-, $\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_+$ and $\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_-$, are easily derived. However, either pair u_+, u_- or $\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_+$, $\left(\underline{\mathbf{a}}_n \bullet \nabla u + \mathbf{b}_n u\right)_-$, are sufficient to define well posed problems in the subdomains, as can be verified by inspection. As a matter of fact when these two pairs are available on Σ, there many other choices one can make to define well-posed local problems. If one decides to define Dirichlet problems in each one of the subdomains, all what is necessary to know is u_+, u_- on Σ and it is enough to seek for $\dot{u}$. Suitable definitions of the operators S and R, are $\langle Sw, v\rangle \equiv \int_\Sigma \mathcal{S}(w,v)dx$ and $\langle Rw, v\rangle \equiv \int_\Sigma \mathcal{R}(w,v)dx$ with

$$\mathcal{S}(w,v) \equiv \dot{v}[\underline{\mathbf{a}}_n \bullet \nabla w] \text{ and } \mathcal{R}(w,v) \equiv -[w]\left(\overline{\underline{\mathbf{a}}_n \bullet \nabla v - \mathbf{b}_n v}^{\bullet}\right) \tag{5.8}$$

With this choice, a function $w \in \left(N_Q \cap N_C \cap N_R\right)$ if and only if

$$\mathcal{L}^* w \equiv -\nabla \bullet (\underline{\underline{\mathbf{a}}} \bullet \nabla w) - \underline{\mathbf{b}} \bullet \nabla w + cw = 0; \quad \text{in } \Omega_i,\ i = 1,\ldots,E \tag{5.9}$$

subjected to the boundary conditions

$$w = 0; \qquad in\ \partial\Omega \tag{5.10a}$$

and jump conditions

$$[w] = 0\ ; \qquad on\ \Sigma \tag{5.10b}$$

In general, the continuity condition that is implied by Eq.(5.10b) can only be fulfilled by functions with support in the union of several subdomains of the original partition. Due to this fact the resulting method is an indirect-overlapping one.

6. Direct methods

Direct methods have been explained in [2,7]. It is possible to apply two different approaches; one which is more direct and the other one which is less direct -in some sense it is only '*semi-direct*'-. In this latter one, the base functions are used to impose a '*compatibility condition*' from which the required information on Σ is obtained. In this manner a very general methodology which subsumes Schwartz methods [8-10] has been derived [7]. This procedure handles only essential information, so that the number of degrees of freedom is minimal.

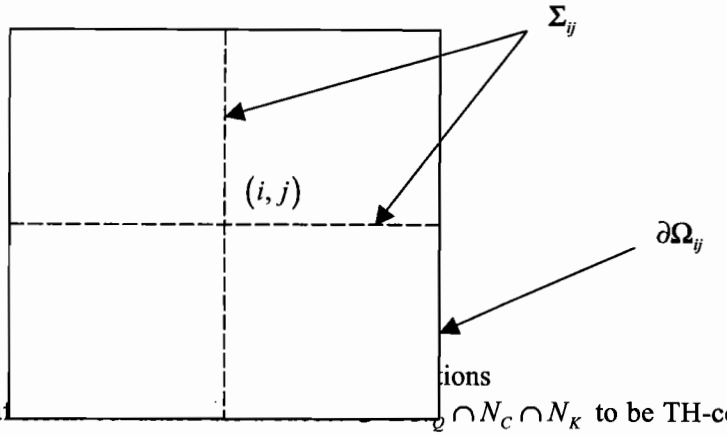
The '*semi-direct*' method has been developed for problems formulated in an arbitrary number of dimensions and handles problems with prescribed jumps and with discontinuous coefficients. It is based on an idea which is easy to explain in one dimension. Thus, consider the one dimensional version of Eq. (5.1), with $f_\Omega \equiv 0$, in an interval of the real line. Let $x_i \in (x_{i-1}, x_{i+1})$, then $u(x_i)$ depends linearly on $u(x_{i-1})$ and $u(x_{i+1})$. Indeed, $u(x_i) = \varphi_i^-(x_i)u(x_{i-1}) + \varphi_i^+(x_i)u(x_{i+1})$, and this equation constitutes a three-diagonal system of equations, whose coefficients can be obtained solving locally, by collocation, a pair of boundary value problems in the interval (x_{i-1}, x_{i+1}): $\mathcal{L}\varphi_i^- = \mathcal{L}\varphi_i^+ = 0$, subjected to $\varphi_i^-(x_{i-1}) = \varphi_i^+(x_{i+1}) = 1$ and $\varphi_i^-(x_{i+1}) = \varphi_i^+(x_{i-1}) = 0$.

The extension of this procedure to several dimensions was explained in [7] and requires the introduction of the concept of conjugate partitions. It was applied to the general elliptic equation of Section 5.

7. TH-complete systems and numerical approximations

Classical approaches for developing TH-complete systems are based on analytical methods and a thorough account may be found in a book by Begehr and Gilbert [17]. The function theoretic method was pioneered by Bergman [18] and Vekua [19], and further developed by Colton [20-22], Gilbert [23,24], Kracht-Kreyszig [25], and others. The author has supplied such systems for Stokes problem [26], Helmholtz equation (in [27] it is shown that a system of plane waves possesses that property) and biharmonic equation [28]. Other means of constructing them are using fundamental solutions and spectral methods, among others (see [17]). However, the most general procedures for constructing TH-complete systems are based on the numerical solution of the 'local' boundary-value problems.

For illustration purposes the subdomains of the partition are assumed to be squares and let Ω_{ij} be the union of four neighboring squares (Fig. 1), while Σ_{ij} will be the inner boundary separating the four subdomains from each other in the interior of Ω_{ij}. To be specific, the construction of TH-complete systems here will only be discussed in connection with the indirect-overlapping method for which the specialized test functions $w \in (N_Q \cap N_C \cap N_K)$ fulfill Eqs.(5.9) and (5.10). In particular, any $w \in (N_Q \cap N_C \cap N_K)$ with support in Ω_{ij} takes zero values on the external boundary $\partial\Omega_{ij}$ of Ω_{ij}. Then, it can be seen that such test function is uniquely determined by its average $\overset{\bullet}{w}$, on Σ_{ij}. This establishes a one-to-one correspondence between $H^0(\Sigma_{ij})$ and $N_Q \cap N_C \cap N_K$; given a function $\overset{\bullet}{w} \in H^0(\Sigma_{ij})$ the construction of $w \in (N_Q \cap N_C \cap N_K)$ such that its restriction to Σ_{ij} is the given $\overset{\bullet}{w} \in H^0(\Sigma_{ij})$ requires solving four boundary value problems, one in each of the subdomains of the partition contained in Ω_{ij}.



On the other hand, a su ions $_Q \cap N_C \cap N_K$ to be TH-complete is that for every Σ_{ij} the set of restrictions to Σ_{ij} of elements of E, span $H^0(\Sigma_{ij})$. Thus, a procedure for constructing TH-complete families of $N_Q \cap N_C \cap N_K$, with local support, is the following: take a subset $\tilde{\mathcal{E}} \subset H^0(\Sigma_{ij})$ which spans $H^0(\Sigma_{ij})$, and for every element of E solve the four boundary value problems, one in each of the subdomains of Ω_{ij}, defined by the differential Eq.(5.9), zero values on the external boundary $\partial\Omega_{ij}$ of Ω_{ij}, and the additional boundary conditions which are implied by Eqs.(5.7a) together with the condition that $\overset{\bullet}{w}$ be the given function on Σ_{ij}.

Any numerical method can be applied, to solve the local boundary-value problems with prescribed jumps. Collocation is quite suitable [1] and has been used by the author and his collaborators in several instances. It must be mentioned that TH-complete systems for one dimensional problems (ordinary differential equations) are finite. However, for multidimensional problems (partial differential equations proper) such systems are infinite and in applications only a finite number can be applied. This introduces an additional error source and when carrying out the error analysis one has to account for inaccuracies due to the approximation of the differential equation and inaccuracies due to the truncation of the TH-complete family.

8. Conclusions

This paper summarizes a general theory of domain decomposition methods (DDM), recently proposed by the author, which subsumes many methods that have been reported in the literature, but which in addition implies certain number of other methods that have not yet been reported or researched. The theory classifies DDM into two quite broad groups: direct and indirect (or Trefftz-Herrera) methods. Most of the research carried out worldwide thus far has been devoted to direct methods while indirect methods have been introduced and developed, almost exclusively, by Herrera and coworkers. It must be mentioned, in addition, that the 'semidirect' method briefly described in this paper, which subsumes Schwarz method, is also fairly new, it was introduced in [7].

There are two main lines of application of the theory; one is to DDM 'per se', and the other one is concerned with the development of new discretization methodologies. Because of the novel character of a good number of the results implied by the general theory of DDM, here presented, they should be researched further since several promising results have already been reported. For example, the methods derived from the theory have been implemented for elliptic equations in several dimensions using collocation and in this manner a non-standard collocation method (Trefftz-Herrera collocation) has been obtained which possesses many advantages over the usual method of orthogonal collocation on Hermite cubics. In particular, a dramatic reduction in the number of degrees of freedom associated with each node is obtained. Indeed, in the standard method of collocation that number is two in one dimension, four in two dimensions and eight in three dimensions, while for some of the new algorithms they are one in all space dimensions. Also, the bandwidth of the associated approximation matrix is also dramatically reduced. In addition, when the 'standard method' of collocation is applied to a differential operator which is symmetric and positive definite this property is not reflected in corresponding properties of the matrix of the global system of equations. On the other hand, such properties are reflected in corresponding properties of the global matrix, when TH-collocation is applied.

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