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1. The Indirect Approach To Domain Decomposition

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1. Introduction. The main objective of DDM is, when a domain Ω and one of its partitions are given, to obtain the solution of a boundary value problem defined on it (the 'global problem'), by solving problems formulated on the subdomains of the partition (the 'local problems'), exclusively. This objective can be achieved if sufficient information about the global solution is known, on the internal boundary (which separates the subdomains from each other and to be denoted by Σ), for defining well-posed problems in each one of the subdomains of the partition. Herrera proposed recently a general and unifying theory [15],[14], in which DDM are interpreted as methods for gathering such information. According to it, one defines an information-target on Σ, referred as the *sought information* [15], and the objective of DDM is to obtain such information. There are two main procedures for gathering the *sought information*, which yield two broad categories of DDM: *direct methods* and *indirect* (or *Trefftz-Herrera*) *methods*. This paper belongs to a sequence of papers [15],[6],[5], [4],[21], included in this Proceedings, in which an overview of Herrera's unified theory is given. In particular, the present paper is devoted to a systematic presentation of indirect methods, and a companion paper deals with direct methods [6].

Herrera *et al.* [18],[9],[16], [10],[11],[17], [13] introduced indirect methods in numerical analysis. They are based on the author's Algebraic Theory of boundary value problems [9],[10],[8]. Numerical procedures such as Localized Adjoint Methods (LAM) and Eulerian-Lagrangian LAM (ELLAM) are representative applications [17],[3]. A large number of transport problems in several dimensions have been treated using ELLAM [20]. Indirect Methods of domain decomposition stem from the following observation: when the method of weighted residuals is applied, the information about the exact solution that is contained in the approximate one is determined by the family of test functions that is used, exclusively [9],[16],[10]. This opens the possibility of constructing and applying a special kind of weighting functions, which have the property of yielding the *sought information* at the internal boundary Σ, exclusively, as it is done in Trefftz-Herrera Methods.

The construction of such weighting functions requires having available an instrument of analysis of the information supplied by different test functions. The natural framework for such analysis is given by Green's formulas. However, the conventional approach to this matter is not sufficiently informative for applications to domain decomposition methods. Indeed, in the usual approach [19], one considers the Green's formula

$$\int_\Omega w\mathcal{L}u dx = \int_\Omega u\mathcal{L}^* w dx \tag{1.1}$$

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where Ω is a given region and, $\mathcal{L}$ and $\mathcal{L}^*$ are a differential operator and its adjoint, respectively. Then, given a family of functions $\left\{w^1, ..., w^N\right\}$, any approximate solution, $\hat{u}$, obtained with the method of weighted residuals, and with such family of test functions, fulfills

$$\int_\Omega w^\alpha \left(\mathcal{L}\hat{u} - f_\Omega\right) dx = \int_\Omega w^\alpha \left(\mathcal{L}\hat{u} - \mathcal{L}u\right) dx = \int_\Omega \left(\hat{u} - u\right) \mathcal{L}^* w^\alpha dx = 0 \tag{1.2}$$

In this manner, the conclusion is reached that the error $u - \hat{u}$ is orthogonal to the space spanned by the family of functions $\left\{\mathcal{L}^* w^1, ..., \mathcal{L}^* w^N\right\}$. However, this result is of little use when dealing with domain decomposition methods. For them, it is necessary to have a theory which is applicable to situations in which both trial and test functions may be discontinuous simultaneously. This was done introducing a kind of Green's formulas (*"Green-Herrera formulas"*) especially developed for operators defined on discontinuous fields (see [9],[16],[10]). They are based on the author's abstract algebraic theory of boundary value problems, which possesses great generality; it was presented in a preliminary form in [8] and, later, further developed [9],[16],[10] and applied to the numerical treatment of differential equations [17],[12]. This kind of Green's formulas have been formulated in a special kind of function-spaces, in which their elements have jump discontinuities across the internal boundary. In particular, a special class of Sobolev spaces is constructed in this manner [2].

2. Notation. Consider a region Ω, with boundary $\partial\Omega$ and a partition $\{\Omega_1, ..., \Omega_E\}$ of Ω. Let

$$\Sigma \equiv \bigcup_{i \neq j} \left(\bar{\Omega}_i \cap \bar{\Omega}_j\right) \tag{2.1}$$

then Σ will be referred as the 'internal boundary' and $\partial\Omega$ as the 'external (or outer) boundary'. For each $i = 1, ..., E$, $D_1\left(\Omega_i\right)$ and $D_2\left(\Omega_i\right)$ will be two linear spaces of functions defined on Ω_i; then the spaces of *trial* (or base) and *test* (or weighting) functions are defined to be

$$\hat{D}_1\left(\Omega\right) \equiv D_1\left(\Omega_1\right) \oplus ... \oplus D_1\left(\Omega_E\right); \tag{2.2}$$

and

$$\hat{D}_2\left(\Omega\right) \equiv D_2\left(\Omega_1\right) \oplus ... \oplus D_2\left(\Omega_E\right); \tag{2.3}$$

respectively. In what follows we write $\hat{D}_1$ and $\hat{D}_2$, instead of $\hat{D}_1\left(\Omega\right)$ and $\hat{D}_2\left(\Omega\right)$, in order to simplify the notation. Functions belonging either to $\hat{D}_1$ and $\hat{D}_2$, are finite sequences of functions belonging to each one of the sub-domains of the partition. It will be assumed that for each $i = 1, ..., E$, and $\alpha = 1,\ 2$, the traces on Σ of functions belonging $D_\alpha\left(\Omega_i\right)$ exist, and the jump and average of test or weighting functions is defined by

$$[u] \equiv u_+ - u_-; \quad and \quad \dot{u} \equiv (u_+ + u_-)/2; \tag{2.4}$$

where u_+ and u_- are the traces from one and the other side of Σ. Here, the unit normal vector to Σ is chosen arbitrarily, but the convention is such that it points towards the positive side of Σ. The special class of Sobolev spaces defined by

$$\hat{\mathbb{H}}^s(\Omega) \equiv \mathbb{H}^s(\Omega_1) \oplus ... \oplus \mathbb{H}^s(\Omega_E); \tag{2.5}$$

has special interest and was considered in [13].

3. Scope. It must be emphasized that the scope of the general theory presented in this paper, Herrera's unified theory of domain decomposition [15],[14], is quite wide, since it is applicable to any linear partial differential equation or system of such equations independently of its type. It handles problems with prescribed jumps on the internal boundary, Σ, and discontinuous equation coefficients, although every kind of equation has its own peculiarities. In particular, we would like to mention explicitly the following:

1. A SINGLE EQUATION
   - (a) Elliptic
     - i. Second Order
     - ii. Higher-Order
       - A. Biharmonic
   - (b) Parabolic
     - i. Heat Equation
   - (c) Hyperbolic
     - i. Wave Equation
2. SYSTEMS OF EQUATIONS
   - (a) Stokes Problems
   - (b) Mixed Methods (Raviart-Thomas)
   - (c) Elasticity

The general form of the boundary value problem with prescribed jumps (BVPJ), to be considered, is

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega; \quad in\ \Omega_i \quad i = 1, ..., E \tag{3.1}$$

$$B_j u = B_j u_\partial \equiv g_j; \quad on \quad \partial\Omega \tag{3.2}$$

and

$$[J_k u] = [J_k u_\Sigma] \equiv j_k; \quad on \quad \Sigma \tag{3.3}$$

where the $B_j's$ and $J_k's$ are certain differential operators (the $j's$ and $k's$ run over suitable finite ranges of natural numbers). Here, in addition, $u_\Omega \equiv \left(u_\Omega^1, ..., u_\Omega^E\right)$, u_∂ and u_Σ are given functions belonging to $\hat{D}_1$ (i.e., *'trial functions'*), which fulfill Eqs.(3.1), (3.2) and (3.3), respectively. Moreover, f_Ω, g_j and j_k may be defined by Eqs. (3.1) to (3.3).

In what follows, it will be assumed that the boundary conditions and jump conditions of this BVPJ can be brought into the point-wise variational form:

$$\mathcal{B}(u,w) = \mathcal{B}(u_\partial,w) \equiv g_\partial(w)\,; \quad \forall w \in \hat{D}_2 \tag{3.4}$$

and

$$\mathcal{J}(u,w) = \mathcal{J}(u_\Sigma,w) \equiv j_\Sigma(w)\,; \quad \forall w \in \hat{D}_2 \tag{3.5}$$

where $\mathcal{B}(u,w)$ and $\mathcal{J}(u,w)$, are bilinear functions defined point-wise.

4. Trefftz-Herrera Approach to DDM. Let us recall a few basic points of Herrera's unified theory (see [15]). The information that one deals with, when formulating and treating partial differential equations (i.e., the BVPJ), is classified in two broad categories: *'data of the problem'* and *'complementary information'*. In turn, three classes of data can be distinguished: data in the interior of the subdomains of the partition (given by the differential equation, which in the BVPJ is fulfilled in the interior of the subdomains, exclusively), the data on the external boundary ($B_j u, \quad on \quad \partial\Omega$) and the data on the internal boundary (namely, $[J_k u], \quad on \quad \Sigma$). The complementary information can be classified in a similar fashion: the values of the sought solution in the interior of the subdomains ($u_i \in D(\Omega_i)$, for $i = 1, ..., E$); the complementary information on the outer boundary (for example, the normal derivative in the case of Dirichlet problems for Laplace's equation); and the complementary information on the internal boundary Σ (for example, the average of the function and the average of the normal derivative across the discontinuity for elliptic problems of second order [5]). In the unified theory of DDM, a target of information, which is contained in the complementary information on Σ, is defined; it is called *'the sought information'*. It is required that the *sought information*, when complemented with the data of the problem, be sufficient for determining uniquely the solution of BVPJ in each one of the subdomains of the partition.

In general, however, the sought information may satisfy this property and yet be redundant, in the sense that if all of it is used simultaneously together with the data of the problem, ill-posed problems are obtained. Consider for example, a Dirichlet problem of an elliptic-type second order equation (see [5]), for which the jumps of the function and of its normal derivative have been prescribed. If for such problem the *sought information* is taken to be the average of the function -i.e., $(u_+ + u_-)/2$- and the average of the normal derivative -i.e., $\frac{1}{2}\partial(u_+ + u_-)/\partial n$, on Σ-, then it may be seen that it contains redundant information. Indeed, $u_+ = \frac{1}{2}(u_+ + u_-) + \frac{1}{2}(u_+ - u_-)$, $u_- = \frac{1}{2}(u_+ + u_-) - \frac{1}{2}(u_+ - u_-)$, and a similar relation holds for the normal derivatives. Therefore, if the *'sought information'* and the *'data of the problem'* are used simultaneously, one may derive not only the value of the BVPJ solution on the boundary of each one of the subdomains, but also the normal derivative, at least in a non-void section of those boundaries. As it is well known, this is an ill-posed problem, because

Dirichlet problem is already well-posed in each one of the subdomains. Thus, the *sought information* contains redundant information in this case.

Generally, in the numerical treatment of partial differential equations, efficiency requires eliminating redundant information. Due to this fact, when the choice of the *sought information* is such that there is a family of well-posed problems -one for each subdomain of the partition- which uses all the sought information, together with all the data of the BVPJ, such choice is said to be *'optimal'*. Once the information-target constituted by the *sought information* has been chosen, it is necessary to design a procedure for gathering it. There are two main ways of proceeding to achieve this goal: *direct methods* and *indirect* (or *Trefftz-Herrera*) *methods*. In the following Sections the general framework for designing indirect procedures is constructed.

Firstly, Green-Herrera formulas, which were originally derived in 1985 [9],[16],[10] will be presented. They are equations that relate the *'data of the problem'* with *'the complementary information'*. Then, a general variational principle of the usual kind, *in terms of the data of the problem*, which applies to any BVPJ, is introduced. Using Green-Herrera formula the variational formulation *in terms of the data of the problem*, is transformed into one *in terms of the complementary information*. Among the complementary information the *sought information* is singled out and the conditions that the test functions must satisfy in order to eliminate all the complementary information, except the *sought information*, are identified. When the variational principle *in terms of the complementary information* is applied, with weighting functions that fulfill such conditions, a variational principle which characterizes the sought information is derived. This principle provides a very general, although somewhat abstract, basis of Trefftz-Herrera Method (this is given by Theorem 7.1 Eq. 7.4).

5. Green-Herrera Formulas. To start, let $\mathcal{L}$ and $\mathcal{L}^*$ be a differential operator and its formal adjoint; then there exists a vector-valued bilinear function $\underline{\mathcal{D}}$, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^*w \equiv \nabla \cdot \underline{\mathcal{D}}(u,w) \tag{5.1}$$

It will also be assumed that there are bilinear functions $\mathcal{B}(u,w)$, $\mathcal{C}(w,u)$, $\mathcal{J}(u,w)$ and $\mathcal{K}(w,u)$, the first two defined on $\partial\Omega$ and the last two on Σ, such that

$$\underline{\mathcal{D}}(u,w) \cdot \underline{n} = \mathcal{B}(u,w) - \mathcal{C}(w,u); \quad \text{ on } \partial\Omega \tag{5.2}$$

and

$$-[\underline{\mathcal{D}}(u,w) \cdot \underline{n}] = \mathcal{J}(u,w) - \mathcal{K}(w,u); \quad \text{ on } \Sigma \tag{5.3}$$

Generally, the definitions of $\mathcal{B}$ and $\mathcal{C}$ depend on the kind of boundary conditions and the "smoothness criterion" of the specific problem considered [9],[16]. For the case when the coefficients of the differential operators are continuous, Herrera has given very general formulas for $\mathcal{J}$ and $\mathcal{K}$ [18]; they are:

$$\mathcal{J}(u,w) = -\underline{\mathcal{D}}([u],\dot{w}) \cdot \underline{n} \quad \text{and} \quad \mathcal{K}(w,u) = \underline{\mathcal{D}}(\dot{u},[w]) \cdot \underline{n} \tag{5.4}$$

Applying the generalized divergence theorem [2], this implies the following Green-Herrera formula [18]:

$$\begin{aligned}&\int_\Omega w\mathcal{L}u dx-\int_{\partial\Omega}\mathcal{B}(u,w)dx-\int_\Sigma\mathcal{J}(u,w)dx\\ &=\int_\Omega u\mathcal{L}^*w dx-\int_{\partial\Omega}\mathcal{C}^*(u,w)dx-\int_\Sigma\mathcal{K}^*(u,w)dx\end{aligned}\tag{5.5}$$

Introduce the following notation:

$$\langle Pu,w\rangle=\int_\Omega w\mathcal{L}u dx;\quad \langle Q^*u,w\rangle=\int_\Omega u\mathcal{L}^*w dx\tag{5.6}$$

$$\langle Bu,w\rangle=\int_{\partial\Omega}\mathcal{B}(u,w)dx;\quad \langle C^*u,w\rangle=\int_{\partial\Omega}\mathcal{C}^*(u,w)dx\tag{5.7}$$

$$\langle Ju,w\rangle=\int_\Sigma\mathcal{J}(u,w)dx;\quad \langle K^*u,w\rangle=\int_\Sigma\mathcal{K}^*(u,w)dx\tag{5.8}$$

With these definitions, each one of P, B, J, Q^*, C^* and K^*, are real-valued bilinear functionals defined on $\hat{D}_1\times\hat{D}_2$, and Eq.(5.5) can be written as

$$\langle(P-B-J)u,w\rangle\equiv\langle(Q^*-C^*-K^*)u,w\rangle;\quad \forall(u,w)\in\hat{D}_1\times\hat{D}_2\tag{5.9}$$

or more briefly

$$P-B-J\equiv Q^*-C^*-K^*;\tag{5.10}$$

6. Variational Formulations of the Problem with Prescribed Jumps. A weak formulation of the BVPJ is

$$\langle(P-B-J)u,w\rangle\equiv\langle f-g-j,w\rangle;\quad \forall w\in\hat{D}_2\tag{6.1}$$

where f, g and $j\in D_2^*$. This equation is equivalent to

$$\langle(Q^*-C^*-K^*)u,w\rangle\equiv\langle f-g-j,w\rangle;\quad \forall w\in\hat{D}_2\tag{6.2}$$

by virtue of Green-Herrera formula of Eq. (5.10). Necessary conditions for the existence of solution of this problem is that there exist $u_\Omega\in\hat{D}_1$, $u_\partial\in\hat{D}_1$ and $u_\Sigma\in\hat{D}_1$, such that:

$$f\equiv Pu_\Omega,\quad g\equiv Bu_\partial\quad and\quad j\equiv Ju_\Sigma\tag{6.3}$$

Thus, it is assumed that such functions exist. From now on, the following notation is adopted: $u\in\hat{D}_1$ will be a solution of the BVPJ, which is assumed to exist and to be unique; therefore, $u\in\hat{D}_1$ fulfills Eq. (6.1). Observe that Eqs. (6.1) and (6.2) supply two different but equivalent variational formulations of the BVPJ. The first one will be referred as the '*variational formulation in terms of the data*', while the second one will be referred as the '*variational formulation in terms of the complementary information*' (this latter variational principle was introduced in [18] with the title "variational formulation in terms of the sought information" but it is more convenient to reserve such name for another formulation that will be introduced later).

Eqs. (6.1) and (6.2), can also be written as equalities between linear funtionals:

$$(P-B-J)u=f-g-j;\tag{6.4}$$

and

$$(Q^* - C^* - K^*)u = f - g - j; \tag{6.5}$$

respectively.

7. Variational Formulation of Trefftz-Herrera Method. A first step to derive Trefftz-Herrera procedures is to use the variational formulation in terms of the complementary information of Eq.(6.2) to establish conditions that a weighting function must fulfill in order to yield information on the internal boundary Σ, exclusively. What is required is to eliminate the terms containing Q^*u and C^*u in that equation. This is achieved if the test functions satisfy $Qw = 0$ and $Cw = 0$, simultaneously, because $\langle Q^*u, w\rangle \equiv \langle Qw, u\rangle$ and $\langle C^*u, w\rangle \equiv \langle Cw, u\rangle$. Thus, in view of Eq. (6.2), one has

$$-\langle K^*u, w\rangle = \langle f - g - j, w\rangle; \quad \forall w \in N_Q \cap N_C \subset \hat{D}_2 \tag{7.1}$$

where N_Q and N_C are null subspaces of the operators Q and C respectively.

Observe that the left-hand side of Eq.(7.1) involves the complementary information on Σ, exclusively, as desired. Generally, the complementary information on Σ, K^*u, is sufficient to define well-posed problems in each one of the subdomains of the domain decomposition, when the boundary data is added to it. However, it can be seen through specific examples that the complementary information K^*u is more than what is essential to achieve this goal and handling excessive information, in general, requires carrying too many degrees of freedom in the computational process, which in many cases is inconvenient. Thus, generally, to develop numerical methods of optimal efficiency, it is better to eliminate part of such information.

The general procedure for carrying out such elimination consists in introducing a '*weak decomposition*' $\{S, R\}$ of the bilinear functional K (for a definition of weak decomposition, see [10]). Then, S and R are bilinear functionals and fulfill

$$K \equiv S + R; \tag{7.2}$$

Then '*the sought information*' is defined to be S^*u, where $u \in \hat{D}_1$ is the solution of the BVPJ. In particular, a function $\tilde{u} \in \hat{D}_1$ is said to '*contain the sought information*' when $S^*\tilde{u}{=}S^*u$.

Let $\tilde{N}_2 \subset \hat{D}_2$ be defined by $\tilde{N}_2 \equiv N_Q \cap N_C \cap N_R$. An auxiliary concept, quite useful for formulating Trefftz-Herrera domain decomposition procedures, which was originally introduced in 1980 [7], is the following (see [1]).

Definition 7.1.- *A subset of weighting functions, $\mathcal{E} \subset \tilde{N}_2 \equiv N_Q \cap N_C \cap N_R$, is said to be TH-complete for S^*, when for any $\hat{u} \in \hat{D}_1$, one has:*

$$\langle S^*\hat{u}, w\rangle = 0, \forall w \in \mathcal{E} \Rightarrow S^*\hat{u} = 0; \tag{7.3}$$

Clearly, a necessary and sufficient condition for the existence of TH-complete systems, is that $\tilde{N}_2 \equiv N_Q \cap N_C \cap N_R$ be, itself, a TH-complete system.

Theorem 7.1 *Let $\mathcal{E} \subset \tilde{N}_2$ be a system of TH-complete weighting functions for S^*, and let $u \in \hat{D}_1$ be the solution of the BVPJ. Then, a necessary and sufficient condition for $\hat{u} \in \hat{D}_1$ to contain the sought information, is that*

$$- \langle S^* \hat{u}, w \rangle = \langle f - g - j, w \rangle ; \quad \forall w \in \mathcal{E} \tag{7.4}$$

Proof. If $u \in \hat{D}_1$ is the solution of the BVPJ, one has

$$- \langle S^* u, w \rangle = \langle f - g - j, w \rangle ; \quad \forall w \in \mathcal{E} \tag{7.5}$$

Hence

$$- \langle S^* (\hat{u} - u), w \rangle = 0; \quad \forall w \in \mathcal{E} \tag{7.6}$$

and, therefore, $S^* \hat{u} = S^* u$. ■

Theorem 7.1, supplies a very **General Formulation of Indirect Methods (or Trefftz-Herrera Methods) of Domain Decomposition** which can be applied to any linear equation or system of such equations. When $u_p \in \hat{D}_1$ is a function satisfying $Pu_p = f$ and $Bu_p = g$ then Eq.(7.4) can be replaced by

$$- \langle S^* \hat{u}, w \rangle = - \langle S^* u_p, w \rangle + \langle J(u_p - u_\Sigma), w \rangle ; \quad \forall w \in \mathcal{E} \tag{7.7}$$

In applications, Eq.(7.7) determines the average of the solution and/or its derivatives on Σ.

8. Unified Approach to DDM: Abstract Formulation. The concepts and notations of the previous Sections, can be used to give an abstract expression to the unified formulation of Domain Decomposition Methods.

In this Section a pair of weak decomposition $\{S_J, R_J\}$ and $\{S, R\}$ of J and K, respectively, will be considered. This assumption implies that [10]

$$J = S_J + R_J \tag{8.1}$$

in addition to Eq. (7.2). Even more, under the above assumption a function $\hat{u} \in \hat{D}_1$ fulfills Eq. (6.4), if and only if

$$(P - B - R_J)\, \hat{u} = Pu_\Omega - Bu_\partial - R_J u_\Sigma \tag{8.2}$$

and

$$S_J \hat{u} = S_J u_\Sigma \tag{8.3}$$

It has interest to consider the case when Eq. (8.3) can be replaced by the condition that $\hat{u}$ contains the sought information; i.e., when Eq. (8.3) can be replaced by

$$S^* \hat{u} = S^* u \tag{8.4}$$

because this leads to a quite general formulation of DDM.

Definition 8.1.- *The pair of weak decompositions $\{S_J, R_J\}$ and $\{S, R\}$ of J and K, respectively, is said to be optimal, when given any $u_\Omega \in \hat{D}_1$, $u_\partial \in \hat{D}_1$, $u_\Sigma \in \hat{D}_1$ and $u_I \in \hat{D}_1$, the problem of finding $\hat{u} \in \hat{D}_1$, such that*

$$(P - B - R_J)\,\hat{u} = Pu_\Omega - Bu_\partial - R_J u_\Sigma \tag{8.5}$$

and

$$S^* \hat{u} = S^* u_I \tag{8.6}$$

is local and well-posed.

Lemma 8.1 *Assume $\hat{u} \in \hat{D}_1$ is solution of the local problems defined by Eqs. (8.5) and (8.6), for some $u_I \in \hat{D}_1$, then the following assertions are equivalent*

i).- u_I contains the sought information,

ii).- $J\hat{u} = Ju$,

iii).- $\hat{u}$ is the solution of the BVPJ.

Proof. First, we show that *ii)* and *iii)* are equivalent. To this end, assuming *ii)* observe that Eq. (8.5) together with *ii)* imply that $\hat{u} \in \hat{D}_1$ is the solution of the BVPJ. Conversely, if $\hat{u} \in \hat{D}_1$ is solution of the BVPJ, then $J\hat{u} = Ju$. The equivalence between *i)* and *iii)* is immediate. Indeed, assume *iii)* then $S^* u_I = S^* \hat{u} = S^* u$; i.e., u_I contains the sought information. If *i)* holds, then

$$(P - B - R_J)\,\hat{u} = (P - B - R_J)\,u \tag{8.7}$$

together with

$$S^* \hat{u} = S^* u \tag{8.8}$$

and *iii)* follows from the uniqueness of solution of the local problems. ■

Definition 8.2 (Steklov-Poincaré Operator).- *Given any $v \in \hat{D}_1$, define $\tau : D_1 \to D_2^*$, by*

$$\tau(v) = J\hat{v} \tag{8.9}$$

where $\hat{v} \in \hat{D}_1$ is the solution of the local boundary value problems with $u_I = v$.

Lemma 8.2 *A function $\hat{u} \in \hat{D}_1$, contains the sought information if and only if*

$$\tau(\hat{u}) = Ju \tag{8.10}$$

Proof. It is immediate in view of the previous Lemma. ■

9. The Second Order Elliptic Equation. As an illustration, consider the BVPJ for an elliptic operator of second order

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b}u) + cu = f_\Omega; \quad in \quad \Omega, \tag{9.1}$$

subjected to the boundary conditions

$$u = u_\partial; \quad on \quad \partial\Omega, \tag{9.2}$$

and the jump conditions

$$[u] = [u_\Sigma] \equiv j_\Sigma^0 \quad and \quad [\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n} = [\underline{\underline{a}} \cdot \nabla u_\Sigma] \cdot \underline{n} \equiv j_\Sigma^1; \quad on \quad \Sigma, \tag{9.3}$$

The numerical treatment of this problem, using both a Direct Method and a Trefftz-Herrera Method, is explained in [6] and [5]. When the differential operator is given as in Eq. (9.1), then,

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla \cdot \underline{\mathcal{D}}(u, w) \tag{9.4}$$

where

$$\underline{\mathcal{D}}(u, w) \equiv u(\underline{a}_n \cdot \nabla w + b_n w) - w\underline{a}_n \cdot \nabla u \tag{9.5}$$

Define the following bilinear functionals:

$$\mathcal{B}(u, w) \equiv u(\underline{a}_n \cdot \nabla w + b_n w) \cdot \underline{n}, \quad \mathcal{C}(w, u) \equiv w\underline{a}_n \cdot \nabla u \tag{9.6}$$

$$\mathcal{J}(u, w) \equiv \dot{w}[\underline{a}_n \cdot \nabla u] - [u]\,\dot{\overline{(\underline{a}_n \cdot \nabla w + b_n w)}} \tag{9.7}$$

$$\mathcal{K}(w, u) \equiv \dot{u}[\underline{a}_n \cdot \nabla w + b_n w] - [w]\,\dot{\overline{(\underline{a}_n \cdot \nabla u)}} \tag{9.8}$$

$$\mathcal{S}_J(u, w) \equiv \dot{w}[\underline{a}_n \cdot \nabla u], \quad \mathcal{R}_J(u, w) \equiv -[u]\,\dot{\overline{(\underline{a}_n \cdot \nabla w + b_n w)}} \tag{9.9}$$

$$\mathcal{S}(w, u) \equiv \dot{u}[\underline{a}_n \cdot \nabla w + b_n w] \quad and \quad \mathcal{R}(w, u) \equiv -[w]\,\dot{\overline{(\underline{a}_n \cdot \nabla u)}} \tag{9.10}$$

In addition, define the bilinear functionals S_J, R_J, S and R in a similar fashion to Eqs. (5.6)-(5.8), by means of corresponding integrals.

Then Green-Herrera formula of Eq. (5.10) holds. Even more, Eqs. (7.2) and (8.1) are fulfilled and the pair $\{S_J, R_J\}$ and $\{S, R\}$ constitute an optimal pair of weak decompositions, because the local problems are well posed. Indeed, Eq.(8.2) is the BVPJ of Eqs.(9.1) to (9.3) except that the jump condition associated with this latter equation has been omitted. However, the jump of the function, of Eq.(9.2), is indeed prescribed. This problem has many solutions. However, with the above definition of $\mathcal{S}$, the *sought information* is the average of the function on the internal boundary Σ. When this information is complemented with the jump of the function, which is the data given by Eq.(9.2), the values of the function on both sides of Σ are determined by the identities

$$u_+ \equiv \dot{u} + 1/2\,[u] \quad \textit{and} \quad u_- \equiv \dot{u} - 1/2\,[u] \tag{9.11}$$

This information together with the boundary conditions in the external boundary permits establishing well-posed problems in each one of the subdomains of the partition.

10. Optimal Interpolation. The Indirect Method yields information on the internal boundary Σ, exclusively. To extend that information into the interior of the subdomains of the partition, it is necessary to solve the local problems [5],[21]. The following results will be useful in applications, to carry out such step.

Let $\tilde{N}_1 \subset \hat{D}_1$ be defined by $\tilde{N}_1 \equiv N_P \cap N_B \cap N_{R_J}$.

Theorem 10.1 *Let $u_P \in \hat{D}_1$ be such that*

$$Pu_P = Pu_\Omega, \quad Bu_P = Bu_\partial \quad \textit{and} \quad R_J u_P = R_J u_\Sigma. \tag{10.1}$$

Then there exists $v \in \tilde{N}_1$ such that

$$-\langle S^* v, w\rangle = \langle S_J\,(u_P - u_\Sigma)\,, w\rangle\,; \quad \forall w \in \tilde{N}_2 \tag{10.2}$$

In addition, define $\hat{u} \in \hat{D}_1$ by $\hat{u} \equiv u_P + v$. Then $\hat{u} \in \hat{D}_1$ contains the sought information. Even more, $\hat{u} \equiv u$, where u is the solution of the BVPJ.

Proof. Take $u \in \hat{D}_1$ as in the Theorem, then this function contains the sought information and, in view of Eq. (10.1), Eq. (10.2) can be applied, with $\hat{u} \equiv u$. Define $v \equiv u - u_P$, then

$$-\langle S^* v, w\rangle = \langle J\,(u_P - u_\Sigma)\,, w\rangle = \langle S_J\,(u_P - u_\Sigma)\,, w\rangle\,; \quad \forall w \in \tilde{N}_2 \tag{10.3}$$

because $R_J\,(u_P - u_\Sigma) = 0$. However, from Eq. (10.1), it follows that $v \in \tilde{N}_1 \equiv N_P \cap N_B \cap N_{R_J}$. When $\tilde{u} \in \hat{D}_1$ is defined as in the Theorem, then it fulfills

$$(P - B - R_J)\,(\hat{u} - u) = 0 \quad \textit{and} \quad S^*(\hat{u} - u) = 0 \tag{10.4}$$

Therefore, $\hat{u} - u = 0$, since the problem of Eqs. (10.4), is well-posed. ■

The Symmetric Case: In this case $\hat{D}_1 = \hat{D}_2 \equiv \hat{D}$, $P = Q$, $B = C$, $J = K$, $S \equiv S_J$ and $R \equiv R_J$. Then $\tilde{N} \equiv \tilde{N}_2 \equiv N_Q \cap N_C \cap N_R = N_P \cap N_B \cap N_{R_J} \equiv \tilde{N}_1$. If it is further assumed that the bilinear functional $-\langle S^* u, w\rangle$ is symmetric and positive definite $\forall u, w \in \tilde{N}$, it can be shown that the quadratic functional $-\langle S^* \tilde{u}, \tilde{u}\rangle - 2\,\langle f - g - j, \tilde{u}\rangle$ attains its minimum over $\tilde{N}$, at $\tilde{u} \in \tilde{N}$, if and only if $\tilde{u} \in \tilde{N}$ contains the sought information.

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