DDM Preprint
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1. Unified Theory of Domain Decomposition Methods

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1. Introduction. Domain Decomposition Methods (DDM) have been derived by Herrera using a unifying concept, which consists in viewing DDM as procedures for gathering information at the internal boundary (Σ) of a partition, sufficient for defining well-posed problems at each one of its subdomains. Two broad categories of Domain Decomposition Methods are identified in this manner: 'direct' and 'indirect (or Trefftz-Herrera)' methods. Direct methods are usually understood as procedures for putting together local solutions, just as bricks, to build the global solution. However, for direct methods the point of view adopted by the unified theory, here presented, is different: the local solutions are used, as means for establishing compatibility relations that the global solution of the problem considered must fulfill. In Trefftz-Herrera methods, on the other hand, local solutions are used in an indirect manner; as specialized test functions with the property of supplying information on Σ, exclusively. Important features of Herrera's unified theory are the use, throughout it, of "fully discontinuous functions" and the treatment of a general boundary value problem with prescribed jumps. The generality of the resulting theory is remarkable, because it is applicable to any partial (or ordinary) differential equation or system of such equations, which is linear, independently of its type and with possibly discontinuous coefficients. The developments that have been carried out, thus far in this framework, have implications along two broad lines: as tools for incorporating parallel processing in the modeling of continuous systems and as an elegant and efficient way of formulating numerical methods from a *domain decomposition* perspective. In addition, the theory supplies a systematic framework for the application of fully discontinuous functions in the treatment of partial differential equations.

This paper is part of a sequence of papers, contained in these Proceedings, devoted to present, and further advance, this unified theory of Domain Decomposition Methods (DDM) and some developments associated with it. DDM have received much attention in recent years[2], mainly because they supply very effective means for incorporating parallel processing in computational models of continuous systems. Another aspect that must be stressed is that it is useful to analyze numerical methods for partial differential equations from a domain-decomposition perspective, since the ideas related to domain decomposition are quite basic for them. Indeed, developing numerical procedures as accurate as desired in small regions is an easy task that can be performed by many numerical schemes and, once this has been done, the remaining problem is essentially the same as that of Domain Decomposition Methods. In this respect, it is useful to recall the main objective of DDM:

Given a domain Ω and one of its partitions (Fig. 1.1), to obtain the solution of a boundary value problem defined on it (the 'global problem'), by solving problems formulated on the subdomains of the partition (the 'local problems'), exclusively. In

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[2]See: International Scientific Committee for Domain Decomposition "Proceedings of 13 conferences on Domain Decomposition Methods", www.ddm.org, 1988-2001

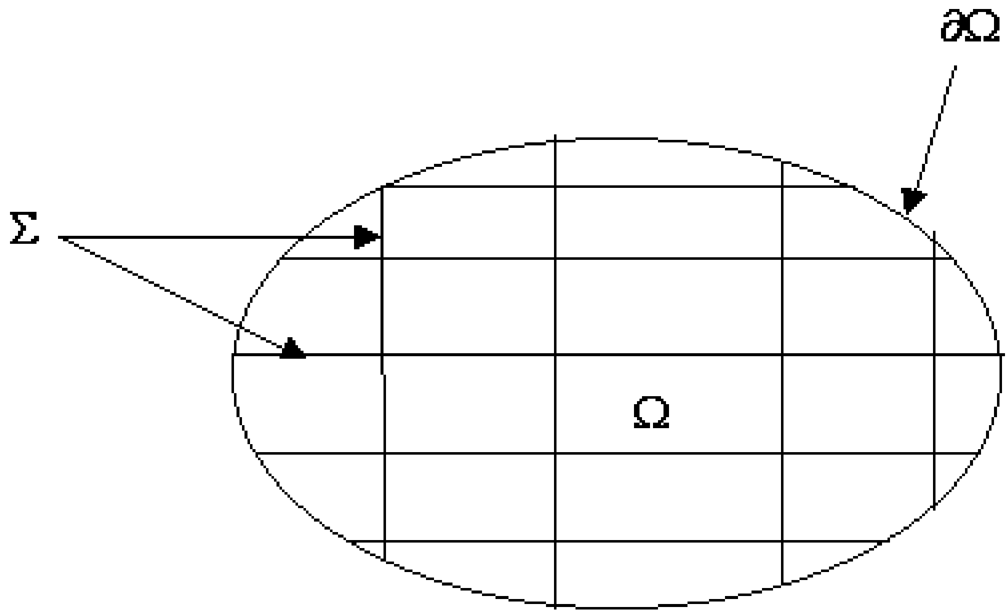

Figure 1.1: Partition of the domain Ω

what follows the subdomains of the partition will be denoted by $\Omega_i (i = 1, ..., E)$ *and the internal boundary, which separates the subdomains from each other, will be* Σ.

Herrera has proposed recently a unified theory of DDM [15],[14], in which most of the known methods may be subsumed, supplying more general formulations of them and hinting new procedures that should be investigated in the future. The sequence of papers mentioned above, intends to present briefly such theory in its different aspects. The present paper contains an exposition of the unified theory. Trefftz-Herrera Method is given in [17], while direct methods are described in [9]. Applications to elliptic equations are presented in [6],[22] and [5] -second order equations are treated in [6] and the biharmonic equation in [5]-.

2. Some Unifying Concepts. Herrera's theory is formulated in function spaces whose elements are generally discontinuous, and the theory supplies systematic procedures for applying discontinuous functions in the numerical treatment of partial differential equations. Such function spaces have the following general form:

$$\hat{D}\left(\Omega\right) \equiv D\left(\Omega_1\right) \oplus ... \oplus D\left(\Omega_E\right); \tag{2.1}$$

If $u \in \hat{D}\left(\Omega\right)$, then $u \equiv (u_1, ..., u_E)$ where $u_i \in D\left(\Omega_i\right)$, $i = 1, ..., E$. Generally, when variational formulations are considered, as in the theory of indirect methods, two such spaces are introduced; namely, the space of *trial or base functions* $\hat{D}_1\left(\Omega\right)$ and the space of *test or weighting functions* $\hat{D}_2\left(\Omega\right)$. When $D\left(\Omega_i\right)$, $i = 1, ..., E$, are Sobolev spaces, a special kind of Sobolev space, $\hat{\mathbb{H}}^s(\Omega)$, is obtained: $\hat{\mathbb{H}}^s(\Omega) \equiv \mathbb{H}^s(\Omega_1) \oplus ... \oplus \mathbb{H}^s(\Omega_E)$. Of course, more complicated combinations are possible.

In addition, the theory deals with a very general boundary value problem, the Boundary Value Problem with prescribed Jumps (the BVPJ), in which, in addition to boundary conditions on the external boundary, $\partial\Omega$, jumps are prescribed across the internal boundary Σ. And it is also applicable when the coefficients of the differential operators are discontinuous across Σ. The general BVPJ considered by the theory is type-independent and has the form

$$\mathcal{L}u = f_\Omega; \quad in\ \Omega_i \quad i = 1, ..., E \tag{2.2}$$

$$B_j u = g_{\partial j}; \quad on \quad \partial\Omega \tag{2.3}$$

$$[J_k u] = j_{\Sigma k}; \quad on \quad \Sigma \tag{2.4}$$

Here $\mathcal{L}$ is a differential operator of any type; in particular it can be elliptic, hyperbolic or parabolic. Furthermore, it can be vector-valued and therefore the theory includes systems of equations and not just a single equation. The solution of the BVPJ will be denoted by $u \equiv (u_1, ..., u_E)$. In this setting, the objective of Domain Decomposition Methods is to find $u_i \in D(\Omega_i)$, for $i = 1, ..., E$. The unified theory is based on the following unifying principle:

Domain Decomposition Methods are procedures for gathering information, on the internal boundary Σ, sufficient for defining well-posed local problems in each one of the subdomains. Then it is possible to reconstruct the solution in the interior of the subdomains, $u_i \in D(\Omega_i)$, for $i = 1, ..., E$ by solving local problems exclusively.

3. The Sought Information. The information that one deals with, when formulating and treating partial differential equations (i.e., the BVPJ), is classified in two broad categories: *'data of the problem'* and *'complementary information'*. In turn, three classes of data can be distinguished: data in the interior of the subdomains of the partition (given by the differential equation, which in the BVPJ is fulfilled in the interior of the subdomains, exclusively), the data on the external boundary ($B_j u, \quad on \quad \partial\Omega$) and the data on the internal boundary (namely, $[J_k u], \quad on \quad \Sigma$). The complementary information can be classified in a similar fashion: the values of the sought solution in the interior of the subdomains ($u_i \in D(\Omega_i)$, for $i = 1, ..., E$); the complementary information on the outer boundary (for example, the normal derivative in the case of Dirichlet problems for Laplace's equation); and the complementary information on the internal boundary Σ (for example, the average of the function and the average of the normal derivative across the discontinuity for elliptic problems of second order [6]). In the unified theory of DDM, a target of information, which is contained in the complementary information on Σ, is defined; it is called *'the sought information'*. It is required that the choice of *the sought information* fulfills the following assumption:

when 'the sought information' is complemented with the data of the problem, there is sufficient information available for defining well-posed problems in each one of the subdomains of the partition.

In general, however, the sought information may satisfy this property and yet be redundant, in the sense that if all of it is used simultaneously together with the data of the problem, ill-posed problems are obtained. Consider for example, a Dirichlet problem of an elliptic-type second order equation (see [6]), for which the jumps of the function and of its normal derivative have been prescribed. If for such problem the *sought information* is taken to be the average of the function -i.e., $(u_+ + u_-)/2$-, and the average of the normal derivative -i.e., $\frac{1}{2}\partial(u_+ + u_-)/\partial n$, on Σ-, then it may be seen that it contains redundant information. Indeed, $u_+ = \frac{1}{2}(u_+ + u_-) + \frac{1}{2}(u_+ - u_-)$, $u_- = \frac{1}{2}(u_+ + u_-) - \frac{1}{2}(u_+ - u_-)$, and a similar relation holds for the normal derivatives. Therefore, if the *'sought information'* and the *'data of the problem'* are used simultaneously, one may derive not only the value of the BVPJ solution on the boundary of each one of the subdomains, but also the normal derivative, at least in a non-void section of those boundaries. As it is well known, this is an ill-posed problem, because Dirichlet problem is already well-posed in each one of the subdomains. Thus, the *sought information* contains redundant information in this case.

Generally, in the numerical treatment of partial differential equations, efficiency requires eliminating redundant information. This fact motivates the following definition:

The sought information is 'optimal' when there is a family of well-posed problems -one for each subdomain of the partition- which uses all the sought information, together with the data of the BVPJ.

Analysis of existing methods reveals that there are some for which the sought information is optimal and others for which this is not the case. In general, except for the simple case of first order equations, methods for which the *sought information* is optimal are overlapping.

4. Direct and Indirect Methods. There are two main procedures for gathering the *sought information* on Σ: *'direct'* and *'indirect* (or *Trefftz-Herrera*)*'* methods. Both of them derive the *sought information*, on Σ, from compatibility conditions that the global solution of the BVPJ must satisfy locally and the local solutions are applied precisely for deriving such compatibility conditions. The global system of equations, for the *sought information*, is constructed in this manner. Trefftz-Herrera methods were introduced in numerical analysis by Herrera *et al.* [10], [16], [11], [4], [18], [12], [13] and [20], and its distinguishing feature is the use of specialized test functions which have the property of yielding any desired information on Σ. The guidelines for the construction of such weighting functions is supplied by a special kind of Green's formulas (Green-Herrera formulas), formulated in spaces of fully discontinuous functions [10],[16],[18], which permit analyzing the information on Σ, contained in approximate solutions. Using Green-Herrera formulas it has been possible to give a very general formulation of Indirect Methods in terms of a variational principle possessing great generality. This is Eq. (?) of reference [6](see also [20]), which corresponds to an Invited Plenary Talk of this Conference that was devoted to a full description of Trefftz-Herrera Methods and is contained in these Proceedings.

Conventional descriptions of Direct Methods present them as procedures for assembling, just as *bricks*, local solutions in order to build the global one. When these methods are formulated using the unified theory approach, direct methods derive the *sought information*, on Σ, from compatibility conditions that the global solution of the BVPJ must satisfy locally [9] and the local solutions are applied precisely for deriving such compatibility conditions. An important difference between direct and Trefftz-Herrera methods is that in the latter local solutions of equations formulated in terms of the adjoint differential operator are used, while in the former such equations are formulated in terms of the original differential operator.

To finish this Section some general remarks are in order. In the methods of the unified theory the only information that is obtained when solving the global problem refers to information on the internal boundary Σ and no information at all is obtained in the interior of the subdomains. If such information is desired, it can be derived solving the well-posed local problems that are obtained in the manner explained before. When the unified theory is applied as a discretization procedure, the process described above for deriving the solution in the interior of the subdomains of the partition, which can be carried out by any numerical method, is referred as *'optimal interpolation'*. This is in agreement with, and supplements, the nomenclature that has been used in some past work, in which the specialized test functions that supply information at the internal boundary exclusively, are referred as *optimal test functions* [3].

5. General Conclusions. An elegant framework for Domain Decomposition Methods, which is quite general and simple, has been presented. The generality of the methodologies must be stressed, since they are applicable to any linear differential equation, or system of such equations and to problems with prescribed jumps and with discontinuous coefficients. In addition, the theory supplies systematic procedures for applying discontinuous functions in the numerical treatment of partial differential equations. Even more, its applicability is type-independent. Thus, it is not only applicable to elliptic equations, but also to hyperbolic and parabolic ones.

Thus far, DDM have been mainly applied as a tool for parallelizing numerical models of continuous systems [21]. However, Herrera's Unified Theory permits developing wide classes of numerical methods with many attractive features [20]. In addition, we claim that this theory subsumes most of the existing methods of domain decomposition. Using its framework Schwarz and Steklov-Poincaré methods were incorporated in this framework in [19] and [20], respectively, while Mixed Methods were derived in [18]. The theory also implies wide generalizations of the Projection Decomposition Method [1]. We suspect that the capacity of using fully discontinuous functions systematically -and the foundations of such capacity is one of the contributions of the theory- permits eliminating Lagrange multipliers in many instances and that it also has a bearing on partitions of unity and its applications. This, however, remains to be shown. Other subjects that should be investigated in the future are the implications of the unified theory on Mortar [2] and FETI methods [7],[8].

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