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Advances in domain decomposition methods

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Domain decomposition methods (DDM) have received much attention in recent years, which was originally motivated mainly because they are the most effective ways for devising parallel algorithms that can benefit from multiprocessor computation, and supercomputers, in the numerical modeling of continuous systems. However, the concept of domain decomposition has always been a fundamental ingredient of the numerical treatment of partial differential equations. Indeed, the procedures in most of such methods start with the introduction of a partition (i.e., a domain decomposition) and, so, the problem of how to connect solutions defined in neighboring regions becomes a central one, although in the past it was not, usually, tackled explicitly. Thus, it is only natural that the clarification of this problem, that the study of domain decomposition methods has brought about, has also enlighten many aspects of numerical methods for partial differential equations. This paper is devoted to present a brief summary of some these developments, from a unified perspective, and their potential in the treatment of water resources problems.

1. INTRODUCTION

Domain decomposition methods (DDM) have received much attention in recent years [1], mainly because they supply very effective means for incorporating parallel computation in mathematical models of continuous systems. In addition, it is useful to analyze numerical methods for partial differential equations (PDE) from a domain-decomposition perspective since the ideas related to domain decomposition are quite basic for them [2–4]. Actually, the implications of DDM in the numerical methods of partial differential equations are very thorough and it is natural that this be so. Developing numerical solutions of PDE as accurate as desired in the "small", is a relatively straightforward task and once this has been done, the remaining problem is how build the global solution given that such capacity is available. When numerical methods for PDE are seen from this perspective, it becomes clear that many of the problems posed by them are addressed by the theory and methods of DDM, whose objective may be summarized as follows:

- *Given a region Ω and a partition of it, to obtain the solution of the boundary value problem in the region where it is formulated ("the global problem"), by solving bound-*

ary value problems formulated in the subdomains of the partition ("the local problems"), exclusively.

2. DDM FROM A UNIFIED PERSPECTIVE

Recently, I.Herrera [2–4] introduced a "unified theory of DDM". The scope of this theory is quite wide since it deals with a very general boundary value problem with prescribed jumps at the internal boundary and it is applicable to any partial (or ordinary) differential equation or system of such equations which is linear, including the case of discontinuous coefficients. It can also be applied to non-linear problems via Newton's method, or alike, in the usual manner. Herrera's unified theory subsumes practically all methods that exist, but leads to more general formulations of them suggesting many new procedures that should be investigated in the future. The basic unifying principle states that:

- *Domain Decomposition Methods are procedures for gathering information about the sought solution in the internal boundary (Σ) -which separates the subdomains of the partition from each other-, sufficient for defining well-posed problems in each one of the subdomains (to be referred as "local well-posed problems"). In this manner, the solution can be reconstructed in each one of the subdomains by solving local problems exclusively.*

In the unified approach, the information involved in the formulation and solution of a boundary value problem for a differential operator, or system of such operators, is classified into two broad categories: "data" of the problem and "complementary information". Generally, the complementary information can be subdivided into three classes: information defined in the interior of the subdomains, information defined on the outer boundary $\partial\Omega$ and complementary information defined on Σ. Then, within the complementary information, a target is defined -to be referred to as "the sought information"-, which consists of complementary information on Σ, exclusively, and with the property of being sufficient for defining well-posed problems in each one of the subdomains of the partition. Thus, among the characteristics of a domain decomposition method one must include the particular choice of the *sought information*.

However, in general, the *sought information* may be redundant, in the sense that although there are subsets of that information that can be used to define local well-posed problems in each one of the subdomains, the application of all of it simultaneously, yields local boundary value problems which are ill-posed. As an example, when considering elliptic equations of second order [5], one may define the *sought information* in such a way that it yields both the values of the function and its normal derivative in the boundary of each one of the subdomains of the partition. Clearly, this is redundant information, because the boundary value problem in which both the function and its normal derivative are prescribed, is ill posed. It is because of these facts that the following definition is adopted:

- *A definition of the sought information is said to be "optimal" when there is a set of well-posed local problems -it is assumed that one and only one of these problems is*

defined in each one of the subdomains of the partition- which are defined using all of the sought information.

In some of the existing methods the sought information is optimal while in others it is non-optimal. In the study of DDM, it is standard to classify such methods into "overlapping" and "non-overlapping". Except for the case of first order equations, when the sought information is optimal, the domain decomposition method is generally overlapping. On the other hand, up to now, DDM have been applied mainly as a parallelization tool. However, the unified theory of Herrera leads to large classes of numerical methods with very attractive features, as it is described below.

3. DIRECT AND INDIRECT METHODS

According to Herrera's unified theory [2-4], there are two main procedures for gathering the sought information on Σ: *'direct'* and *'indirect'*, or *'Trefftz-Herrera Method'*. Different domain decomposition methods that exist and others that may be developed in the future, use either one of these strategies in an explicit or in an implicit manner. Methods available at present, except for some that have been developed by I. Herrera and his collaborators, use a direct approach for gathering such information and this is done in an implicit manner in most cases.

Trefftz-Herrera Method, is a more fully developed version of Localized Adjoint Method (LAM), that was introduced in the Water Resources Literature by I.Herrera long ago [6-8]. In 1989, the ELLAM Group -M.E. Celia, R.E. Ewing, I. Herrera and T.F. Russell- was integrated with the purpose of applying LAM to advection dominated transport [9,10]. The ELLAM Group presented the Eulerian-Lagrangian Localized Adjoint Method in a sequence of two papers: the first one -whose first author is Celia, entitled "An Eulerian-Lagrangian Localized Adjoint Method for the Advection-Diffusion Equation" [11]- devoted to exhibit its numerical implementation, and the second one to explain its numerical underpinnings -whose first author is I. Herrera, entitled "Eulerian-Lagrangian Localized Adjoint Method: The Theoretical Framework" [12]. The theory presented in this latter paper is an early version of what, under further development, became Trefftz-Herrera Method of Domain Decomposition [13–18]. The distinguishing feature of Trefftz-Herrera Methods (the indirect approach), as developed by I. Herrera and his collaborators, is the use of specialized test functions that yield the *sought information*, exclusively. An updated and fairly well integrated presentation of the theory is given in [5,14].

In standard approaches, Direct Methods consist in piecing together, just as *'bricks'*, the local solutions of the differential equations in order to build the global solution. However, in Herrera's theory a more sophisticated point of view is adopted, in which direct methods are seen as procedures that use the local solutions as means for establishing compatibility conditions that the *sought information* on the internal boundary, must fulfill. Thus, in the unified theory, to derive both direct and indirect methods, local solutions are constructed and applied to establish relations that the *sought information* must fulfill locally. The global matrix is constructed in this manner [2-4]. The solution of the corresponding global system of equations yields the 'sought information' on Σ only, but no information about the solution of the problem in the interior of the subdomains is obtained. An important difference is that direct methods derive the information on Σ using solutions of equations

formulated in terms of the original differential operator -i.e., that occurring in the BVPJ-, while in Trefftz-Herrera approach the specialized test functions, which perform this task, fulfill equations formulated in terms of the adjoint differential operators.

4. DDM AS DISCRETIZATION AND PARALLELIZATION TOOLS

As mentioned, Herrera's unified theory of DDM subsumes practically all methods of domain decomposition that exist at present [4], but generally implies generalized versions of them suggesting new procedures that should be investigated in the future. In previous publications it has been indicated how to incorporate Schwarz Methods [17], Steklov-Poincaré [5] and Mixed Methods in this framework. Regarding this latter class of methods, Herrera et al. [12] presented a straight-forward derivation of Raviart and Thomas results on which hybrid methods are based. The incorporation of the Projection Decomposition Method [21] in the unified theory framework is more or less straightforward, but its implications in the corresponding procedures, as well as those of the methods already mentioned, should be investigated more thoroughly. A similar comment applies to mortar methods. In addition, the future of the methodologies derived from the unified theory as a parallelization tool of time dependent problems governed either by hyperbolic or parabolic equations -and this is the case of the ELLAM method- is quite promising, although work in this area is only beginning (see [20], in this Proceedings).

When the unified theory is applied as a discretization tool, a very large class of numerical methods possessing special features, which offer significant advantages, is obtained. In particular, the base functions have to be defined on Σ, exclusively. It must be mentioned that a similar feature is exhibited by the Projection Decomposition Method [21], but the unified theory permits handling in this manner a considerably wider range of problems. The numerical algorithms yield information on Σ, exclusively. In a manner similar to finite difference approaches, frequently this information needs to be interpolated to the interior of the subdomains of the partition. This can be done solving local problems, since the information available in the interior boundary is sufficient for formulating well-posed local problem. In the past, the weighting functions which are applied in Trefftz-Herrera method have been called Optimal Test Functions and it is natural to call the above procedure of interpolation, Optimal Interpolation Method.

Most of the discretization procedures developed thus far, using the unified approach, have been based on Trefftz-Herrera method. For elliptic equations of second order [5], when the special test functions, which are applied, are exact, the precision of the algorithms depend exclusively on the degree of the interpolating polynomials on the internal boundary Σ. In particular, algorithms in which test functions that are piece-wise linear, piece-wise quadratic or piece-wise cubic on Σ, yield errors that are $O(h^2), O(h^3)$, and $O(h^4)$, respectively. Any numerical procedure may be applied for producing the local solutions and this property is preserved, as long as such numerical method is compatible with the order of accuracy of the interpolation functions used on Σ. In particular, if collocation is applied, a non-standard method of collocation is obtained, which possesses several attractive features. Indeed, in the standard method of collocation the number of degrees of freedom associated with each node is two in one dimension, four in two dimensions and eight in three dimensions. For some of the new algorithms, on the other

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