General Theory of Domain Decomposition: Indirect Methods

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According to a general theory of domain decomposition methods (DDM), recently proposed by Herrera, DDM may be classified into two broad categories: direct and indirect (or Trefftz-Herrera methods). This article is devoted to formulate systematically indirect methods and apply them to differential equations in several dimensions. They have interest since they subsume some of the best-known formulations of domain decomposition methods, such as those based on the application of Steklov-Poincaré operators. Trefftz-Herrera approach is based on a special kind of Green's formulas applicable to discontinuous functions, and one of their essential features is the use of weighting functions which yield information, about the sought solution, at the internal boundary of the domain decomposition exclusively. A special class of Sobolev spaces is introduced in which boundary value problems with prescribed jumps at the internal boundary are formulated. Green's formulas applicable in such Sobolev spaces, which contain discontinuous functions, are established and from them the general framework for indirect methods is derived. Guidelines for the construction of the special kind of test functions are then supplied and, as an illustration, the method is applied to elliptic problems in several dimensions. A nonstandard method of collocation is derived in this manner, which possesses significant advantages over more standard procedures. © 2002 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 18: 296–322, 2002; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/num.10008

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I. INTRODUCTION

Domain decomposition methods have received much attention in recent years [1], mainly because they supply very effective means for incorporating supercomputers in the numerical modeling of continuous systems, because they are one of the most significant ways for devising parallel algorithms that can benefit from multiprocessor computing. Some of the best known formulations

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of domain decomposition methods are based on the analysis of transmission conditions at subdomain interfaces, which in turn make use of Steklov-Poincaré operators. A recent, systematic and well-organized account of such approach is contained in [2].

This article belongs to a series of articles in which a research on a general theory of domain decomposition methods, proposed by Herrera in a previous article [3], is being reported. The basic unifying concept of that theory, consists in interpreting domain decomposition methods as procedures for obtaining information about the sought solution at the "internal boundary" (Σ), which separates the subdomains from each other, sufficient for defining well-posed problems in each one of the subdomains (to be referred as "local problems"). In this manner, the solution can be reconstructed by solving such kind of problems exclusively. There are two general procedures that can be followed for gathering the information on Σ [3]; they are referred as "direct" and "indirect" (or Trefftz-Herrera) methods. In standard approaches, direct methods piece together, just as "bricks," the local solutions of the differential equations to build the global solution. A slightly more innovative point of view is obtained when they are interpreted as procedures for gathering information at the internal boundary, and when such point of view is adopted, direct methods use the local solutions as an auxiliary tool to formulate compatibility conditions, to be satisfied by the sought solution, from which the required information on Σ can be derived (see [4]).

An introductory presentation of direct methods from this point of view was given in [4], and the present article is devoted to make an integrated exposition of indirect methods of domain decomposition in their present state of development. Trefftz-Herrera formulation has special interest because it subsumes the Steklov-Poincaré operators approach—thus, offering an alternative for its derivation—and also, because it is quite general and systematic. One-dimensional problems have already been treated in [5], by the indirect method, and the first multidimensional numerical applications are presented here. The distinguishing feature of Trefftz-Herrera methods is the use of specialized test (or weighting) functions in order to obtain the information on Σ. They stem from the observation that when the method of weighted residuals is applied, the information about the exact solution that is contained in the approximate one is determined exclusively by the family of test functions that is used [6]. Then a special kind of weighting functions, which have the property of yielding information at the internal boundary exclusively, is identified and applied. Even more, among all the information that one might consider on Σ, a target information is defined (to be referred as the "sought information") and families of test functions that yield only the sought information are applied.

The development of indirect methods requires a framework suitable for identifying the information yielded by different weighting functions, in which guidelines for constructing the special kind of functions mentioned before can be established. The natural setting is based on Green's formulas, but the conventional approach to this matter is not suitable for application to domain decomposition methods. Indeed, in the conventional approach [7], one considers the Green's formula

$$\int_{\Omega} w\mathcal{L}u dx = \int_{\Omega} u\mathcal{L}^* w dx, \tag{1.1}$$

where Ω is a given region, and $\mathcal{L}$ and $\mathcal{L}^*$ are a differential operator and its adjoint, respectively. Then, given a family of functions $\{w^1, \ldots, w^N\}$, any approximate solution, $\hat{u}$, obtained with the method of weighted residuals and this family as test functions, fulfills

$$\int_{\Omega} w^{\alpha}(\mathcal{L}\hat{u} - f_{\Omega})dx = \int_{\Omega} w^{\alpha}(\mathcal{L}\hat{u} - \mathcal{L}u)dx = \int_{\Omega} (\hat{u} - u)\mathcal{L}^* w^{\alpha} dx = 0. \tag{1.2}$$

Thus, the conclusion is reached that the error $u - \hat{u}$ is orthogonal to the space spanned by the family of functions $\{\mathcal{L}^* w^1, \ldots, \mathcal{L}^* w^N\}$. However, this result is of little use when dealing with domain decomposition methods. For them, it is necessary to have a theory that is applicable to situations in which both trial and test functions may be discontinuous simultaneously. This was done introducing a kind of Green's formulas (''Green-Herrera formulas'') especially developed for operators defined on discontinuous fields [6]. They are based on an abstract algebraic theory possessing great generality, which was presented in a preliminary form in [8] and further developed in [6] and [9]. It was later applied to the numerical treatment of differential equations [10–13] and can be formulated in a special class of Sobolev spaces, in which their functions have jump discontinuities [14].

Starting from a usual kind of variational formulation that involves the data of the problem—and referred here as ''the variational formulation in terms of the data,'' a nonstandard kind of variational formulation—here referred as ''variational formulation in terms of the complementary information''—is derived, when Green-Herrera formula is applied. This latter formulation is a fundamental piece in Trefftz-Herrera domain-decomposition methods because they supply effective means for analyzing the information contained in approximate solutions, which allows the design of strategies for the construction of weighting functions that yield precisely the desired information. In particular, the conditions that the weighting functions must fulfill in order to yield the information on the internal boundary exclusively, are easily derived. However, in general, all the information that one can get on Σ would be excessive for defining local well-posed problems, and numerically, it may be inconvenient—at least, in some cases—to handle excessive information. Thus, it is necessary to define the ''target information'' that will be sought in Σ. This choice generally will be influenced by the fact that, given a differential operator, there are several kinds of boundary conditions that can be imposed locally to define well-posed problems in the subregions, and each choice leads to a different domain-decomposition algorithm. Using Green-Herrera formulas, it is possible to establish a very systematic procedure for identifying the conditions that the weighting functions must fulfill in order to yield the ''sought information'' and also to characterize such information. This latter objective is accomplished by means of a variational principle, which is shown to be a necessary and sufficient condition for the sought information. This principle constitutes a rather general—although, somewhat abstract—formulation of indirect methods, which subsumes the Steklov-Poincaré approach.

When discontinuous functions are admitted as base functions, one is led to formulate a general boundary value problem in which jumps of the sought solution, and its derivatives are prescribed on the internal boundary of the partitions. Consideration of this problem is required in order to have complete freedom in the use of the base functions (one might say, in more mathematical terms, in order for the methodology to be ''closed''). The methodology obtained in this manner is very systematic and possesses great generality because it is applicable to any boundary value problem with prescribed jumps of linear differential equations or systems of such equations, which are otherwise arbitrary. This includes operators of any type (elliptic, parabolic, and hyperbolic) and with possibly discontinuous coefficients. Thus, it must be stressed that the variational principle characterizing the sought information, presented in Section 4 and which constitutes the general formulation of TH-domain decomposition, is applicable to this very general problem. The basic theory and the ideas stemming from it have had already a good number of applications, among others being localized adjoint methods (LAM), Eulerian-Lagrangian LAM (ELLAM [12, 15–17]) (also the method of Begehr and Gilbert [18]), and a general method for solving diffraction problems in elasticity and other fields [19].

In this article, the numerical implementation for elliptic equations in two and more dimensions is discussed with considerable detail. It is appealing to use collocation for solving the local problems, and in this manner a nonstandard collocation method is obtained that possesses many advantages over the usual method of orthogonal collocation on Hermite cubics. In particular, an inconvenience of this latter method is that the matrices for the global problem are nonsymmetric, even when the differential operator is symmetric; however, in the new methods the matrix of the global problem is positive definite and symmetric when the differential operator is. Also, a dramatic reduction in the number of degrees of freedom associated with each node is obtained. Indeed, in the standard method of collocation, that number is two in one dimension, four in two dimensions, and eight in three dimensions; for some of the new algorithms, however, there is only one degree of freedom in all space dimensions. A final comment worth mentioning refers to the fact that the treatment of problems with prescribed jumps is not more complicated than those without them; as a matter of fact, the global matrix is exactly the same for both problems.

The article is organized as follows. Section 2 is devoted to Preliminary Notions and Notations. The scope of the theory and the general boundary value problem with prescribed jumps, to which the theory is devoted, is introduced in Section 3. Trefftz-Herrera domain decomposition is systematically presented, in an abstract and very general manner, in Section 4. In Section 5, such theory is applied to the elliptic equation of second order in an arbitrary number of dimensions. TH-complete systems of test functions are discussed in Section 6. The numerical application of the theory is discussed in Section 7, whereas Section 8 is devoted to conclusions.

II. NOTATION

Consider a region Ω, with boundary $\partial\Omega$ and a partition $\{\Omega_1, \ldots, \Omega_E\}$ of Ω (or a "domain decomposition": Fig. 1). Let

$$\Sigma \equiv \bigcup_{i \neq j} (\bar{\Omega}_i \cap \bar{\Omega}_j); \tag{2.1}$$

then Σ will be referred as the "internal boundary" and $\partial\Omega$ as the "external (or outer) boundary." For each $i = 1, \ldots, E$, $D_1(\Omega_i)$ and $D_2(\Omega_i)$ will be two linear spaces of functions defined on Ω_i; then the spaces of trial (or base) and test (or weighting) functions are defined to be

$$\hat{D}_1(\Omega) \equiv D_1(\Omega_1) \oplus \cdots \oplus D_1(\Omega_E); \tag{2.2}$$

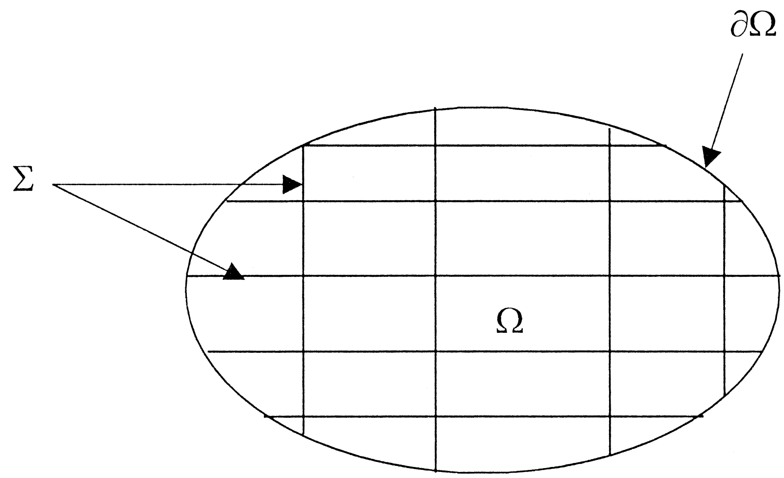

FIG. 1. Partition of the domain Ω.

and

$$\hat{D}_2(\Omega) \equiv D_2(\Omega_1) \oplus \cdots \oplus D_2(\Omega_E), \tag{2.3}$$

respectively. In what follows, we write $\hat{D}_1$ and $\hat{D}_2$, instead of $\hat{D}_1(\Omega)$ and $\hat{D}_2(\Omega)$, in order to simplify the notation. Functions belonging either to $\hat{D}_1$ and $\hat{D}_2$ are finite sequences of functions belonging to each one of the subdomains of the partition. It will be assumed that for each $i = 1, \ldots, E$, and $\alpha = 1, 2$, the traces on Σ of functions belonging $D_\alpha(\Omega_i)$ exist, and the jump and average of test or weighting functions is defined by

$$[u] \equiv u_+ - u_-; \qquad \text{and} \qquad \dot{u} \equiv (u_+ + u_-)/2, \tag{2.4}$$

where u_+ and u_- are the traces from one and the other side of Σ. Here, the unit normal vector to Σ is chosen arbitrarily, but the convention is such that it points towards the positive side of Σ.

The case when for each $i = 1, \ldots, E$, and each $\alpha = 1, 2$, $D_\alpha(\Omega_i) \equiv H^s(\Omega_i)$, with $s \geq 0$ has special interest and will be considered in Section 5. If one defines

$$\hat{H}^S(\Omega) \equiv H^S(\Omega_1) \oplus \cdots \oplus H^S(\Omega_E), \tag{2.5}$$

then $\hat{D}_1 = \hat{D}_2 \equiv \hat{H}^S(\Omega)$. This is the special class of Sobolev spaces that was considered in [13].

III. SCOPE

It must be emphasized that the scope of the general theory presented in this article, "Herrera's unified theory of domain decomposition" [4, 5], is quite wide, because it is applicable to any linear partial differential equation or system of such equations independently of its type. It handles problems with prescribed jumps in the internal boundary, Σ, and discontinuous equation coefficients, although every kind of equation has its own peculiarities. In particular, we would like to mention explicitly the following:

A. A Single Equation

1. Elliptic
   - i) Second order
   - ii) Higher-order
     - Biharmonic

2. Parabolic
   - i) Heat equation

3. Hyperbolic
   - i) Wave equation

B. Systems of Equations

i) Stokes Problems
ii) Mixed Methods (Raviart-Thomas)
iii) Elasticity

The general form of the boundary value problem with prescribed jumps (BVPJ), to be considered, is

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega; \qquad \text{in } \Omega_i, i = 1, \ldots, E \tag{3.1}$$

$$B_j u = B_j u_\partial \equiv g_j; \qquad \text{on } \partial\Omega \tag{3.2}$$

and

$$[J_k u] = [J_k u_\Sigma] \equiv j_k; \qquad \text{on } \Sigma \tag{3.3}$$

where the B_j's and J_k's are certain differential operators (the j's and k's run over suitable finite ranges of natural numbers). Here, in addition, $u_\Omega \equiv (u_\Omega^1, \ldots, u_\Omega^E)$, u_∂ and u_Σ are given functions belonging to $\hat{D}_1$ (i.e., ''trial functions''), which fulfill Eqs. (3.1), (3.2), and (3.3), respectively. Moreover, f_Ω, g_j, and j_k may be defined by Eqs. (3.1), (3.2), and (3.3), respectively.

In what follows, it will be assumed that the boundary conditions and jump conditions of this BVPJ can be brought into the point-wise variational form:

$$\mathcal{B}(u, w) = \mathcal{B}(u_\partial, w) \equiv g_\partial(w); \qquad \forall w \in \hat{D}_2 \tag{3.4}$$

and

$$\mathcal{J}(u, w) = \mathcal{J}(u_\Sigma, w) \equiv j_\Sigma(w); \qquad \forall w \in \hat{D}_2 \tag{3.5}$$

where $\mathcal{B}(u, w)$ and $\mathcal{J}(u, w)$ are bilinear functions defined point-wise.

IV. TREFFTZ-HERRERA APPROACH TO DDM

The description of Trefftz-Herrera approach to Domain Decomposition Methods, presented in the Introduction is here recalled, because in this Section such methods are derived with more detail. Subsection 4.1 is devoted to Green-Herrera formulas, which are applicable in function-spaces whose members are generally discontinuous. Subsection 4.2 presents two variational formulations: one in terms of the ''data'' of the problem—this variational principle is of the same kind as those usually applied in the discussion of finite elements—and the other one is in terms of the ''complementary information''—this kind of principles is not usually considered in the numerical treatment of partial differential equations. The ''sought information in the internal boundary'' is included in the complementary information and by manipulation of this latter principle a variational principle characterizing the sought information is derived in Subsection 4.3. This yields a rather general formulation, although somewhat abstract, of TH-domain decomposition.

A. Green-Herrera Formulas

To start, let $\mathcal{L}$ and $\mathcal{L}^*$ be a differential operator and its formal adjoint; then there exists a vector-valued bilinear function $\mathcal{D}(u, w)$, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^*w \equiv \nabla \cdot \underline{\mathcal{D}}(u, w). \tag{4.1}$$

It will also be assumed that there are bilinear functions $\mathcal{B}(u, w), \mathcal{C}(w, u), \mathcal{J}(u, w)$, and $\mathcal{K}(w, u)$, the first two defined on $\partial\Omega$ and the last two on Σ, such that

$$\underline{\mathcal{D}}(u, w) \cdot \underline{n} = \mathcal{B}(u, w) - \mathcal{C}(w, u); \qquad \text{on } \partial\Omega \tag{4.2}$$

and

$$-[\underline{\mathcal{D}}(u,w)]\cdot\underline{n}=\mathcal{J}(u,w)-\mathcal{K}(w,u);\qquad \text{on } \Sigma. \tag{4.3}$$

Generally, the definitions of $\mathcal{B}$ and $\mathcal{J}$ depend on the kind of boundary conditions and the "smoothness criterion" of the specific problem considered [6, 8]. For the case when the coefficients of the differential operators are continuous, Herrera [3, 6, 10, 12] has given very general formulas for $\mathcal{J}$ and $\mathcal{K}$, which fulfill Eq. (4.3); they are

$$\mathcal{J}(u,w)\equiv-\underline{\mathcal{D}}([u],\dot{w})\cdot\underline{n},\qquad \text{and}\qquad \mathcal{K}(w,u)\equiv\underline{\mathcal{D}}(\dot{u},[w])\cdot\underline{n}. \tag{4.4}$$

Applying the generalized divergence theorem [13], this implies the following Green-Herrera formula [3, 8, 10]:

$$\begin{aligned}\int_\Omega w\mathcal{L}u\,dx-\int_{\partial\Omega}\mathcal{B}(u,w)dx-\int_\Sigma\mathcal{J}(u,w)dx\\ =\int_\Omega u\mathcal{L}^*w\,dx-\int_{\partial\Omega}\mathcal{C}^*(u,w)dx-\int_\Sigma\mathcal{K}^*(u,w)dx.\end{aligned} \tag{4.5}$$

Introduce the following notation:

$$\langle Pu,w\rangle=\int_\Omega w\mathcal{L}u\,dx;\qquad \langle Q^*u,w\rangle=\int_\Omega u\mathcal{L}^*w\,dx; \tag{4.6}$$

$$\langle Bu,w\rangle=\int_{\partial\Omega}\mathcal{B}(u,w)dx;\qquad \langle C^*u,w\rangle=\int_{\partial\Omega}\mathcal{C}^*(u,w)dx; \tag{4.7}$$

$$\langle Ju,w\rangle=\int_\Sigma\mathcal{J}(u,w)dx;\qquad \langle K^*u,w\rangle=\int_\Sigma\mathcal{K}^*(u,w)dx. \tag{4.8}$$

With these definitions, each one of P, B, J, Q^*, C^*, and K^*, are real-valued bilinear functionals defined on $\hat{D}_1\times\hat{D}_2$, and Eq. (4.5) can be written as

$$\langle(P-B-J)u,w\rangle\equiv\langle(Q^*-C^*-K^*)u,w\rangle;\qquad \forall(u,w)\in\hat{D}_1\times\hat{D}_2 \tag{4.9}$$

or more briefly

$$P-B-J\equiv Q^*-C^*-K^*. \tag{4.10}$$

B. Variational Formulations of the Problem with Prescribed Jumps

A weak formulation of the BVPJ is

$$\langle(P-B-J)u,w\rangle\equiv\langle Pu_\Omega-Bu_\partial-Ju_\Sigma,w\rangle;\qquad \forall w\in\hat{D}_2 \tag{4.11}$$

From now on, the following notation is adopted: $u\in\hat{D}_1$ will be a function that fulfills Eq. (4.11), which is assumed to be unique. This equation can also be written as an equality between linear functionals, if f, g, and $j\in D_2^*$ are defined by $f\equiv Pu_\Omega$, $g\equiv Bu_\partial$, and $j\equiv Ju_\Sigma$:

$$(P-B-J)u=f-g-j. \tag{4.12}$$

This equation is equivalent to

$$(Q^*-C^*-K^*)u=f-g-j; \tag{4.13}$$

by virtue of Eq. (4.10). Thus, Eqs. (4.12) and (4.13) supply two different but equivalent variational formulations of the BVPJ. The first one will be referred as the "variational formulation in terms of the data," whereas the second one will be referred as the "variational formulation in terms of the complementary information" (this latter variational principle was introduced in [6] with the title "variational formulation in terms of the sought information," but, as will be seen, it is more convenient to reserve such name for another formulation that will be introduced later in this Section). Notice that Eqs. (4.12) and (4.13), alternatively, may be written as

$$\langle (P - B - J)u, w\rangle = \langle f - g - j, w\rangle; \qquad \forall w \in \hat{D}_2 \tag{4.14}$$

and

$$\langle (Q - C - K)^* u, w\rangle = \langle f - g - j, w\rangle; \qquad \forall w \in \hat{D}_2 \tag{4.15}$$

respectively. These equations exhibit more clearly their variational character.

C. Internal Boundary Information: Variational Formulations

A first step to derive Trefftz-Herrera procedures is to use the variational formulation in terms of the complementary information of Eq. (4.13) to establish conditions that a weighting function must fulfill in order to yield information on the internal boundary Σ, exclusively. By inspection of Eq. (4.15), it is seen that such test functions must have the property of annihilating Q^*u and C^*u in that equation. A weighting function possesses this ability if and only if $Cw = 0$ and $Qw = 0$, because

$$-\langle K^* u, w\rangle = \langle f - g - j, w\rangle; \qquad \forall w \in N_Q \cap N_C \subset \hat{D}_2. \tag{4.16}$$

Observe that the left-hand side of Eq. (4.16) involves the complementary information on Σ, exclusively, as desired. Generally, the complementary information on Σ, K^*u, is sufficient to define well-posed problems in each one of the subdomains of the domain decomposition, when the boundary data is added to it. However, it can be seen through specific examples that the complementary information K^*u is more than what is essential to achieve this goal and handling excessive information, in general, leads to handling too many degrees of freedom, which is computationally inconvenient. Thus, to develop numerical methods of optimal efficiency, it is necessary to eliminate part of such information. The general procedure for carrying out such elimination consists in introducing a "strong decomposition" $\{S, R\}$ of the bilinear functional K (for a definition of strong decomposition, see [9]). Then, S and R are bilinear functionals and fulfill

$$K \equiv S + R. \tag{4.17}$$

Then the sought information is defined to be S^*u, where $u \in \hat{D}_1$ is the solution of the BVPJ. In particular, a function $\tilde{u} \in \hat{D}_1$ is said to contain the sought information when $S^*\tilde{u} = S^*u$.

Let $\tilde{N} \subset \hat{D}_2$ be defined by $\tilde{N} \equiv N_Q \cap N_C \cap N_R$. An auxiliary concept, quite useful for formulating Trefftz-Herrera domain decomposition procedures, is the following (see [20]).

Definition 4.1. *A subset of weighting functions, $\mathcal{E} \subset \tilde{N} \equiv N_Q \cap N_C \cap N_R$, is said to be TH-complete for S^*, when for any $\hat{u} \in \hat{D}_1$, one has*

$$\langle S^*\hat{u}, w\rangle = 0, \qquad \forall w \in \mathcal{E} \Rightarrow S^*\hat{u} = 0. \tag{4.18}$$

Clearly, a necessary and sufficient condition for the existence of TH-complete systems, is that $\tilde{N} \equiv N_Q \cap N_C \cap N_R$ be, itself, a TH-complete system.

Theorem 4.1. *Let $\mathcal{E} \subset \tilde{N}$ be a system of TH-complete weighting functions for S^*, and let $u \in \hat{D}_1$ be the solution of the BVPJ. Then, a necessary and sufficient condition for $\hat{u} \in \hat{D}_1$ to contain the sought information, is that*

$$-\langle S^*\hat{u}, w\rangle = \langle f - g - j, w\rangle; \qquad \forall w \in \mathcal{E}. \tag{4.19}$$

Proof. If $u \in \hat{D}_1$ is the solution of the BVPJ, one has

$$-\langle S^* u, w\rangle = \langle f - g - j, w\rangle; \qquad \forall w \in \mathcal{E}. \tag{4.20}$$

Hence,

$$-\langle S^*(\hat{u} - u), w\rangle = 0; \qquad \forall w \in \mathcal{E}, \tag{4.21}$$

and, therefore, $S^*\hat{u} = S^*u$. ■

Theorem 4.1, supplies a very General Formulation of Indirect Methods (or Trefftz-Herrera Methods) of Domain Decomposition, which can be applied to any linear equation or system of such equations. When $u_P \in \hat{D}_1$ is a function satisfying $Pu_P = f$ and $Bu_P = g$, then Eq. (4.19) can be replaced by

$$-\langle S^*\hat{u}, w\rangle = -\langle S^*u_P, w\rangle + \langle J(u_P - u_\Sigma), w\rangle; \qquad \forall w \in \mathcal{E} \tag{4.22}$$

In applications, Eq. (4.22) determines the average of the solution and/or its derivatives on Σ. It is applicable to any linear differential equation or system of such equations independently of its type (elliptic, parabolic, or hyperbolic), including the case when the coefficients are discontinuous. In particular, when the differential operator is elliptic symmetric and positive definite and the sought information, S^*, is suitably chosen, the Steklov-Poincaré formulation can be derived from it.

D. The Symmetric Case

In this subsection it is assumed that $\hat{D}_1 = \hat{D}_2 \equiv \hat{D}, P = Q, B = C$, and $J = K$. Then $\tilde{N} \equiv N_Q \cap N_C \cap N_R = N_P \cap N_B \cap N_R$, and it is further assumed that the bilinear functional $-\langle S^*u, w\rangle$ is symmetric and positive definite when $\forall u, w \in \tilde{N}$. Even more, the hypotheses is taken that given any $u \in \hat{D}$, there is a $\tilde{u} \in \tilde{N}$ such that $S^*\tilde{u} = S^*u$. This latter hypotheses is tantamount to assume existence of solution of a local boundary value problem, as can be seen through specific examples. When this assumptions are fulfilled, it can be shown that the quadratic functional $-\langle S^*\hat{u}, \hat{u}\rangle - \langle f - g - j, \hat{u}\rangle$ attains its minimum over $\tilde{N}$, at $\tilde{u} \in \tilde{N}$, if and only if $\tilde{u} \in \tilde{N}$ contains the sought information.

V. ELLIPTIC EQUATIONS

As an illustration, in this section the methodology will be presented in detail in connection with the second-order linear differential equation of elliptic type, when the problem is defined in a space of an arbitrary number of dimensions. The procedures are applicable to any kind of boundary conditions for which the problem is well posed, but only boundary conditions of Dirichlet type will be considered here.

Using the notions and notations introduced in Section 2, for each $i = 1, \ldots, E$ and each $\alpha = 1, 2$, it will be assumed that $D_\alpha(\Omega_i) \equiv H^2(\Omega_i)$, so that $\hat{D}_1 = \hat{D}_2 \equiv \hat{H}^2(\Omega)$. Then the boundary value problem with prescribed jumps to be considered is

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b}u) + cu = f_\Omega \tag{5.1}$$

subjected to the boundary conditions

$$u = u_\partial \qquad \text{on } \partial\Omega, \tag{5.2}$$

and the jump conditions

$$[u] = [u_\Sigma] \equiv j_\Sigma^0 \qquad \text{and} \qquad [\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n} = [\underline{\underline{a}} \cdot \nabla u_\Sigma] \cdot \underline{n} \equiv j_\Sigma^1; \qquad \text{on } \Sigma. \tag{5.3}$$

Here, it will be assumed that $\mathcal{L}$ is coercive and that the BVPJ possesses one and only one solution.

When $\mathcal{L}$ is given by Eq. (5.1), the adjoint differential operator $\mathcal{L}^*$ is

$$\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw, \tag{5.4}$$

whereas

$$\underline{\mathcal{D}}(u, w) \equiv \underline{\underline{a}} \cdot (u\nabla w - w\nabla u) + \underline{b}uw. \tag{5.5}$$

For Dirichlet boundary conditions, a possible choice for $\mathcal{B}$ is

$$\mathcal{B}(u, w) \equiv (\underline{a}_n \cdot \nabla w)u. \tag{5.6}$$

In such case, Eq. (4.2) implies

$$\mathcal{C}^*(u, w) \equiv w(\underline{a}_n \cdot \nabla u - b_n u). \tag{5.7}$$

Above $\underline{a}_n = \underline{\underline{a}} \cdot \underline{n}$ and $b_n = \underline{b} \cdot \underline{n}$. The developments that follow apply even if the coefficients of the differential operator are discontinuous. In particular, when the coefficients are continuous, the second of the jump conditions of Eq. (5.3), in the presence of the first one, is equivalent to

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right]; \qquad \text{on } \Sigma. \tag{5.8}$$

Define

$$\mathcal{J}^0(u, w) \equiv -[u](\dot{\overline{\underline{\underline{a}} \cdot \nabla w + \underline{b}w}}) \cdot \underline{n} \qquad \text{and} \qquad \mathcal{J}^1(u, w) \equiv \dot{w}[\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n}, \tag{5.9}$$

together with

$$\mathcal{K}^0(w, u) \equiv \dot{u}[\underline{\underline{a}} \cdot \nabla w + \underline{b}w] \cdot \underline{n} \qquad \text{and} \qquad \mathcal{K}^1(w, u) \equiv -[w](\dot{\overline{\underline{\underline{a}} \cdot \nabla u}}) \cdot \underline{n}. \tag{5.10}$$

Then, for the case of continuous coefficients, application of Eqs. (4.4) yields

$$\mathcal{J}(u, w) \equiv \mathcal{J}^0(u, w) + \mathcal{J}^1(u, w), \qquad \text{and} \qquad \mathcal{K}(w, u) \equiv \mathcal{K}^0(w, u) + \mathcal{K}^1(w, u). \tag{5.11}$$

It is relevant to observe that, although in the above developments continuous coefficients were assumed, when the definitions of Eqs. (5.6), (5.7), (5.9), (5.10), and (5.11) are adopted, Eqs. (4.2)–(4.5) are fulfilled even if the coefficients of the differential operator $\mathcal{L}$ are discontinuous. Then, it is not difficult to verify that as was mentioned before, the theoretical developments that follow are applicable also to that case.

For the BVPJ defined by Eqs. (5.1)–(5.3), the variational formulations of Section 4, in "terms of the data" and in "terms of the complementary information," become available when the definitions of Eqs. (4.6)–(4.8) are introduced in Eqs. (4.14) and (4.15). If the notations

$$\langle J^0 u, w\rangle \equiv \int_\Sigma \mathcal{J}^0(u, w)\, dx \qquad \text{and} \qquad \langle J^1 u, w\rangle \equiv \int_\Sigma \mathcal{J}^1(u, w)\, dx; \tag{5.12}$$

$$\langle K^0 w, u\rangle \equiv \int_\Sigma \mathcal{K}^0(w,u)\,dx \qquad \text{and} \qquad \langle K^1 w, u\rangle \equiv \int_\Sigma \mathcal{K}^1(w,u)\,dx \tag{5.13}$$

are adopted, then $J = J^0 + J^1$ and $K = K^0 + K^1$, according to Eqs. (4.8). There are several options for the definition of the ''sought information'' of Section 4. To illustrate them, some of the choices that are possible are given next.

Choice 1. $S = K$ and $R = 0$. For this choice, a function $\tilde{u} \in \hat{H}^2(\Omega)$ ''contains the sought information,'' if and only if

$$\dot{\tilde{u}} \equiv \dot{u} \qquad \text{and} \qquad (\overline{\underline{\underline{a}} \cdot \nabla \tilde{u}})^{\cdot} \cdot \underline{n} = (\overline{\underline{\underline{a}} \cdot \nabla u})^{\cdot} \cdot \underline{n};\ \text{on}\ \Sigma, \tag{5.14}$$

where $u \in \hat{H}^2(\Omega)$ is the solution of the BVPJ. In the case of continuous coefficients, the second of these equations can be replaced by

$$\dot{\overline{\frac{\partial \tilde{u}}{\partial n}}} = \dot{\overline{\frac{\partial u}{\partial n}}}.$$

A function $w \in \mathcal{E} \subset N_Q \cap N_C \cap N_R \subset \hat{H}^2(\Omega)$ fulfills

$$\mathcal{L}^* w = 0,\ \text{in}\ \Omega \qquad \text{and} \qquad w = 0,\ \text{on}\ \partial\Omega \tag{5.15}$$

Observe that no condition is imposed on Σ, and therefore the method can be applied in a nonoverlapping domain decomposition.

Choice 2. $S = K^0$ and $R = K^1$. The sought information, in this case, is the average of the function across Σ, i.e., $\dot{\tilde{u}} \equiv \dot{u}$, and

$$-\langle S^* \tilde{u}, w\rangle \equiv -\int_\Sigma \dot{\tilde{u}}[\underline{\underline{a}} \cdot \nabla w] \cdot \underline{n} dx. \tag{5.16}$$

A function $w \in \mathcal{E} \subset N_Q \cap N_C \cap N_R \subset \hat{H}^2(\Omega)$, if and only if, it fulfills

$$[w] = 0, \qquad \text{on}\ \Sigma, \tag{5.17}$$

in addition to satisfying Eq. (5.15). Because of this continuity condition that the specialized test functions must satisfy, the application of this option of the method requires an overlapping domain decomposition. For Choice 2, the local well-posed problems, can be formulated as follows: assuming $\dot{u}$, on Σ, has been determined, one can apply the identities

$$u_+ = \dot{u} + \frac{1}{2}\,[u] \qquad \text{and} \qquad u_- = \dot{u} - \frac{1}{2}\,[u];\ \text{on}\ \Sigma \tag{5.18}$$

to obtain the values of the sought solution on both sides of Σ, because the jump $[u]$ is a datum. This information, when it is complemented with the prescribed boundary values on the outer boundary, in those elements for which this is required, is enough for formulating a Dirichlet boundary-value problem in each one of the subdomains of the partition. Solving them, the solution $u \in \hat{H}^2(\Omega)$ is recovered in the whole region Ω. On the other hand, for Choice 1, applying Eq. (5.16) together with

$$(\underline{a}_n \cdot \nabla u)_+ = (\overline{\underline{a}_n \cdot \nabla u})^{\cdot} + \frac{1}{2}\,[\underline{a}_n \cdot \nabla u];$$

$$\text{and} \qquad (\underline{a}_n \cdot \nabla u)_- = (\overline{\underline{a}_n \cdot \nabla u})^{\cdot} - \frac{1}{2}\,[\underline{a}_n \cdot \nabla u], \tag{5.19}$$

one can obtain not only the values of the sought solution but also the fluxes $\underline{a}_n \cdot \nabla u$, on both sides of Σ. This information is redundant; indeed, if all of it were used to formulate local boundary-value problems, they would be ill-posed. Numerically, this is a handicap of Choice 1, because handling excessive information is in general inconvenient.

It is interesting to observe that due to Eq. (5.17), for Choice 2, any function of a TH-complete system $\mathcal{E} \subset N_Q \cap N_C \cap N_R \subset \hat{H}^2(\Omega)$ belongs to $C^0(\Omega)$, in spite of the fact that the sought solution $u \in \hat{H}^2(\Omega)$ itself cannot be $C^0(\Omega)$, when $[u_\Sigma] \neq 0$, on Σ. Thus, the boundary value problem with prescribed jumps has been solved using exclusively weighting functions taken from a smaller space. Indeed, the original problem has been formulated on a space of fully discontinuous functions [i.e., this is Wheeler's $C^{-1}(\Omega)$ space], whereas the specialized weighting functions are sought on a space of continuous functions [i.e., a $C^0(\Omega)$ space]. Observe also that in a C^0 spline formulation the functions of the space $\hat{H}^2(\Omega)$ are C^0 from the start, and one necessarily has $[u] = 0$, on Σ. Thus, the formulation here presented is more general, because one may consider problems for which $[u] \neq 0$, on Σ, is prescribed. It is timely to mention that when implementing numerically the corresponding algorithms, the global matrix is exactly the same for problems in which the functions are smooth (zero jumps) and for problems with non vanishing jumps, as will be seen in Section 7.

When $\underline{b} \equiv 0$ and $c \geq 0$, the differential operator $\mathcal{L}$ is symmetric and positive definite and the following relations hold:

$$P \equiv Q, \qquad B \equiv C, \qquad J \equiv K \tag{5.20}$$

Even more, for Choice 2, $S = K^0 = J^1$ and $R = K^1 = J^0$ and the notation $\tilde{N} \equiv N_Q \cap N_C \cap N_R = N_Q \cap N_C \cap N_{K^1} = N_P \cap N_B \cap N_{J^0}$ will be adopted. It can be verified that

$$-\langle S^* \hat{u}, w \rangle \equiv -\int_\Sigma \hat{u}[\underline{\underline{a}} \cdot \nabla w] \cdot \underline{n} dx \equiv \int_\Omega \{\nabla w \cdot a \cdot \nabla \hat{u} + cw\hat{u}\} dx, \tag{5.21}$$

when $\hat{u} \in \tilde{N} \subset \hat{H}^2(\Omega)$ and $w \in \tilde{N} \subset \hat{H}^2(\Omega)$, so that $-S^* \equiv -K^{0*}$ is a symmetric and positive definite bilinear functional on $\tilde{N} \subset \hat{H}^2(\Omega)$.

VI. TH-COMPLETE SYSTEMS OF TEST FUNCTIONS

Discussions of TH-complete systems, in the context of the general theory, may be found in [4, 11]. Here attention will be restricted to the elliptic problem of Section 5, when the sought information is the average $\dot{u}$ across Σ (this is Choice 2 of Section 5). In this case the test functions $w \in \tilde{N} \equiv N_Q \cap N_C \cap N_R$ are continuous, vanish on $\partial\Omega$, and fulfill

$$\mathcal{L}^* w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw = 0; \qquad \text{in } \Omega - \Sigma, \tag{6.1}$$

In what follows it will be assumed that the region Ω is a rectangular region and the partition is also a rectangular one (Fig. 2). Then one can associate with each internal node (x_i, y_j), four rectangles $\{\Omega_{ij}^1, \ldots, \Omega_{ij}^4\}$ (Fig. 3), and the notations $\Omega_{ij}, \partial\Omega_{ij}$, and Σ_{ij} are adopted for the interior of the union of the four rectangle closures, the boundary of Ω_{ij}, and the intersection $\Sigma \cap \Omega_{ij}$, respectively. In a manner similar to what is done in direct decomposition methods (see Quarteroni and Valli [2]), let $\Lambda(\Sigma_{ij})$ be a space of functions defined on Σ_{ij} and vanishing on $\partial\Omega_{ij}$, defined by

$$\Lambda(\Sigma_{ij}) \equiv \{\eta \in H^{3/2}(\Sigma_{ij}) | \eta = v|_{\Sigma_{ij}} \text{ for a suitable } v \in H_0^2(\Omega_{ij})\}.$$

Given any function $w_{\Sigma_{ij}} \in \Lambda(\Sigma_{ij})$, one can associate uniquely a function $w_{ij} \in \tilde{N}$ whose support is Ω_{ij} and its restriction to Σ_{ij} is $w_{\Sigma_{ij}}$. Thus, in this manner a mapping of such subset of $H^{3/2}(\Sigma_{ij})$

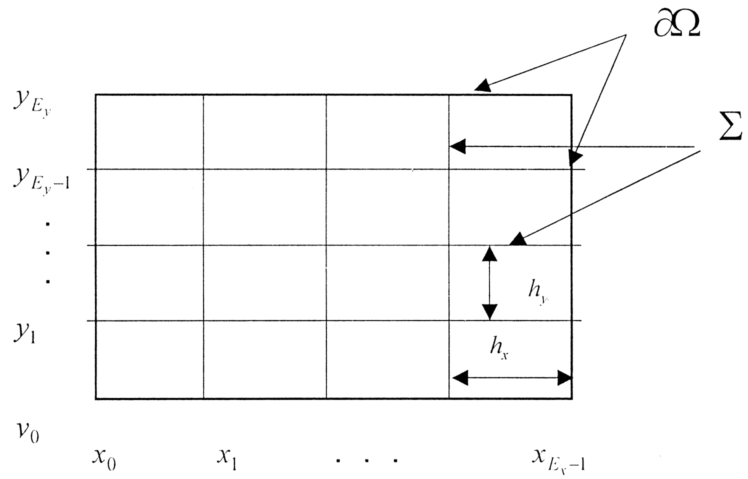

FIG. 2. Partition of the domain $\Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$ in rectangular $E_x \times E_y$ elements, where $h_x = x_i - x_{i-1}; i = 1, \ldots, E_x$ and $h_y = y_j - y_{j-1}; j = 1, \ldots, E_y$.

into $\tilde{N}$ is defined. Let $\mathcal{E}^r_{i,j} \subset H^{3/2}(\Sigma_{ij})$ be a family of functions that spans $H^{3/2}(\Sigma_{ij})$, and let $\mathcal{E}_{i,j} \subset \tilde{N}$ be the transform of $\mathcal{E}^r_{i,j}$ by means of the mapping just described. Define

$$\mathcal{E} \equiv \bigcup_{(i,j)\in\mathcal{S}} \mathcal{E}_{i,j} \subset \tilde{N}, \tag{6.2}$$

where $\mathcal{S}$ is the set of indexes for which (x_i, y_j) is a node of the partition. Then, it can be shown that $\mathcal{E} \equiv \cup_{(i,j)\in\mathcal{S}} \mathcal{E}_{i,j} \subset \tilde{N}$ is a TH-complete system.

In what follows, the local families of functions $\mathcal{E}^r_{i,j}$ will be taken to be families of piecewise polynomials defined on $\Sigma_{i,j}$. This kind of TH-complete families were first described in [11]. According to Fig. 3, Σ_{ij} is the union of four intervals and using the numbering of internal boundaries of Fig. 3, associated with each node (x_i, y_j), five classes of weighting functions will be constructed:

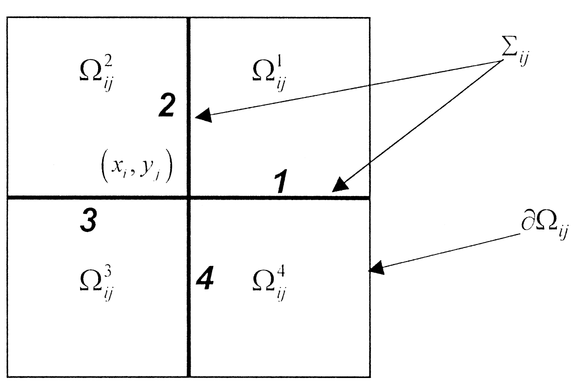

FIG. 3. Subregion Ω_{ij} associated with the node (x_i, y_j).

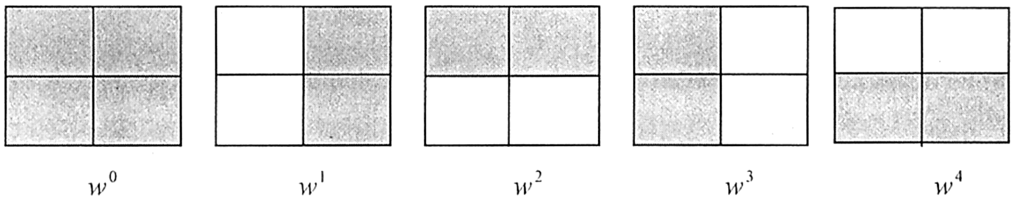

FIG. 4. Support of five classes of weighting functions.

Class 0. This is made of only one function, which is linear in each one of the four interior boundaries between the rectangles of Fig. 4, and such that $(x_i, y_j) = 1$.

Class 1. The restriction to interval "1," of Fig. 4 is a polynomial that vanishes at the end points of interval "1." There is one such polynomial for each degree (G) greater than one.

Classes 2–4. Classes 2, 3, and 4 are defined replacing interval "1" by the interval of the corresponding number in the definition of Class 1.

The support of the function of Class "0," is the whole square Ω_{ij}, whereas those associated with Classes "1" to "4," have as support the rectangles illustrated in Fig. 4. Observe that the set of polynomials of degree one that vanish at the end points of the interval "1" is void, and the same is true for the other intervals ("2" to "4").

It must be observed that the above construction does not lead directly to a system of linearly independent functions. This is due to the fact that each interval is counted twice in the above construction. Thus, for example, the four rectangles illustrated in Fig. 3 are those associated with the node which limits interval "1" by the left. When one considers the four rectangles associated with the node that limits interval "1" by the right, then the same interval becomes interval "3." However, a slight modification of the procedure just explained allows avoiding such repetition.

VII. THE NUMERICAL IMPLEMENTATION

As mentioned before, Theorem 4.1 constitutes a very general formulation, although somewhat abstract, of Trefftz-Herrera method of domain decomposition and the different algorithms in specific applications are derived directly from that theorem. However, Theorem 4.1 is an exact result and if it were possible to apply it fully, an exact solution would be obtained. The approximate nature of numerical solutions derived using TH-domain decomposition (TH-DD) stems from two sources: one of them is due to the fact that generally the differential equation is only fulfilled in an approximate manner and the second one comes from the fact that TH-complete systems for problems in several dimensions are infinite, and in numerical implementations one can apply only finite families of test functions. In particular, with reference to the families of functions introduced in the previous section, one may construct algorithms in which only polynomials of degree less or equal to G, where G is a given number, are kept in each one of the Classes "1" to "4." In general, each choice of G will give rise to a different kind of algorithm.

In this section the following notations are used: $H_i^0(x)$ is the one-dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}), which takes the value 1 at node x_i and zero at nodes x_{i-1} and x_{i+1}, and its first derivative is zero at all nodes x_{i-1}, x_i, and x_{i+1}. Similarly, $H_i^1(x)$ is the one-dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}),

TABLE I. Definitions of functions $B_{ij}^{\beta}(x, y)$, $\beta = 0, \ldots, 4$, where $H_i^0(x)$, $H_i^1(x)$, $H_j^0(y)$ and $H_j^1(y)$ are Hermite cubic polynomials in x and y, respectively.

	Ω_{ij}^1	Ω_{ij}^2	Ω_{ij}^3	Ω_{ij}^4
$B_{ij}^0(x,y)$	$\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)\left(\frac{y-y_{j+1}}{y_j-y_{j+1}}\right)$	$\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j+1}}{y_j-y_{j+1}}\right)$	$\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)$	$\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)$
$B_{ij}^1(x,y)$	$H_i^1(x)H_j^0(y)$	0	0	$H_i^1(x)H_j^0(y)$
$B_{ij}^2(x,y)$	$H_i^0(x)H_j^1(y)$	$H_i^0(x)H_j^1(y)$	0	0
$B_{ij}^3(x,y)$	0	$H_i^1(x)H_j^0(y)$	$H_i^1(x)H_j^0(y)$	0
$B_{ij}^4(x,y)$	0	0	$H_i^0(x)H_j^1(y)$	$H_i^0(x)H_j^1(y)$

which takes the value zero at nodes x_{i-1}, x_i, and x_{i+1}, and its first derivative takes the value 1 at node x_i and zero at the other nodes x_{i-1} and x_{i+1}.

A. The Weighting Functions

In the numerical implementations reported in this article, two families of test functions were constructed:

$$\mathcal{F} \equiv \{w_{ij}^0, w_{ij}^1, w_{ij}^2\} \quad \text{and} \quad \widehat{\mathcal{F}} \equiv \{\widehat{w}_{ij}^0, \widehat{w}_{ij}^1, \widehat{w}_{ij}^2, \widehat{w}_{ij}^3, \widehat{w}_{ij}^4\}. \tag{7.1}$$

Here, $w_{ij}^0 \equiv \widehat{w}_{ij}^0$ is the unique function belonging to Class "0," of Section 6, and $\widehat{w}_{ij}^\alpha$ is a function of Class "α," for each $\alpha = 1, \ldots, 4$, which fulfills, at interval "α," the boundary condition $\widehat{w}_{ij}^\alpha(x, y_j) = H_i^1(x)$, for $\alpha = 1, 3$, and $\widehat{w}_{ij}^\alpha(x_i, y) = H_j^1(y)$, for $\alpha = 2, 4$. In addition, one defines

$$w_{ij}^1(x,y) \equiv \widehat{w}_{ij}^1(x,y) + \widehat{w}_{ij}^3(x,y) \quad \text{and} \quad w_{ij}^2(x,y) \equiv \widehat{w}_{ij}^2(x,y) + \widehat{w}_{ij}^4(x,y). \tag{7.2}$$

It can be seen that the supports of w_{ij}^1 and w_{ij}^2 are the whole rectangle Ω_{ij}, and they are $C^1(\Omega_{ij})$ and fulfill the conditions $w_{ij}^1(x, y_j) = H_i^1(x)$ at the interval $x_{i-1} \le x \le x_{i+1}$ together with $w_{ij}^2(x_i, y) = H_j^1(y)$ at the interval $y_{j-1} \le y \le y_{j+1}$.

The family $\widehat{\mathcal{F}}$ was first constructed, and the family $\mathcal{F}$ was then derived by application of Eq. (7.2). The family $\widehat{\mathcal{F}}$ was built by solving local boundary value problems in each one of the subregions $\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\}$, separately. This was done introducing a set of functions $\{B_{ij}^0, B_{ij}^1, B_{ij}^2, B_{ij}^3, B_{ij}^4\}$, which fulfills the boundary conditions, and adding to it a linear combination of a family of functions $\{N_{ij}^1, N_{ij}^2, N_{ij}^3, N_{ij}^4\}$, which vanish on the boundary of each one of the subregions $\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\}$, in order to fulfill the differential equation. At each one of the four rectangles the local differential equations were solved using orthogonal collocation at four Gaussian points.

The functions $\{B_{ij}^0, B_{ij}^1, B_{ij}^2, B_{ij}^3, B_{ij}^4\}$ and $\{N_{ij}^1(x,y), \ldots, N_{ij}^4(x,y)\}$ have different expressions in each one of those rectangles; they are given in Tables I and II.

The general expression for functions of family $\widehat{\mathcal{F}}$ is (for $\alpha = 0, \ldots, 4$):

$$\widehat{w}_{ij}^\alpha(x,y) = B_{ij}^\alpha(x,y) + \sum_{\beta=1}^{4} C_{ij}^{\alpha\beta} N_{ij}^\beta(x,y), \tag{7.3}$$

where the coefficients $C_{ij}^{\alpha\beta}$ are piecewise constant in Ω_{ij} (Fig. 3), generally taking different values at each one of the rectangles $\{\Omega_{ij}^1, \Omega_{ij}^2, \Omega_{ij}^3, \Omega_{ij}^4\}$. They were obtained solving the system of collocation equations

$$\sum_{\beta=1}^{4} C_{ij}^{\alpha\beta} \mathcal{L}^* N_{ij}^{\beta}(x^p, y^p) = \mathcal{L}^* B_{ij}^{\alpha}(x^p, y^p); \qquad p = 1, \ldots, 4. \tag{7.4}$$

Generally, for nodes on the outer boundary $\partial\Omega$, some of the functions of the family $\mathcal{F}$ do not vanish on $\partial\Omega$ and therefore do not belong to $\tilde{N} = N_Q \cap N_C \cap N_R$ and cannot be used as weighting functions when applying the variational principle of Eq. (4.19).

B. The Base Functions

With respect to the base functions to be applied, except for the fact that their number has to be equal to that of weighting functions, there is considerable freedom of choice since they need only be defined on Σ. However, it is frequently advantageous to use the restriction of the test functions to the internal boundary Σ as base functions; in particular, in the symmetric and positive definite case this leads to symmetric and positive matrices, a fact that is specially advantageous when iterative methods are applied to solve the global system of equations. When the base functions are chosen in this manner, the general expression for the approximate solution of the BVPJ to be used on Σ is

$$\dot{\hat{u}}(x, y) = \sum_{(k,l)\in\eta} \sum_{\nu=0}^{NF-1} U_{kl}^{\nu} w_{kl}^{\nu}(x, y) + \sum_{(r,s)\in\eta_\partial} u_{\partial rs} B_{rs}^{0}(x, y) + \sum_{(k,l)\in\eta_I} \frac{\sigma}{2} [u_\Sigma]_{kl} B_{kl}^{0}(x, y). \tag{7.5}$$

Here, η is the collection of nodes for which the set of weighting functions, which vanish identically on $\partial\Omega$, is not void; η_∂ is the set of nodes lying on the external boundary but excluding the nodes at the corners; η_I is the set of internal nodes; and σ is the sign of the side of the internal boundary Σ. In addition, $u_{\partial rs} = u_\partial(x_r, y_s)$ and NF is the number of functions associated with the corresponding node.

C. The Algorithms

The algorithms that were developed in the work reported in this article, stem from direct application of the variational principle of Eq. (4.19) with Choice 2 of Section 5. According to Eq. (5.16), one has

$$-\int_\Sigma \dot{u}[\underline{n} \cdot \underline{\underline{a}} \cdot \nabla w] d\underline{x} = \int_\Omega w f_\Omega d\underline{x} - \int_{\partial\Omega} u_\partial \underline{n} \cdot \underline{\underline{a}} \cdot \nabla w d\underline{x} + \int_\Sigma j_\Sigma^0 \dot{\overline{(\underline{n} \cdot \underline{\underline{a}} \cdot \nabla w + b_n w)}} d\underline{x} - \int_\Sigma j_\Sigma^1 \dot{w} d\underline{x}, \qquad \forall w \in \mathcal{E}; \tag{7.6}$$

TABLE II. Definitions of functions $N_{ij}^{\beta}(x, y)$, $\beta = 1, \ldots, 4$, where $H_i^1(x)$ and $H_j^1(y)$ are Hermite cubic polynomials in x and y, respectively.

	Ω_{ij}^1	Ω_{ij}^2	Ω_{ij}^3	Ω_{ij}^4
$N_{ij}^1(x, y)$	$H_i^1(x)H_j^1(y)$	$H_{i-1}^1(x)H_j^1(y)$	$H_{i-1}^1(x)H_{j-1}^1(y)$	$H_i^1(x)H_{j-1}^1(y)$
$N_{ij}^2(x, y)$	$H_{i+1}^1(x)H_j^1(y)$	$H_i^1(x)H_j^1(y)$	$H_i^1(x)H_{j-1}^1(y)$	$H_{i+1}^1(x)H_{j-1}^1(y)$
$N_{ij}^3(x, y)$	$H_i^1(x)H_{j+1}^1(y)$	$H_{i-1}^1(x)H_{j+1}^1(y)$	$H_{i-1}^1(x)H_j^1(y)$	$H_i^1(x)H_j^1(y)$
$N_{ij}^4(x, y)$	$H_{i+1}^1(x)H_{j+1}^1(y)$	$H_i^1(x)H_{j+1}^1(y)$	$H_i^1(x)H_j^1(y)$	$H_{i+1}^1(x)H_j^1(y)$

Applying the expression for the approximate solution $\dot{\hat{u}}(x, y)$ of Eq. (7.5), the global system of equations is derived:

$$M^{\mu\nu}_{ijkl}U^{\nu}_{kl} = F^{\mu}_{ij}; \qquad (k,l) \text{ and } (i,j) \in \eta, \quad \mu,\nu = 0,\ldots,NF-1 \tag{7.7}$$

where the matrix of the system is given by

$$M^{\mu\nu}_{ijkl} = -\int_{\Sigma} w^{\nu}_{kl}[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w^{\mu}_{ij}]d\underline{x}, \tag{7.8}$$

and the right hand side is

$$F^{\mu}_{ij} = \int_{\Omega} w^{\mu}_{ij} f_{\Omega} d\underline{x} - \int_{\partial\Omega} u_{\partial}\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w^{\mu}_{ij} d\underline{x} + \sum_{(r,s)\in\eta_{\partial}} u_{\partial rs}\int_{\Sigma} B^{0}_{rs}[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w^{\mu}_{ij}]d\underline{x}$$
$$+\int_{\Sigma} j^{0}_{\Sigma}\overline{\dot{(\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w^{\mu}_{ij} + b_n w^{\mu}_{ij})}}d\underline{x} - \int_{\Sigma} w^{\mu}_{ij} j^{1}_{\Sigma} d\underline{x} + \sum_{(k,l)\in\eta_I}\frac{\sigma}{2}[u_{\Sigma}]_{kl}\int_{\Sigma} B^{0}_{kl}[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w^{\mu}_{ij}]d\underline{x}$$

$$(k,l) \text{ y } (i,j)\in\eta, \quad \mu,\nu = 0,\ldots,NF-1 \tag{7.9}$$

In the work reported in this article two algorithms were developed and tested:

Algorithm I. The family of test functions contained only one member: w^{0}_{ij} associated with each internal node. This leads to an algorithm in which only one degree of freedom is associated with each internal node. Even more, the global matrix is nine-diagonal, and when so is the differential operator, symmetric and positive definite (this corresponds to $\underline{b} \equiv 0$ and $c \geq 0$).

Algorithm II. The complete family $\mathcal{F}$ of three functions (or less, at those nodes in which some of the functions of this family do not satisfy the required zero boundary condition on the external boundary) was applied at each node, including boundary nodes. This leads to an algorithm in which three, or less, degrees of freedom are associated with each node. Even more, the global matrix is block nine-diagonal and, when so is the differential operator, symmetric and positive definite (this corresponds to $\underline{b} \equiv 0$ and $c \geq 0$). The blocks are 3×3.

D. The Numerical Experiments

The numerical experiments that were performed consisted in applying the Algorithms I and II, of Subsection 7.3, for solving the BVPJ Eqs. (5.1)–(5.3), subjected to Dirichlet boundary conditions. The examples treated correspond to several choices of the coefficients in Eq. (5.1), which are given in Table III. The analytical solutions are given in Table IV, for each one of them. In all cases the domain of definition was the unit square $[0,1]\times[0,1]$, except for Example 2 in which the domain of definition of the problem was the square $[1,2]\times[1,2]$. The boundary values that were imposed were those which are implied by the analytical solutions. Only in Examples 6 and 7 the jumps imposed were different from zero, and for these latter examples, they are given in Table IV.

The numerical results are summarized in Figs. 5–11. Each one of the examples was solved in a uniform rectangular partition ($E = E_x = E_y$) of the domain (Fig. 2) using Algorithm I and, subsequently, Algorithm II, for which the weighting functions are piecewise linear and piecewise cubic, respectively, on Σ. The convergence rate of the error—measured in terms of the norm $\|\ \|_{\infty}$—is $O(h^2)$ and $O(h^4)$, respectively, as shown in those figures.

TABLE III. Coefficients and right-hand term of the examples treated.

Example	$\underline{\underline{a}}$	$\underline{b}$	c	f_Ω
1	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	1	$(1 - x^2 - y^2)e^{xy}$
2	$a_{11} = a_{22} = xy$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	0	0
3	$a_{11} = 1 + x^2$ $a_{22} = 1 + y^2$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	0	$6(y^2 - x^2)$
4	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	0	$-2(4\pi)^2 \cos(4\pi x)\sin(4\pi y)$
5	$a_{11} = 1 + x^2$ $a_{22} = 1 + y^2$ $a_{12} = a_{21} = 0$	$b_1 = y - 2,$ $b_2 = x - 2$	$2(x + y)$	$-(x^4 + y^4)e^{xy}$
6	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 1$	0	0
7	$a_{11} = a_{22} =$ $= \begin{cases} 1; & 0 \le y \le \frac{1}{2} \\ 4; & \frac{1}{2} < y \le 1 \end{cases}$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	1	$(1 - x^2 - y^2)e^{xy};\ 0 \le y \le \frac{1}{2}$ $(1 - 4x^2 - 4y^2)e^{xy};\ \frac{1}{2} < y \le 1$

TABLE IV. Analytic solution for each one of the examples.

Example	Exact solution
1	e^{xy}
2	$x^2 - y^2$
3	$x^2 - y^2$
4	$\cos(4\pi x)\sin(4\pi y)$
5	e^{xy}
6	$e^x + e^y - 2;\ y < \frac{1}{2}$ $e^x + e^y;\ y = \frac{1}{2}$ $e^x + e^y + 2;\ y > \frac{1}{2}$ with jump conditions: $j^0_\Sigma(x, 0.5) = 4;\ x \in [0, 1]$
7	e^{xy} with jump conditions: $j^1_\Sigma(x, 0.5) = 3xe^{x/2};\ x \in [0, 1]$

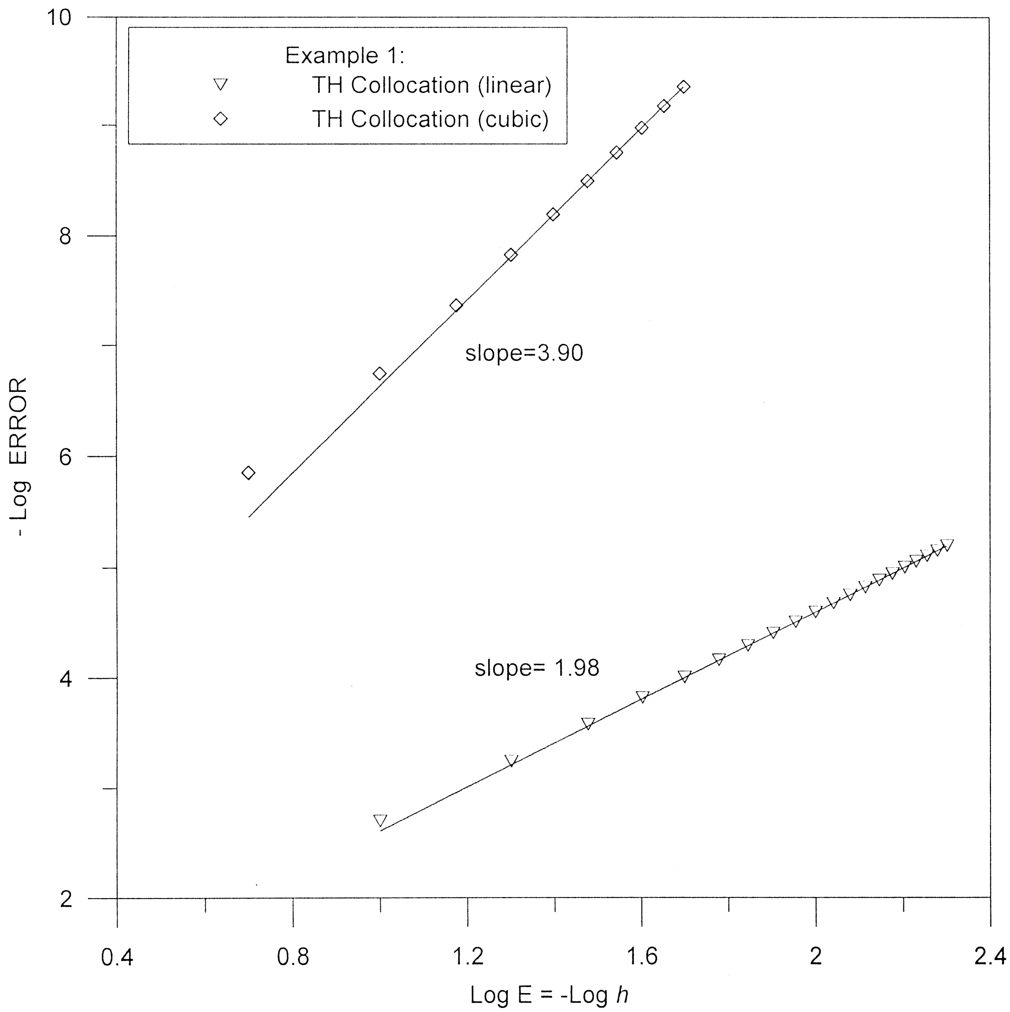

FIG. 5. Example 1. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

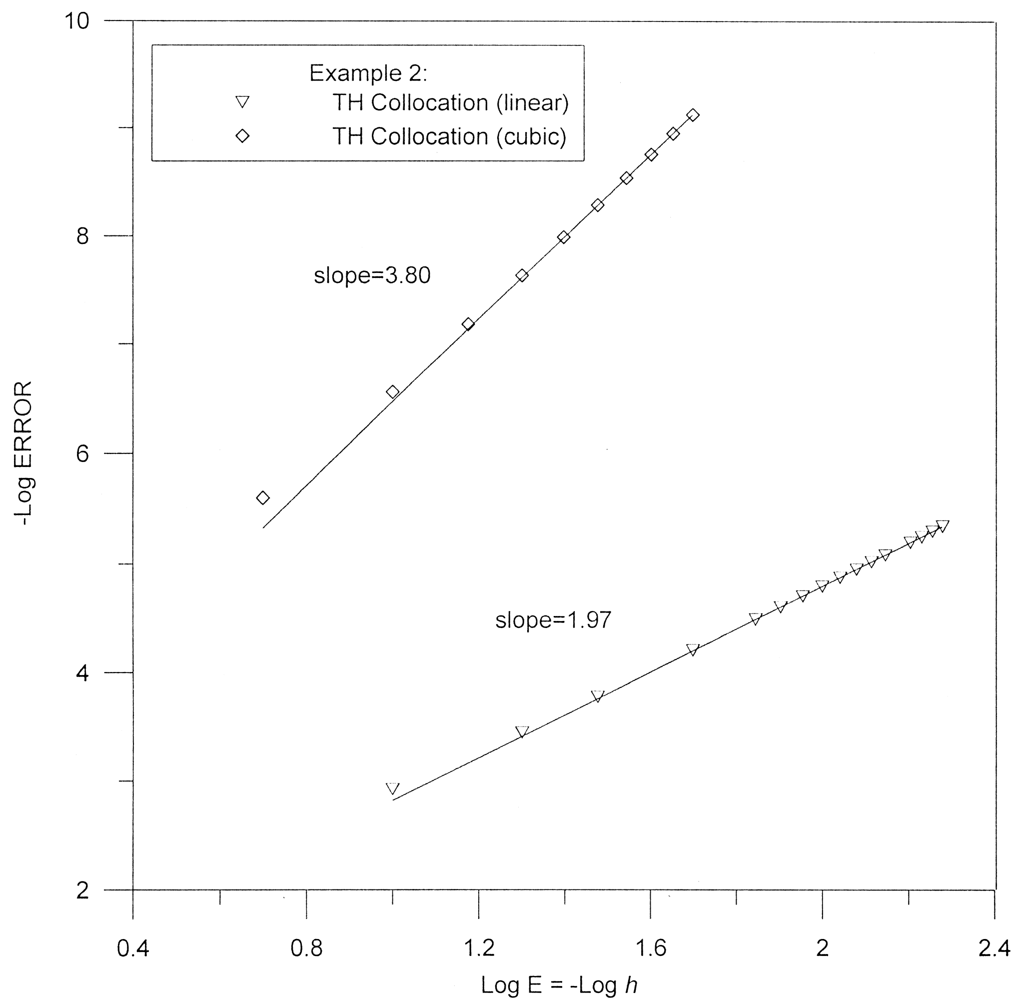

FIG. 6. Example 2. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

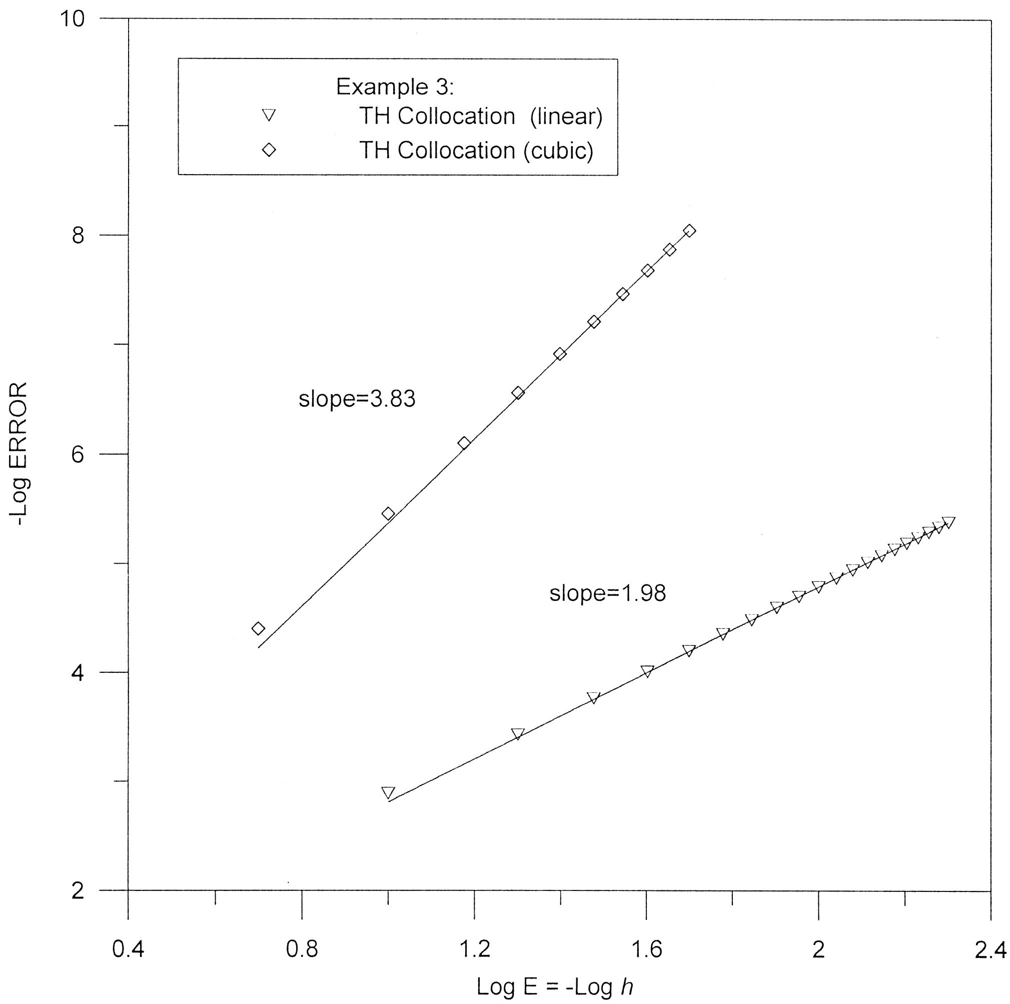

FIG. 7. Example 3. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

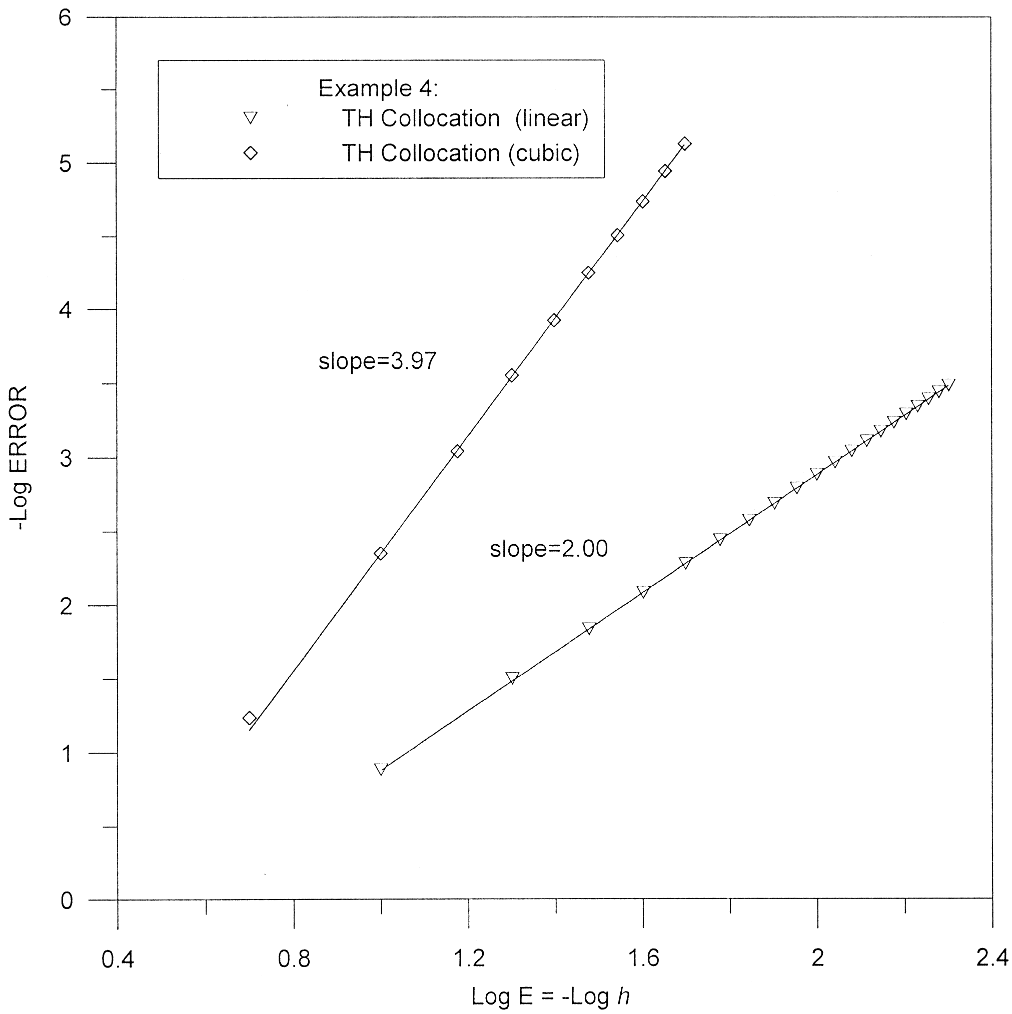

FIG. 8. Example 4. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

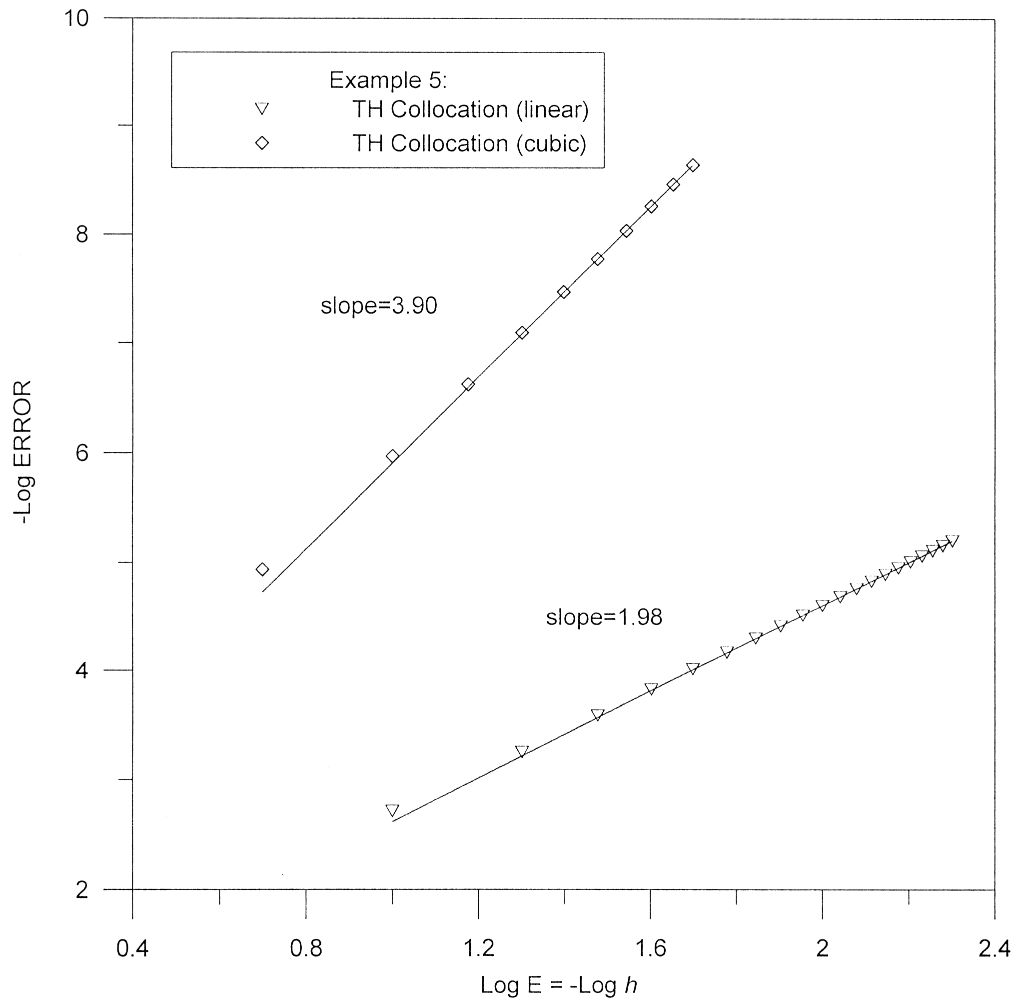

FIG. 9. Example 5. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

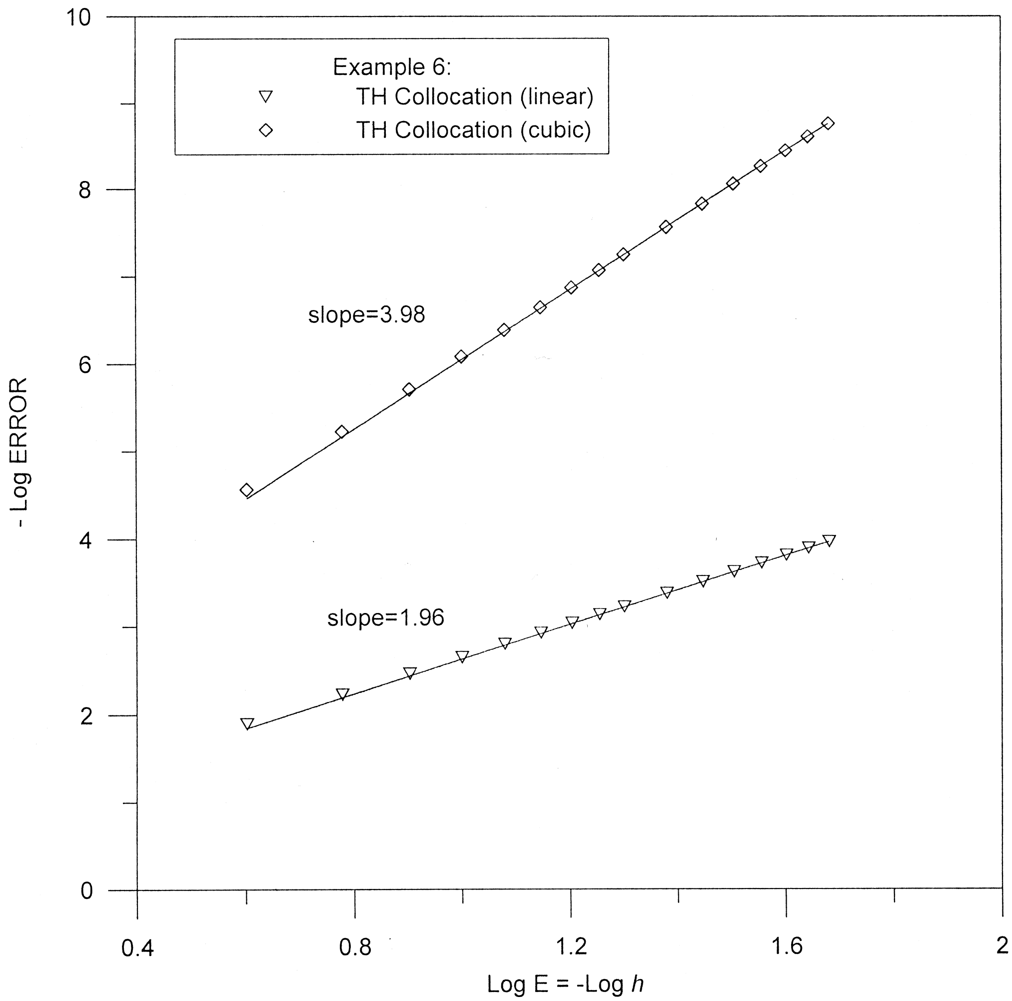

FIG. 10. Example 6. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

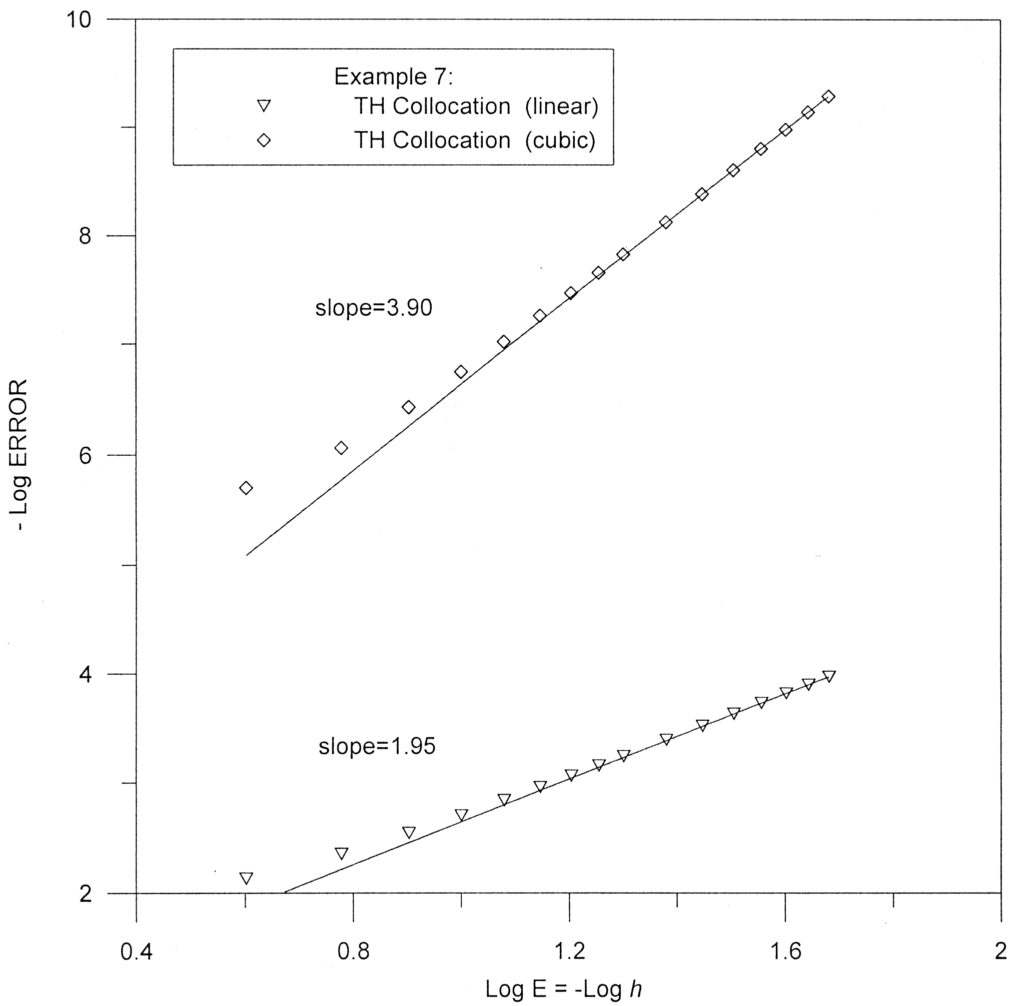

FIG. 11. Example 7. Convergence rate of Trefftz-Herrera collocation method using linear and cubic weighting functions.

VIII. CONCLUSIONS

In the general theory of domain decomposition methods, introduced by Herrera et al. in previous articles [3, 4], two broad classes of procedures were identified: direct and indirect, or Trefftz-Herrera, methods. In the present article, this latter kind of method has been presented in a more complete and systematic manner than in previous publications. As it has been exhibited here, the theory of indirect methods is quite systematic and possesses outstanding generality. One important feature of indirect methods is that they subsume the Steklov-Poincaré approach, as has been indicated here and will be discussed more thoroughly elsewhere. When the numerical procedure that is used for producing the local solutions is collocation, a nonstandard method of collocation is obtained that possesses several attractive features. Indeed, a dramatic reduction in the number of degrees of freedom associated with each node is obtained: in the standard method of collocation that number is two in one dimension, four in two dimensions, and eight in three dimensions, whereas for some of the new algorithms they are only one in all space dimensions, which is due to the relaxation in the continuity conditions required by indirect methods. Also, the global matrix is symmetric and positive definite when so is the differential operator, whereas in the standard method of collocation, using Hermite cubics, this does not happen. In addition, it must be mentioned that the boundary value problem with prescribed jumps at the internal boundaries can be treated as easily as the smooth problem, i.e., that with zero jumps, because the solution matrix and the order of precision is the same for both problems. It must be observed also that, when the indirect method is applied, the error of the approximate solution stems from two sources: the approximate nature of the test functions and the fact that TH-complete systems of test functions, which are infinite for problems in several dimensions, are approximated by finite families of such functions. In particular, when Hermite cubics are used to approximate the local solutions, in the problems treated in this article, the error is $O(h^4)$, if the test functions are piecewise cubic on Σ, and it is $O(h^2)$ when the test functions are only piecewise linear, on that interior boundary. Finally, the construction of the test functions is quite suitable to be computed in parallel.

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