Computer Assisted Mechanics and Engineering Sciences, 10: 495–514, 2003.

Unified theory of Trefftz methods and numerical implications

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(Received September 11, 2003)

In the 1st International Workshop, devoted to Trefftz Method, the author presented an indirect approach to Trefftz Method (Trefftz–Herrera Method), while in the Second one some of the basic ideas of how to integrate different approaches to Trefftz method were introduced. The present Plenary Lecture, corresponding to the 3rd International Workshop of this series, is devoted to show that Trefftz Method, when formulated in an suitable framework, is a very broad concept capable of incorporating and unifying many numerical methods for partial differential equations. In this manner, the unified theory of Trefftz Method that was announced in the second publication of this series, has been developed. It includes Direct Trefftz Methods (Trefftz–Jirousek) and Indirect Trefftz Methods (Trefftz–Herrera). At present, the unified theory is fully developed and an overview is given here, as well as a brief description of its numerical implications.

1. Introduction

Trefftz Method is a very broad concept capable of incorporating and unifying many numerical methods for partial differential equations [1, 2]. In the first publication of this series [3] (see also [4]), Trefftz–Herrera method was introduced and, for the first time, presented as a domain decomposition procedure. Later, in the second one [1, 5], the author announced a unified theory of Trefftz methods. Since then, considerable progress has been made both in the systematic formulation of Trefftz–Herrera method [6, 7] and in the conceptualization of the unified theory of Trefftz methods [2, 8]. The present publication, which corresponds to the Third International Workshop on Trefftz Method, is devoted to describe Herrera's unified theory of Trefftz Method, as it is today, including recent numerical applications [6–12].

Let us start by explaining, very concisely, the main ideas of such framework. Consider a boundary-value problem – or an initial-boundary-value problem – for a partial differential equation, or system of such equations, formulated in a domain Ω (Fig. 1), with boundary $\partial\Omega$. When an initial-boundary-value problem is discussed, Ω is a space-time domain and $\partial\Omega$ contains the space domain, at the initial time. Thus, in the theory, for initial-boundary-value problems the initial conditions are incorporated as part of the boundary conditions. The general problem to be discussed consists in, given a partition $\Pi \equiv \{\Omega_1, \dots, \Omega_E\}$ of Ω, establishing procedures that permit solving the *global problem*, defined in Ω, by solving exclusively *local problems*, defined in each one of the subdomains of the partition, Ω_i, $i = 1, \dots, E$. The notion of *internal boundary*, to be denoted by Σ (Fig. 1), is extensively used in the developments. This is the boundary that separates the subdomains of the partition from each other. The basic and unifying strategy of the theory consists in defining well-posed *local problems*, which the restrictions of global solution must satisfy in each one of the subdomains of the partition. To this end, a target of information on Σ – the *sought information* – is chosen beforehand. And a search is carried out for obtaining the *sought information*. Different formulations of Trefftz method correspond to different procedures for gathering the *sought information*.. Two very broad categories of searching procedures are identified by the theory: 'direct' and 'indirect' (or Trefftz–Herrera) methods [1, 2, 6, 10].

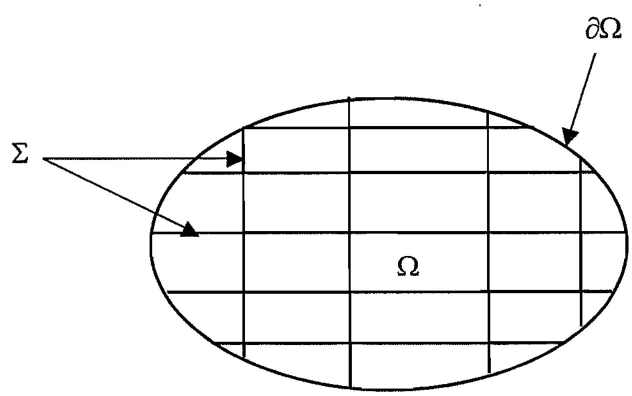

Fig. 1. Partition of the domain Ω

An essential property that the *sought information* must possess is to be sufficient for defining well-posed local problems in each one of the subdomains. But a choice of the *sought information* may satisfy this condition and yet contain additional information – referred as *redundant information* – that is not required for defining the well-posed *local problems*. When this happens, the corresponding method necessarily handles more information than necessary, which is numerically inconvenient. Thus, it is generally better to define the *sought information* in such a manner that only essential information is included and *redundant information* is eliminated. When the definition of the *sought information* enjoys this property, it is said to be an *optimal definition*. Once the *sought information* is known, the solution can be reconstructed by solving *local problems* exclusively. This latter process is called *optimal interpolation* [13]. This terminology is justified by the fact that, when the definition of the *sought information* is *optimal*, it is the only interpolation procedure that is compatible with the available information.

In the usual interpretation of direct methods – referred as Trefftz–Jirousek method [5] – as they were originally introduced by Trefftz [14] and later extensively developed by Jirousek and his collaborators [15–17], they are seen as techniques for building the global solution by putting together, just as 'bricks', the *local solutions*. In Herrera's unified theory of Trefftz method, however, a slightly more sophisticated point of view is adopted, since the local solutions of the differential operator are only used to establish compatibility relations that the *sought information* must fulfill [10]. These relations give rise to the global system of equations from which the *sought information* is obtained.

In Trefftz–Herrera methods, on the other hand, a system of weighting functions of a special kind – the *optimal test functions* [18] – with the property of yielding the *sought information* in the internal boundary, exclusively, is developed and applied [6]. The idea of constructing such test functions stems from the observation that in the method of weighted residuals the information about the exact solution that the approximate one contains, depends on the system of weighting functions that is applied (see, for example, [19]), exclusively. However, in order to establish the conditions, which the *optimal test functions* must fulfill, it is necessary to have a procedure for analyzing such dependence. Green–Herrera formula [20–22] constitutes such basic tool. Using it, necessary and sufficient conditions for a weighting function to be an *optimal test function* are established [6]. Also, a characterization of the *sought information* in terms of a variational principle is supplied, which holds when the *optimal test functions* are applied [6, 7]. This principle constitutes a very general, although abstract, formulation of *Trefftz–Herrera methods*. In general, the *optimal test functions* fulfill the adjoint differential equation in each one of the subdomains of the partition. Thus, an important difference between *direct* and *indirect* methods is that, in the latter, the global matrix is derived using local solutions of the adjoint differential equation while, in the former, the same is done using local solutions of the original differential equation. Techniques to fabricate the *optimal* *test functions* have also been developed [2], but many open questions remain and the field, calling for future research, is quite wide.

In the unified theory discontinuous trial and test functions are systematically applied, and the general problem that is considered is a boundary value problem with prescribed jumps (BVPJ). More precisely, in addition to the boundary conditions, which are prescribed in the outer boundary of the domain, jumps are prescribed at the internal boundary, Σ. Thus, a systematic framework for the use of discontinuous functions in numerical methods for partial differential equations is also supplied. In addition, the theory is applicable to any linear differential equation or system of such equations, independently of its type, and the coefficients of the differential operators involved may be discontinuous.

The general problem of Trefftz Methods, when formulated in the above manner parallels that of Domain Decomposition Methods (DDM) [23], but the resulting theory is quite general, elegant and systematic. It subsumes many numerical methods for partial differential equations and constitutes a powerful tool for analyzing such methods. It is very effective, both as a discretization methodology and as a domain decomposition procedure [11], and the area of its potential applications is quite broad.

2. Function spaces

Consider a region Ω, with boundary $\partial\Omega$ and a partition $\{\Omega_1, \ldots, \Omega_E\}$ of Ω (Fig. 1). Let

$$\Sigma \equiv \bigcup_{i \neq j} \left(\overline{\Omega}_i \cap \overline{\Omega}_j \right), \tag{1}$$

then Σ will be referred as the *internal boundary* and $\partial\Omega$ as the *external (or outer) boundary.* For each $i = 1, \ldots, E$, $D_1(\Omega_i)$ and $D_2(\Omega_i)$ will be two linear spaces of functions defined on Ω_i; then the space of *trial (or base) functions* and that of *test (or weighting) functions* are defined to be

$$D_1 \equiv D_1(\Omega_1) \oplus \ldots \oplus D_1(\Omega_E) \tag{2}$$

and

$$D_2 \equiv D_2(\Omega_1) \oplus \ldots \oplus D_2(\Omega_E), \tag{3}$$

respectively. Observe that members of either D_1 or D_2 are finite sequences of functions, each one of them defined at one sub-domain of the partition. The fact that no matching condition across Σ is required of such sequences generally implies that the function-spaces D_1 and D_2 contain discontinuous functions. It will be assumed that for each $i = 1, \ldots, E$, and $\alpha = 1, 2$, the traces on Σ of elements of $D_\alpha(\Omega_i)$ exist, and the *jump* and the *average* of test or weighting functions are defined by

$$[u] \equiv u_+ - u_- \quad \text{and} \quad \dot{u} \equiv \frac{u_+ + u_-}{2}. \tag{4}$$

Here u_+ and u_- are the traces from one and the other side of Σ. Throughout this paper, the unit normal vector to Σ, $\underline{n}$, is chosen arbitrarily but the convention is such that it points towards the positive side of Σ.

The case when for each $i = 1, \ldots, E$, and each $\alpha = 1, 2$, $D_\alpha(\Omega_i) \equiv H^s(\Omega_i)$, with $s \geq 0$ has special interest. If one defines

$$\widehat{H}^s(\Omega) \equiv H^s(\Omega_1) \oplus \ldots \oplus H^s(\Omega_E), \tag{5}$$

then $D_1 = D_2 \equiv \widehat{H}^s(\Omega)$. This is the special class of Sobolev spaces that was considered in [24].

3. Trefftz problem

The general boundary value problem with prescribed jumps (BVPJ), considered in the unified theory of Trefftz methods, consists in finding $u \equiv \left(u^1, \ldots, u^E\right) \in D_1$ such that

$$\mathcal{L}u = \mathcal{L}u_\Omega \equiv f_\Omega \quad \text{in} \quad \Omega_i \qquad i = 1, \ldots, E\,, \tag{6}$$

$$B_j u = B_j u_\partial \equiv g_j \quad \text{on} \quad \partial\Omega \tag{7}$$

and

$$[J_k u] = [J_k u_\Sigma] \equiv j_k \quad \text{on} \quad \Sigma\,. \tag{8}$$

The notation used here is similar to that of [25]: the $B_j's$ and $J_k's$ are certain differential operators (the $j's$ and $k's$ run over suitable finite ranges of natural numbers). In addition, here as in what follows $u_\Omega \equiv \left(u_\Omega^1, \ldots, u_\Omega^E\right)$, $u_\partial \equiv \left(u_\partial^1, \ldots, u_\partial^E\right)$ and $u_\Sigma \equiv \left(u_\Sigma^1, \ldots, u_\Sigma^E\right)$ are given functions belonging to D_1 (i.e., *trial functions*), which fulfill Eqs. (6), (7) and (8), respectively. Moreover, f_Ω, g_j and j_k may be defined by Eqs. (6) to (8). Let, $u \equiv \left(u^1, \ldots, u^E\right) \in D_1$, be the solution of the BVPJ, which is assumed to be unique. Then, *Trefftz problem* consists in finding the sequence $u \equiv \left(u^1, \ldots, u^E\right) \in D_1$; i.e., it consists in finding $u^i \in D_1^i$, $i = 1, \ldots, E$.

As an example, consider the BVPJ for Laplace equation

$$\mathcal{L}u \equiv -\nabla \cdot \nabla u = f_\Omega\,, \tag{9}$$

subjected to Dirichlet boundary conditions

$$u = u_\partial \quad \text{on} \quad \partial\Omega\,, \tag{10}$$

and the jump conditions

$$[u] = [u_\Sigma] \equiv j_\Sigma^0 \quad \text{and} \quad [\partial u/\partial n] \cdot \underline{n} = [\partial u_\Sigma/\partial n] \cdot \underline{n} \equiv j_\Sigma^1 \quad \text{on} \quad \Sigma\,. \tag{11}$$

4. The sought information and redundant information

Some notions related with the concept of *sought information*, of the general framework of the unified theory that was explained in the Introduction, are here discussed and illustrated for the case of Laplace equation. Two broad categories of information, relating to the solution $u \equiv \left(u^1, \ldots, u^E\right) \in D_1$ of Trefftz problem, will be distinguished. They are

i. Data of the problem. This is information that is prescribed beforehand. In particular: f_Ω, g_j and j_k, in Eqs. (6) to (8).

ii. Complementary information. Any information that is not prescribed is included under these terms.

Within the *complementary information*, several classes are distinguished. Firstly, *complementary information in Ω*, such as the values of the solution in the interior of each one of the subdomains. Secondly, *complementary information in the outer boundary, $\partial\Omega$*. The normal derivative at the outer boundary belongs to this category, when considering the Dirichlet problem for Laplace equation of Section 3. Thirdly, *complementary information in the internal boundary, Σ* – and this class of *complementary information* plays an important role in the unified theory of Trefftz methods. In the example of Section 3, the jumps of the function $[u]$ and that of its normal derivative $[\partial u/\partial n]$ are data of the problem, on Σ. In this case, the average of the function and that of its normal derivative – $\dot{u} \equiv \frac{1}{2}\left(u_+ + u_-\right)$ and $\widehat{\partial u/\partial n}$, respectively – are *complementary information on Σ*. A choice for the target of information on Σ – the *sought information* – may be the pair $\left\{\dot{u}, \overbrace{\partial u/\partial n}^{\cdot}\right\}$ everywhere on Σ. This pair possesses the required property of being sufficient, when complemented by the data of the problem, for defining a well-posed problem at each one of the subdomains, separately. Indeed, using the identities

$$u_+ = \dot{u} + \frac{1}{2}[u] \quad \text{and} \quad (\partial u/\partial n)_+ = \overbrace{\partial u/\partial n}^{\cdot} + \frac{1}{2}[\partial u/\partial n] \,, \tag{12}$$

the values of u and $(\partial u/\partial n)$ on both sides of Σ can be derived, since the definitions of the positive and negative sides of Σ can be interchanged and, both, $[u]$ and $[\partial u/\partial n]$ are data of the problem. In addition, it can be seen that the knowledge of these values together with Eqs. (6) and (7) permit formulating well-posed boundary value problems in each one of the subdomains of the partition. However, knowledge of the average $\dot{u}$, or alternatively $\overbrace{\partial u/\partial n}^{\cdot}$, would be sufficient for this purpose. Thus, either $\overbrace{\partial u/\partial n}^{\cdot}$ or $\dot{u}$ is redundant information. Therefore, if the *sought information* is defined to be the pair $\left\{\dot{u}, \overbrace{\partial u/\partial n}^{\cdot}\right\}$ everywhere on Σ, such definition is not *optimal*. On the contrary, if the *sought information* is defined to be the average $\dot{u}$ everywhere on Σ, eliminating the redundant information $\overbrace{\partial u/\partial n}^{\cdot}$, such definition is indeed *optimal*. Similarly, if the *sought information* is defined to be $\overbrace{\partial u/\partial n}^{\cdot}$, on Σ, such definition is also *optimal*.

5. The direct Trefftz Method

The direct approach, using the point of view of the unified theory of Trefftz Method, was first presented in [10] from which we draw. Consider the one-dimensional version of the BVPJ of a second order elliptic equation. The notations are those of Section 2, with $\Omega \equiv (0,l)$, $\Omega_i \equiv (x_{i-1}, x_i)$, $D_1(\Omega_i) = D_2(\Omega_i) \equiv H^2(\Omega_i)$ and $i = 1, \ldots, E$. Then

$$\mathcal{L}u \equiv -\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \frac{\mathrm{d}}{\mathrm{d}x}(bu) + cu = f_\Omega \quad \text{in} \quad \Omega_i \equiv (x_{i-1}, x_i)\,, \quad i = 1, \ldots, E\,. \tag{13}$$

The boundary and jump conditions are:

$$u(0) = g_{\partial 0}, \qquad u(l) = g_{\partial l}, \qquad [u] = j_i^0 \equiv [u_\Sigma] \quad \text{and}$$
$$\left[\frac{\mathrm{d}u}{\mathrm{d}x}\right] = j_i^1 \equiv \left[\frac{\mathrm{d}u_\Sigma}{\mathrm{d}x}\right], \qquad i = 1, \ldots, E-1\,. \tag{14}$$

It will be assumed that the Dirichlet problem is well-posed in Ω and in each one of the subdomains Ω_i.

In every subinterval (x_{i-1}, x_{i+1}), define the function $u^i(x)$ to be the restriction of $u(x)$ to Ω_i. Then, for every $i = 1, \ldots, E-1$, $u^i(x)$, is the unique solution of a boundary value problem with prescribed jumps defined in the subinterval (x_{i-1}, x_{i+1}), which is derived from the following conditions:

$$\mathcal{L}u^i = f_\Omega \quad \text{in} \quad (x_{i-1}, x_{i+1})\,, \qquad i = 1, \ldots, E-1\,, \tag{15}$$

$$[u^i]_i = j_i^0\,, \qquad \left[\frac{\mathrm{d}u^i}{\mathrm{d}x}\right]_i = j_i^1\,, \qquad i = 1, \ldots, E-1\,, \tag{16}$$

$$u^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2} j_{i-1}^0\,, \qquad i = 2, \ldots, E-1\,, \tag{17}$$

$$u^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2} j_{i+1}^0\,, \qquad i = 1, \ldots, E-2\,, \tag{18}$$

$$u^1(0) = u(0) = g_{\partial 0} \quad \text{and} \quad u^{E-1}(l) = u(l) = g_{\partial l}\,. \tag{19}$$

Let the functions $u_H^i(x)$ and $u_P^i(x)$ be defined in (x_{i-1}, x_{i+1}) by the following conditions:

$$\mathcal{L}u_H^i = 0 \quad \text{in} \quad (x_{i-1}, x_{i+1})\,, \qquad i = 1, \ldots, E\,, \tag{20}$$

$$[u_H^i]_i = \left[\frac{\mathrm{d}u_H^i}{\mathrm{d}x}\right]_i = 0\,, \qquad i = 1, \ldots, E-1\,, \tag{21}$$

$$u_H^i(x_{i-1}+) = u(x_{i-1}+) = \dot{u}(x_{i-1}) + \frac{1}{2} j_{i-1}^0\,, \qquad i = 2, \ldots, E-1\,, \tag{22}$$

$$u_H^i(x_{i+1}-) = u(x_{i+1}-) = \dot{u}(x_{i+1}) - \frac{1}{2} j_{i+1}^0\,, \qquad i = 1, \ldots, E-2\,, \tag{23}$$

$$u_H^1(x_0) = u(0) = g_{\partial 0}\,, \tag{24}$$

and

$$u_H^{E-1}(x_E) = u(l) = g_{\partial l}\,, \tag{25}$$

together with

$$\mathcal{L}u_P^i = f_\Omega \quad \text{in} \quad (x_{i-1}, x_i) \quad \text{and} \quad (x_i, x_{i+1}), \quad \text{separately,} \quad \text{for} \quad i = 1, \ldots, E-1\,, \tag{26}$$

$$u_P^i(x_{i-1}+) = u_P^i(x_{i+1}-) = 0 \quad \text{for} \quad i = 1, \ldots, E-1\,, \tag{27}$$

$$[u_P^i]_i = j_i^0 \quad \text{and} \quad \left[\frac{\mathrm{d}u_P^i}{\mathrm{d}x}\right]_i = j_i^1\,, \qquad i = 1, \ldots, E-1\,, \tag{28}$$

Then, it can be verified that

$$u^i(x) = u_H^i(x) + u_P^i(x)\,, \qquad i = 1, \ldots, E-1\,. \tag{29}$$

Even more:

$$u_H^i(x) = u_H^i(x_{i-1}-)\phi_-^i(x) + u_H^i(x_{i+1}+)\phi_+^i(x)\,, \tag{30}$$

when $\phi_-^i(x)$ and $\phi_+^i(x)$ are defined by the conditions:

$$\mathcal{L}\phi_+^i = 0\,, \qquad \phi_+^i(x_{i-1}) = 0\,, \qquad \phi_+^i(x_{i+1}) = 1\,, \tag{31}$$

$$\mathcal{L}\phi_-^i = 0\,, \qquad \phi_-^i(x_{i-1}) = 1\,, \qquad \phi_-^i(x_{i+1}) = 0\,, \tag{32}$$

together with

$$[\phi_+^i]_i = [\phi_-^i]_i = \left[\frac{\mathrm{d}\phi_+^i}{\mathrm{d}x}\right]_i = \left[\frac{\mathrm{d}\phi_-^i}{\mathrm{d}x}\right]_i = 0\,. \tag{33}$$

From Eqs. (29), (30), (22) and (23), it follows that

$$\dot{u}(x_i) - \dot{u}_P^i(x_i) = \dot{u}_H^i(x_i) = \left\{\dot{u}(x_{i-1}) + \frac{1}{2} j_{i-1}^0\right\}\phi_-^i(x_i) + \left\{\dot{u}(x_{i+1}) - \frac{1}{2} j_{i+1}^0\right\}\phi_+^i(x_i)\,. \tag{34}$$

Hence

$$-\rho_-^i \dot{u}_{i-1} + \dot{u}_i - \rho_+^i \dot{u}_{i+1} = \mu_i\,, \qquad i = 2, \ldots, E-2\,, \tag{35}$$

$$\dot{u}_i - \rho_+^i \dot{u}_{i+1} = \mu_i\,, \qquad i = 1\,, \tag{36}$$

and

$$-\rho_-^i \dot{u}_{i-1} + \dot{u}_i = \mu_i\,, \qquad i = E-1\,, \tag{37}$$

where

$$\rho_-^i = \phi_-^i(x_i)\,, \qquad \rho_+^i = \phi_+^i(x_i)\,, \qquad i = 1, \ldots, E-1\,, \tag{38}$$

$$\mu_i = \frac{\rho_-^i}{2} j_{i-1}^0 + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2} j_{i+1}^0\,, \qquad i = 2, \ldots, E-2\,, \tag{39}$$

$$\mu_i = \rho_-^i g_{\partial 0} + \dot{u}_P^i(x_i) - \frac{\rho_+^i}{2} j_{i+1}^0\,, \qquad i = 1\,, \tag{40}$$

and

$$\mu_i = \frac{\rho_-^i}{2} j_{i-1}^0 + \dot{u}_P^i(x_i) + \rho_+^i g_{\partial l}\,, \qquad i = E-1\,. \tag{41}$$

Equations (35) to (37) constitute an $E-1$ three-diagonal system of equations, which can be solved for $\dot{u}_i$ $(i = 1, \ldots, E-1)$. Once the averages $\dot{u}_i$ $(i = 1, \ldots, E-1)$ are known, all that is required to reconstruct the exact solution of the BVPJ is to apply *optimal interpolation.* To this end use is made of the identities

$$u(x_i+) \equiv \dot{u}_i + \frac{1}{2}[u]_i = \dot{u}_i + \frac{1}{2} j_i^0 \quad \text{and} \quad u(x_i-) \equiv \dot{u}_i - \frac{1}{2}[u]_i = \dot{u}_i - \frac{1}{2} j_i^0\,. \tag{42}$$

When these values are complemented with the prescribed boundary values of Eq. (14), well-posed *local problems* in each one of the subintervals of the partition can be defined. Using the previous developments, one can apply Eqs. (29) and (30), to obtain $u(x)$ in the interior of the subintervals of the partition. Up to now, all the developments have been exact. However, the construction of the functions ϕ_-^i, ϕ_+^i and u_P^i, $i = 1, \ldots, E-1$, requires resorting to numerical approximations. Although any numerical method can be used for this purpose, collocation was applied in [10] giving rise to a non-standard method of collocation (Trefftz–Herrera Collocation).

6. Trefftz–Herrera approach to DDM

Trefftz–Herrera indirect method has a long history [20–22, 26, 27], although its interpretation and development as a domain decomposition procedure is more recent [3, 4]. In particular, it has been known by a variety of names. An early version of it is known as Localized Adjoint Method (LAM) [22]. It was combined with the method of characteristics to treat advection-dominated transport; the resulting procedure has been quite successful and has been extensively applied and it is known as Eulerian–Lagrangian LAM (ELLAM) [18]. In particular, the presentation given in this Section is based on [6]. Given a differential operator, $\mathcal{L}$, and its formal adjoint, $\mathcal{L}^*$, the vector-valued bilinear function $\underline{\mathcal{D}}(u,w)$, is such that

$$w\mathcal{L}u - u\mathcal{L}^*w \equiv \nabla \cdot \underline{\mathcal{D}}(u,w)\,. \tag{43}$$

The bilinear functions $\mathcal{B}(u,w)$, $\mathcal{C}(w,u)$, $\mathcal{J}(u,w)$ and $\mathcal{K}(w,u)$, are such that

$$\underline{\mathcal{D}}(u,w) \cdot \underline{n} \equiv \mathcal{B}(u,w) - \mathcal{C}(w,u) \quad \text{on} \quad \partial\Omega\,, \tag{44}$$

$$-[\underline{\mathcal{D}}(u,w)] \cdot \underline{n} \equiv \mathcal{J}(u,w) - \mathcal{K}(w,u) \quad \text{on} \quad \Sigma\,. \tag{45}$$

Then, the following Green–Herrera formula holds:

$$\int_{\Omega} w\mathcal{L}u\mathrm{d}x - \int_{\partial\Omega} \mathcal{B}(u,w)\mathrm{d}x - \int_{\Sigma} \mathcal{J}(u,w)\mathrm{d}x = \int_{\Omega} u\mathcal{L}^{*}w\mathrm{d}x - \int_{\partial\Omega} \mathcal{C}^{*}(u,w)\mathrm{d}x - \int_{\Sigma} \mathcal{K}^{*}(u,w)\mathrm{d}x\,. \tag{46}$$

The choice of B and J depends on the kind of boundary and jump conditions considered. When the coefficients of the differential operators are continuous, J and K, are:

$$\mathcal{J}(u,w) \equiv -\underline{D}([u],\dot{w})\cdot\underline{n}, \quad \text{and} \quad \mathcal{K}(w,u) \equiv \underline{D}(\dot{u},[w])\cdot\underline{n}\,. \tag{47}$$

Introduce the following notation:

$$\langle Pu,w\rangle = \int_{\Omega} w\mathcal{L}u\mathrm{d}x\,, \qquad \langle Q^{*}u,w\rangle = \int_{\Omega} u\mathcal{L}^{*}w\mathrm{d}x\,, \tag{48}$$

$$\langle Bu,w\rangle = \int_{\partial\Omega} \mathcal{B}(u,w)\mathrm{d}x\,, \qquad \langle C^{*}u,w\rangle = \int_{\partial\Omega} \mathcal{C}^{*}(u,w)\mathrm{d}x\,, \tag{49}$$

$$\langle Ju,w\rangle = \int_{\Sigma} \mathcal{J}(u,w)\mathrm{d}x\,, \qquad \langle K^{*}u,w\rangle = \int_{\Sigma} \mathcal{K}^{*}(u,w)\mathrm{d}x\,. \tag{50}$$

With these definitions, each one of P, B, J, Q^*, C^* and K^*, are real-valued bilinear functionals defined on $\widehat{D}_1 \times \widehat{D}_2$, and Eq. (46) can be written as

$$\langle (P-B-J)\,u,w\rangle \equiv \langle (Q^{*}-C^{*}-K^{*})\,u,w\rangle\;, \qquad \forall\,(u,w)\in \widehat{D}_1\times\widehat{D}_2\,. \tag{51}$$

Or more briefly, as an identity between two bilinear functionals:

$$P-B-J \equiv Q^{*}-C^{*}-K^{*}\,. \tag{52}$$

From now on: $u \in D_1$ is the solution of the BVPJ, while f, g and $j \in D_2^*$ are defined by $f \equiv Pu_{\Omega}$, $g \equiv Bu_{\partial}$ and $j \equiv Ju_{\Sigma}$. It is assumed that a weak formulation of the BVPJ is

$$\langle (P-B-J)\,u,w\rangle \equiv \langle f-g-j,w\rangle\;, \qquad \forall w\in D_2\,. \tag{53}$$

Eq. (53) can also be written as

$$\langle (Q-C-K)^{*}\,u,w\rangle = \langle f-g-j,w\rangle\;, \qquad \forall w\in D_2\,. \tag{54}$$

These equations supply two equivalent variational formulations of the BVPJ. They are the *variational formulation in terms of the data* and the *variational formulation in terms of the complementary information*, respectively. According to their definitions, the linear functionals Q^*u, C^*u and K^*u, represent *complementary information* in $\Omega_i\,(i=1,\ldots,E)$, $\partial\Omega$ and Σ, respectively. As explained in the Introduction, in the unified theory of Trefftz Methods one chooses a target of information on Σ. Such target could be K^*u itself. However, generally such choice would lead to a definition of *complementary information* that is not *optimal*, since usually K^*u contains *redundant information.* Thus, to develop more efficient numerical methods, one needs to eliminate *redundant information.* To this end introduce a decomposition $\{S,R\}$ of the bilinear functional K, such that

$$K \equiv S+R\,. \tag{55}$$

Here, S is such that a function $\widetilde{u}\in\widehat{D}_1$ *contains the sought information* when $S^*\widetilde{u}=S^*u$, with $u\in\widehat{D}_1$ the solution of the BVPJ. Then, $w\in D_2$, is an *optimal test function* when $w\in\widetilde{N}\equiv N_Q\cap N_C\cap N_R$, where N_Q, N_C and N_R are the null-subspaces of Q, C and R, respectively. At this point it is convenient to introduce an auxiliary concept:

Definition 1. *A subset of optimal weighting functions,* $\mathcal{E} \subset \widetilde{N} \equiv N_Q \cap N_C \cap N_R$, *is said to be TH-complete for* S^*, *when for any* $\hat{u} \in \widehat{D}_1$, *one has:*

$$\langle S^*\hat{u}, w\rangle = 0\,, \qquad \forall w \in \mathcal{E} \Rightarrow S^*\hat{u} = 0\,. \tag{56}$$

Using this concept it is possible to give a rather concise formulation of *Trefftz–Herrera Methods.* This is given in the next Theorem.

Theorem 1. *Let* $\mathcal{E} \subset \widetilde{N}$ *be a system of optimal weighting functions, TH-complete for* S^*. *Then, a necessary and sufficient condition for* $\hat{u} \in D_1$ *to contain the sought information, is that*

$$-\langle S^*\hat{u}, w\rangle = \langle f - g - j, w\rangle\ , \qquad \forall w \in \mathcal{E}\,. \tag{57}$$

Proof is given in [6], by substitution in Eq. (54).

Theorem 1, supplies a **General Formulation of Indirect Trefftz–Herrera Methods** which can be applied to any linear equation or system of such equations, independently of its type (elliptic, parabolic or hyperbolic), including the case when the coefficients are discontinuous. In particular, when the differential operator is symmetric and positive definite, then the bilinear form S^*, is also positive definite on the linear subspace of optimal test functions, $\widetilde{N} \equiv N_Q \cap N_C \cap N_R$.

7. Trefftz–Herrera method in several dimensions

Throughout this Section the results of Section 6, will be illustrated with the BVPJ of the general second order elliptic equation

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + \nabla \cdot (\underline{b}u) + cu = f_\Omega\,. \tag{58}$$

Here $\underline{\underline{a}}$ is symmetric and positive definite. The boundary and jump conditions are

$$u = u_\partial \quad \text{on} \quad \partial\Omega\,, \tag{59}$$

$$[u] = [u_\Sigma] \equiv j^0_\Sigma \quad \text{and} \quad [\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n} = [\underline{\underline{a}} \cdot \nabla u_\Sigma] \cdot \underline{n} \equiv j^1_\Sigma \quad \text{on} \quad \Sigma\,. \tag{60}$$

The developments that follow apply even if the coefficients of the differential operator are discontinuous. In particular, when the coefficients are continuous the second of the jump conditions of Eq. (11), in the presence of the first one, is equivalent to

$$\left[\frac{\partial u}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right] \quad \text{on} \quad \Sigma\,. \tag{61}$$

When $\mathcal{L}$ is given by Eq. (9), the adjoint differential operator $\mathcal{L}^*$ is:

$$\mathcal{L}^*w \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla w) - \underline{b} \cdot \nabla w + cw\,, \tag{62}$$

while

$$\underline{\mathcal{D}}(u,w) \equiv \underline{\underline{a}} \cdot (u\nabla w - w\nabla u) + \underline{b}uw\,. \tag{63}$$

For Dirichlet boundary conditions, a possible choice for $\mathcal{B}$ is:

$$\mathcal{B}(u,w) \equiv (\underline{a}_n \cdot \nabla w)u\,. \tag{64}$$

In such case, Eq. (44) implies

$$\mathcal{C}^*(u,w) \equiv w(\underline{a}_n \cdot \nabla u - b_n u)\,. \tag{65}$$

Above $\underline{\underline{a}}_n = \underline{\underline{a}} \cdot \underline{n}$ and $b_n = \underline{b} \cdot \underline{n}$. Apply Eq. (45), to define

$$\mathcal{J}(u,w) \equiv \mathcal{J}^0(u,w) + \mathcal{J}^1(u,w) \quad \text{and} \quad \mathcal{K}(w,u) \equiv \mathcal{K}^0(w,u) + \mathcal{K}^1(w,u)\,. \tag{66}$$

With

$$\mathcal{J}^0(u,w) \equiv -[u](\dot{\overline{\underline{\underline{a}} \cdot \nabla w + \underline{b}w}}) \cdot \underline{n} \quad \text{and} \quad \mathcal{J}^1(u,w) \equiv \dot{w}[\underline{\underline{a}} \cdot \nabla u] \cdot \underline{n}\,, \tag{67}$$

and

$$\mathcal{K}^0(w,u) \equiv \dot{u}[\underline{\underline{a}} \cdot \nabla w + \underline{b}w] \cdot \underline{n} \quad \text{and} \quad \mathcal{K}^1(w,u) \equiv -[w](\dot{\overline{\underline{\underline{a}} \cdot \nabla u}}) \cdot \underline{n}\,. \tag{68}$$

Observe that $J = J^0 + J^1$ and $K = K^0 + K^1$, if

$$\langle J^0 u, w\rangle \equiv \int_\Sigma \mathcal{J}^0(u,w)\,\mathrm{d}x \quad \text{and} \quad \langle J^1 u, w\rangle \equiv \int_\Sigma \mathcal{J}^1(u,w)\,\mathrm{d}x\,, \tag{69}$$

$$\langle K^0 w, u\rangle \equiv \int_\Sigma \mathcal{K}^0(w,u)\,\mathrm{d}x \quad \text{and} \quad \langle K^1 w, u\rangle \equiv \int_\Sigma \mathcal{K}^1(w,u)\,\mathrm{d}x\,. \tag{70}$$

There are several options for the definition of the *sought information*. Here, our target of information will be the average of the solution across Σ, $\dot{u}$. Such information is enough to define well posed problems in each one of the subdomains of the partition, when it is complemented with the data of the BVPJ. Even more, this definition of *sought information* is *optimal*, For this choice, $S = K^0$ and $R = K^1$:

$$-\langle S^*\widetilde{u}, w\rangle \equiv -\int_\Sigma \dot{\widetilde{u}}[\underline{\underline{a}} \cdot \nabla w] \cdot \underline{n}\mathrm{d}x\,, \tag{71}$$

and the system of equations for the sought information is

$$-\int_\Sigma \dot{\widetilde{u}}[\underline{\underline{a}}_n \cdot \nabla w]\mathrm{d}x = \langle f - g - j, w\rangle \qquad \forall w \in \mathcal{E} \subset \widetilde{N} \equiv N_Q \cap N_C \cap N_R\,. \tag{72}$$

Here, $w \in N_Q \Leftrightarrow \mathcal{L}^* w = 0$, on $\Omega_i\,(i = 1, \ldots, E)\,, w \in N_C \Leftrightarrow w = 0$, on $\partial\Omega$, and $w \in N_R \Leftrightarrow [w] = 0$, on Σ. In addition,

$$\langle f, w\rangle \equiv \int_\Omega w f_\Omega \mathrm{d}x\,, \qquad \langle g, w\rangle \equiv \int_{\partial\Omega} \mathcal{B}(u_\partial, w)\,\mathrm{d}x \quad \text{and} \quad \langle j, w\rangle \equiv \int_\Sigma \mathcal{J}(u_\Sigma, w)\,\mathrm{d}x\,. \tag{73}$$

Because of the continuity condition that the *optimal test functions* must satisfy, the application of this procedure leads to an overlapping DDM; i.e., the support of test functions includes more than one subdomain of the partition.

When $\underline{b} \equiv 0$ and $c \geq 0$, the differential operator $\mathcal{L}$ is symmetric and positive definite and the following relations hold:

$$P \equiv Q, \quad B \equiv C, \quad J \equiv K\,. \tag{74}$$

Even more, when $\hat{u} \in \widetilde{N}$ and $w \in \widetilde{N}$, it can be verified that

$$-\langle S^*\hat{u}, w\rangle \equiv -\int_\Sigma \hat{u}[\underline{\underline{a}} \cdot \nabla w] \cdot \underline{n}\mathrm{d}x \equiv \int_\Omega \{\nabla w \cdot a \cdot \nabla \hat{u} + c w \hat{u}\}\mathrm{d}x\,. \tag{75}$$

Therefore, S^* is a symmetric and positive definite bilinear functional on $\widetilde{N}$.

8. TH-COLLOCATION

For simplicity, here the procedure is only explained for the case of vanishing jumps; i.e., $j^0_\Sigma = j^1_\Sigma = 0$ (see [6], for more general results). The system of equations for the *sought information* – which has been chosen to be the average of the function, across Σ; i.e., the function on Σ, since it is continuous – are derived by direct application of the variational principle of Eq. (72), with a suitable TH-family of *optimal test functions*, $\mathcal{E} \subset D_2$:

$$-\int_\Sigma u\left[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w\right]\mathrm{d}\underline{x} = \int_\Omega w f_\Omega \mathrm{d}\underline{x} - \int_{\partial\Omega} u_{\partial}\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w \mathrm{d}\underline{x} \qquad \forall w \in \mathcal{E}\,. \tag{76}$$

8.1. The weighting functions

For one-dimensional problems like the one considered in Section 5, TH-complete systems are finite and Eq. (76) can be applied using a whole TH-complete system. On the other hand, in numerical applications to multidimensional problems one can not apply TH-complete systems in full since, in that case, they are infinite. Instead, one truncates a TH-complete family and applies Eq. (76) with a finite subset of it, only. The numerical construction of *optimal test functions*, for the case when the region Ω is a rectangle and the subdomains of the partition are also rectangles, Fig. 2, proceeds as follows. One can associate with each internal node (x_i, y_j), four rectangles $\left\{\Omega^1_{ij}, \ldots, \Omega^4_{ij}\right\}$, Fig. 3, and the notations Ω_{ij}, $\partial\Omega_{ij}$ and Σ_{ij} are adopted for the interior of the union of the four rectangle closures, the boundary of Ω_{ij} and the intersection $\Sigma \cap \Omega_{ij}$, respectively. Associated with each node, (x_i, y_j), several optimal test functions can be constructed. The support of each one of them is contained in Ω_{ij}. They vanish identically on $\partial\Omega_{ij}$ and, so, they are determined uniquely by their restrictions to Σ_{ij}.

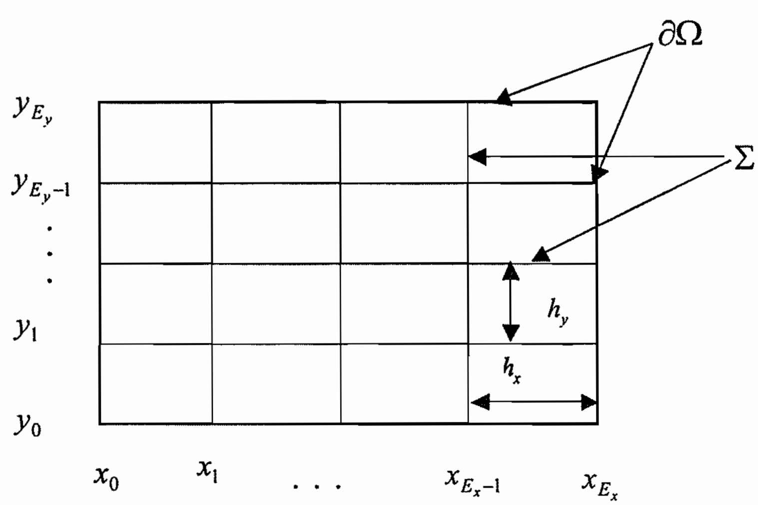

Fig. 2. Partition of the domain $\Omega = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$ in rectangular $E_x \times E_y$ elements, where $h_x = x_i - x_{i-1}\,,\quad i = 1, \ldots, E_x$ and $h_y = y_j - y_{j-1}\,,\quad j = 1, \ldots, E_y$

In [6], associated with each node (x_i, y_j) three optimal test functions $w^\alpha_{ij}(x, y)$, $\alpha = 0, 1, 2$, were built. The general expression, separately in each one of the four basic subdomains, for the optimal test functions is (for $\alpha = 0, 1, 2$, $\gamma = 1, \ldots, 4$):

$$w^\alpha_{ij}(x, y) = B^\alpha_{ij}(x, y) + \sum_{\beta=1}^{4} C^{\alpha\beta}_{ij}(\gamma)\, N^\beta_{ij}(x, y; \gamma)\,, \qquad (x, y) \in \Omega^\gamma_{ij}\,. \tag{77}$$

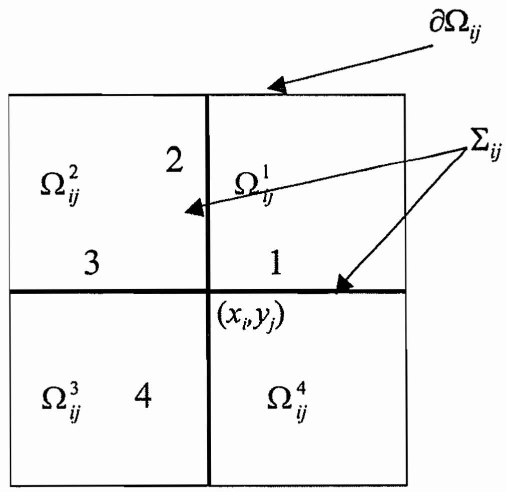

Fig. 3. Subregion $\Psi^\nu = B^\nu + \sum_{j=1}^4 C_j^\nu N^j$ associated with the node (x_i, y_j)

Here, $C_{ij}^{\alpha\beta}(\gamma)$, $\gamma = 1, \ldots, 4$, are suitable real numbers. The values of w_{ij}^α, on Σ_{ij}, are determined by the functions $B_{ij}^\alpha(x, y)$, which were taken successively, as piecewise linear and cubic, on Σ. In addition, they vanished on $\partial\Omega_{ij}$. The functions $N_{ij}^\beta(x, y; \gamma)$, $\beta = 1, \ldots, 4$, with support in Ω_{ij}^γ for each $\gamma = 1, \ldots, 4$, were introduced in order to be able fulfill the adjoint differential equation, $\mathcal{L}^* w_{ij}^\alpha = 0$, at four Gaussian collocation points, of each basic rectangle, without modifying the values at their boundaries. For each $\gamma = 1, \ldots, 4$, $N_{ij}^\beta(x, y; \gamma)$ was taken to be a bi-cubic polynomial vanishing identically on the boundaries of the basic subdomains. The functions $B_{ij}^\alpha(x, y)$ and $N_{ij}^\beta(x, y; \gamma)$ are given in Tables 1 and 2.

Table 1. Definitions of functions $B_{ij}^\beta(x, y), \beta = 0, \ldots, 4$, where $H_i^0(x)$, $H_i^1(x)$, $H_j^0(y)$ and $H_j^1(y)$ are Hermite cubic polynomials in x and y, respectively

	Ω_{ij}^1	Ω_{ij}^2	Ω_{ij}^3	Ω_{ij}^4
$B_{ij}^0(x,y)$	$\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)\left(\frac{y-y_{j+1}}{y_j-y_{j+1}}\right)$	$\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j+1}}{y_j-y_{j+1}}\right)$	$\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)$	$\left(\frac{x-x_{i+1}}{x_i-x_{i+1}}\right)\left(\frac{y-y_{j-1}}{y_j-y_{j-1}}\right)$
$B_{ij}^1(x,y)$	$H_i^1(x)H_j^0(y)$	$H_i^1(x)H_j^0(y)$	$H_i^1(x)H_j^0(y)$	$H_i^1(x)H_j^0(y)$
$B_{ij}^2(x,y)$	$H_i^0(x)H_j^1(y)$	$H_i^0(x)H_j^1(y)$	$H_i^0(x)H_j^1(y)$	$H_i^0(x)H_j^1(y)$

Table 2. Definitions of functions $N_{ij}^\beta(x, y), \beta = 1, \ldots, 4$, where $H_i^1(x)$ and $H_j^1(y)$ are Hermite cubic polynomials in x and y, respectively

γ	1	2	3	4
$N_{ij}^1(x,y;\gamma)$	$H_i^1(x)H_j^1(y)$	$H_{i-1}^1(x)H_j^1(y)$	$H_{i-1}^1(x)H_{j-1}^1(y)$	$H_i^1(x)H_{j-1}^1(y)$
$N_{ij}^2(x,y;\gamma)$	$H_{i+1}^1(x)H_j^1(y)$	$H_i^1(x)H_j^1(y)$	$H_i^1(x)H_{j-1}^1(y)$	$H_{i+1}^1(x)H_{j-1}^1(y)$
$N_{ij}^3(x,y;\gamma)$	$H_{i+1}^1(x)H_{j+1}^1(y)$	$H_i^1(x)H_{j+1}^1(y)$	$H_i^1(x)H_j^1(y)$	$H_{i+1}^1(x)H_j^1(y)$
$N_{ij}^4(x,y;\gamma)$	$H_i^1(x)H_{j+1}^1(y)$	$H_{i-1}^1(x)H_{j+1}^1(y)$	$H_{i-1}^1(x)H_j^1(y)$	$H_i^1(x)H_j^1(y)$

8.2. Solution approximation on Σ

In Trefftz–Herrera indirect method one retrieves information about the solution of the BVPJ on the internal boundary Σ, exclusively. On Σ, one has to distinguish two sources of information, about the solution of the BVPJ; the optimal test functions, when they are used in the variational principle of Eq. (76), and the boundary conditions, at intersection points of $\partial\Omega$ and Σ. In addition, the solution approximation has to be defined there only. There is considerable freedom of choice for the base functions to be used on Σ, except for the fact that their number has to be equal to that of the weighting functions. This is required in order for the system of Eqs. (76) to be determined. However, it is frequently advantageous to use, as base functions, the restriction of the test functions to the internal boundary, Σ. In particular, for the symmetric and positive definite case this leads to symmetric and positive matrices. When the base functions are chosen in this manner, the general expression for the approximate solution of the BVPJ, to be used on Σ, is

$$\tilde{u}(x,y) = \sum_{(k,l)\in\eta} \sum_{\nu=0}^{NF-1} U_{kl}^{\nu} w_{kl}^{\nu}(x,y) + \sum_{(r,s)\in\eta_\partial} u_\partial(x_r,y_s)\, w_{rs}^{0}(x,y), \qquad (x,y)\in\Sigma\,. \tag{78}$$

The reader is warned that Eq. (78) holds at the internal boundary Σ, only. Here, η is the collection of nodes for which the set of weighting functions that vanish identically on $\partial\Omega$ is not void. In addition, η_∂ is the set of nodes lying on the external boundary. More precisely, η_∂ is the set of pairs (r,s) such that $(x_r,y_s)\in\partial\Omega$. Corner-nodes need not be included, in the second sum of Eq. (78), because when (x_r,y_s) is a corner then $w_{rs}^{\alpha}\equiv 0$, on Σ. In addition, NF is the number of functions associated with the node (x_k,y_l). Although this number generally varies with the particular node considered, this variation is not incorporated in the notation for the sake of simplicity. Finally, U_{kl}^{ν} are coefficients that are determined by application of the variational principle of Eq. (76). In Eq. (78) the two sources of information, mentioned before, are clearly separated. Indeed, in the right-hand side member of that equation, the coefficients of the second sum contain information supplied by the Dirichlet boundary conditions, while the coefficients of the first one contain information that will be derived by application of the optimal test functions in the variational principle of Eq. (76). This is explained next.

8.3. The system of equations

Applying the expression for the approximate solution of Eq. (78), the global system of equations is obtained:

$$M_{ijkl}^{\mu\nu} U_{kl}^{\nu} = F_{ij}^{\mu}\,, \quad (k,l) \text{ and } (i,j)\in\eta, \qquad \mu,\nu = 0,\ldots,NF-1\,. \tag{79}$$

The matrix of the system is given by

$$M_{ijkl}^{\mu\nu} = -\int_{\Sigma} w_{kl}^{\nu}\left[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w_{ij}^{\mu}\right]\mathrm{d}\underline{x}\,. \tag{80}$$

The right hand side of Eq. (79) is

$$F_{ij}^{\mu} = \int_{\Omega} w_{ij}^{\mu} f_{\Omega}\mathrm{d}\underline{x} - \int_{\partial\Omega} u_\partial \underline{n}\cdot\underline{\underline{a}}\cdot\nabla w_{ij}^{\mu}\mathrm{d}\underline{x} + \sum_{(r,s)\in\eta_\partial} u_{\partial rs}\int_{\Sigma} w_{rs}^{0}\left[\underline{n}\cdot\underline{\underline{a}}\cdot\nabla w_{ij}^{\mu}\right]\mathrm{d}\underline{x}\,,$$

$$(k,l) \text{ and } (i,j)\in\eta, \qquad \mu,\nu = 0,\ldots,NF-1\,. \tag{81}$$

When the differential operator, is symmetric and positive definite (this corresponds to $\underline{b} \equiv 0$ and $c \geq 0$), then so is this global matrix of Eq. (80). In [6] two algorithms were developed and tested:

Algorithm I.

The family of test functions contains only one member, whose values on the internal boundary are piecewise linear. Only one degree of freedom is associated with each internal node and a nine-diagonal global matrix. This matrix is symmetric and positive definite when so is the differential operator.

Algorithm II.

The full family of three test functions (or less, at those nodes in which some of the functions of this family do not satisfy the required zero boundary condition on the external boundary) was applied at each node, including boundary nodes. This leads to an algorithm in which the optimal test functions are piecewise cubic on the internal boundary and the global matrix is block nine-diagonal. The blocks are 3 by 3.

8.4. The error

For problems in several dimensions the error of numerical solutions, derived using Trefftz–Herrera method stems from two sources. Firstly, the differential equations are only fulfilled in an approximate manner in the interior of the subdomains of the partition. Secondly, an additional error is introduced by truncation of the TH-complete systems. For the above-mentioned algorithms the error introduced by the differential equation is $O\left(h^4\right)$; the truncation errors are $O\left(h^2\right)$ and $O\left(h^4\right)$, for Algorithms I and II, respectively, and so are the overall errors are $O\left(h^2\right)$ and $O\left(h^4\right)$.

8.5. Optimal interpolation

To extend the information on the internal boundary to the whole domain, Ω, one solves the local problems by collocation. When the differential operator is symmetric, the optimal test functions are used for this purpose.

8.6. The numerical experiments

The numerical experiments that were performed in [6] consisted in applying two algorithms (I and II), for solving the BVPJ of Eqs. (58) to (60). In Algorithm I, linear base functions on Σ are used, while in Algorithm II the whole set of cubic piecewise polynomials, which are $C^1(\Sigma)$ are applied. The examples treated correspond to several choices of the coefficients in Eq. (58), which are given in Table 3. In addition, Table 4 exhibits the analytical solutions for each one of them.

In all cases the domain of definition was the unit square $[0,1] \times [0,1]$, except for Example 2, in which the problem domain was the square $[1,2] \times [1,2]$. The analytical solutions imply the boundary values that were imposed. Only in Examples 6 and 7 the jumps conditions were different to zero and they are indicated in Table 4.

The numerical results are summarized in Figs. 4 to 10. Each one of the examples was solved in a uniform rectangular partition ($E = E_x = E_y$) of the domain (Fig. 2), using Algorithm I and, subsequently, Algorithm II. The convergence rate of the error –measured in terms of the norm $\|\|_\infty$ – is $O\left(h^2\right)$ and $O\left(h^4\right)$ respectively, as shown in those figures.

Table 3. Coefficients and right hand term of the examples treated

Example	$\underline{\underline{a}}$	$\underline{b}$	$\bar{c}$	f_Ω
1	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	1	$(1 - x^2 - y^2)e^{xy}$
2	$a_{11} = a_{22} = xy$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	0	0
3	$a_{11} = 1 + x^2$ $a_{22} = 1 + y^2$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	1	$6(y^2 - x^2)$
4	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	1	$-2(4\pi)^2 \cos(4\pi x) \sin(4\pi y)$
5	$a_{11} = 1 + x^2$ $a_{22} = 1 + y^2$ $a_{12} = a_{21} = 0$	$b_1 = y - 2$ $b_2 = x - 2$	$2(x + y)$	$-(x^4 + y^4)e^{xy}$
6	$a_{11} = a_{22} = 1$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 1$	0	0
7	$a_{11} = a_{22} =$ $= \begin{cases} 1 & 0 \leq y \leq 1/2 \\ 4 & 1/2 \leq y \leq 1 \end{cases}$ $a_{12} = a_{21} = 0$	$b_1 = b_2 = 0$	0	$(1 - x^2 - y^2)e^{xy} \quad 0 \leq y \leq 1/2$ $(1 - 4x^2 - 4y^2)e^{xy} \quad 1/2 \leq y \leq 1$

Table 4. Coefficients and right hand term of the examples treated

Example	Exact solution
1	e^{xy}
2	$x^2 - y^2$
3	$x^2 - y^2$
4	$\cos(4\pi x) \sin(4\pi y)$
5	e^{xy}
6	$e^x + e^y \; y < 1/2$ $e^x + e^y \; y = 1/2$ $e^x + e^y + 2 \; y > 1/2$ with jump conditions: $j^0_\Sigma(x, 0.5) = 4; \quad x \in [0, 1]$
7	e^{xy} with jump conditions: $j^1_\Sigma(x, 0.5) = 3xe^{x/2}; \quad x \in [0, 1]$

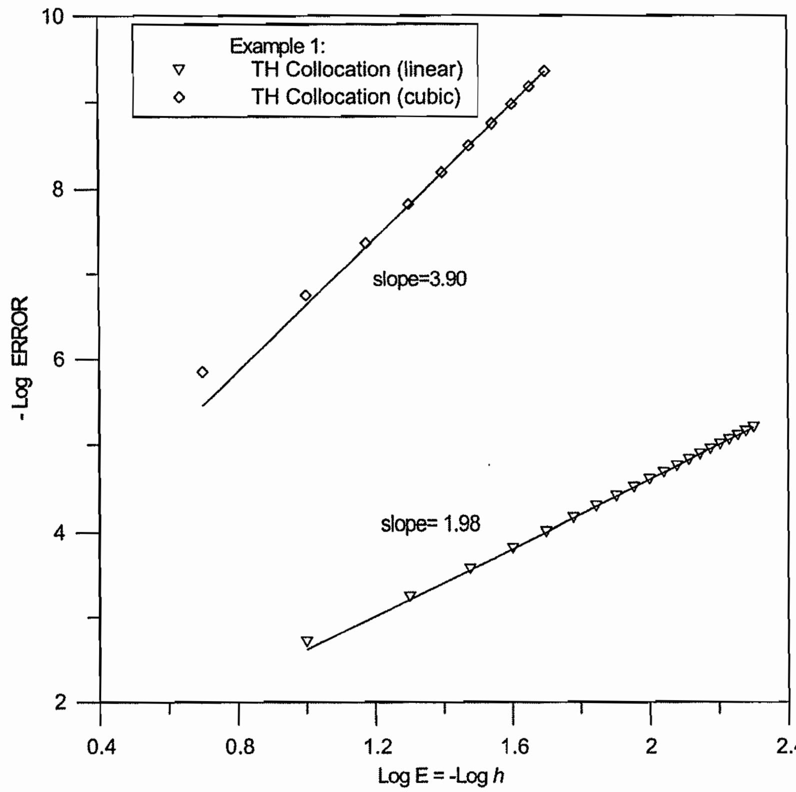

Fig. 4. Example 1: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

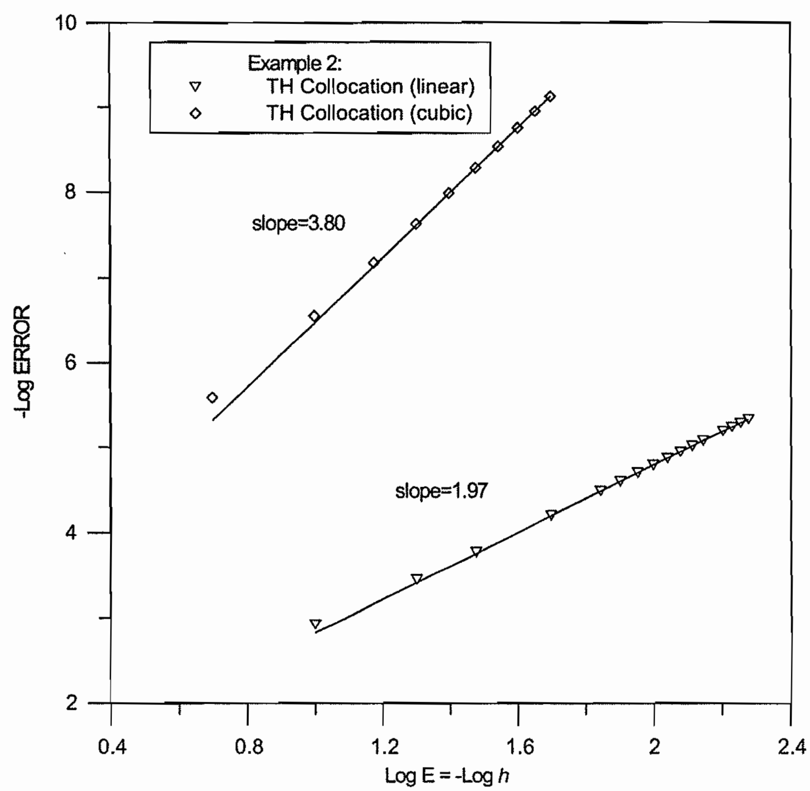

Fig. 5. Example 2: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

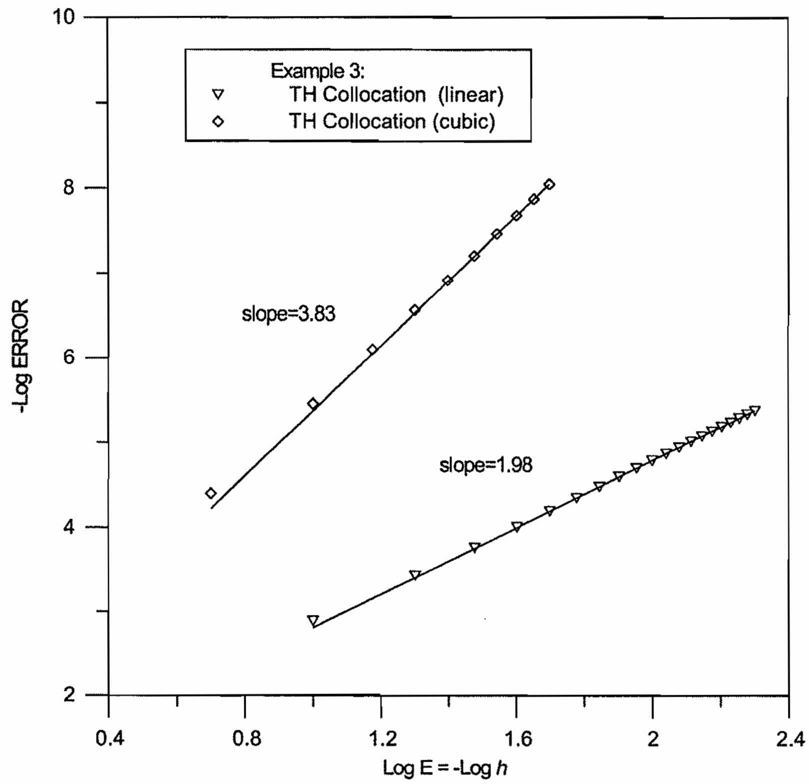

Fig. 6. Example 3: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

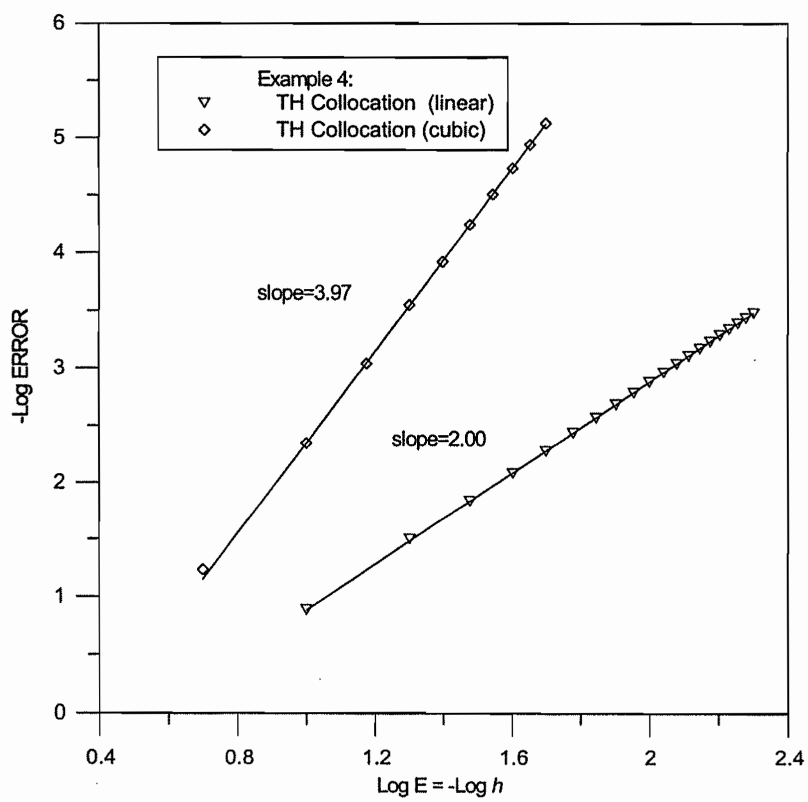

Fig. 7. Example 4: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

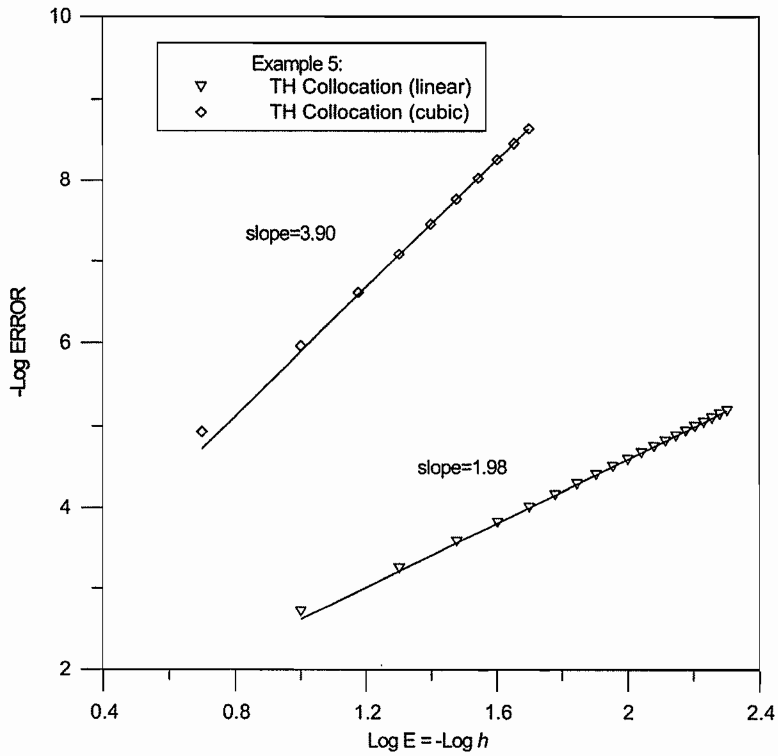

Fig. 8. Example 5: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

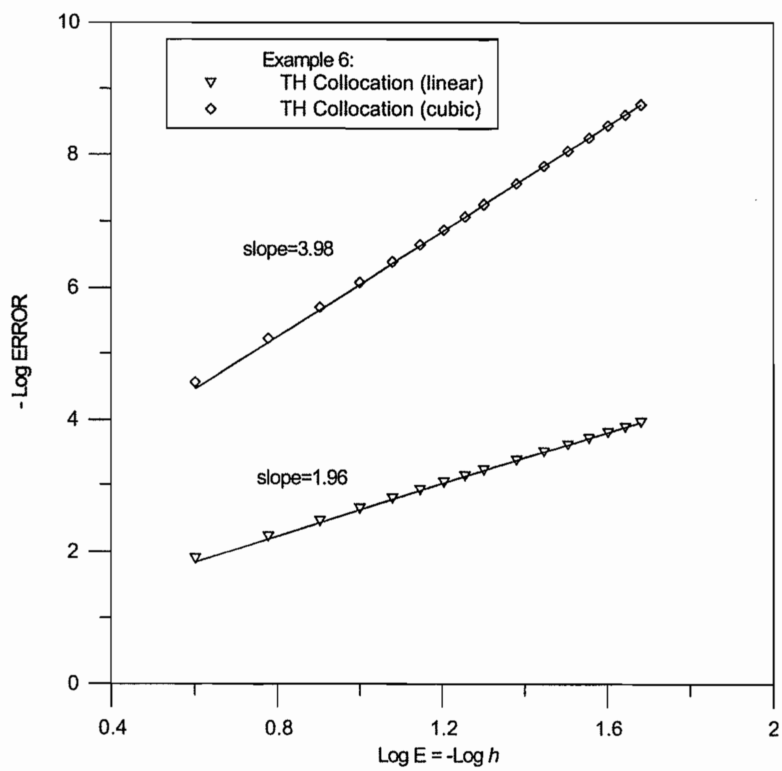

Fig. 9. Example 6: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

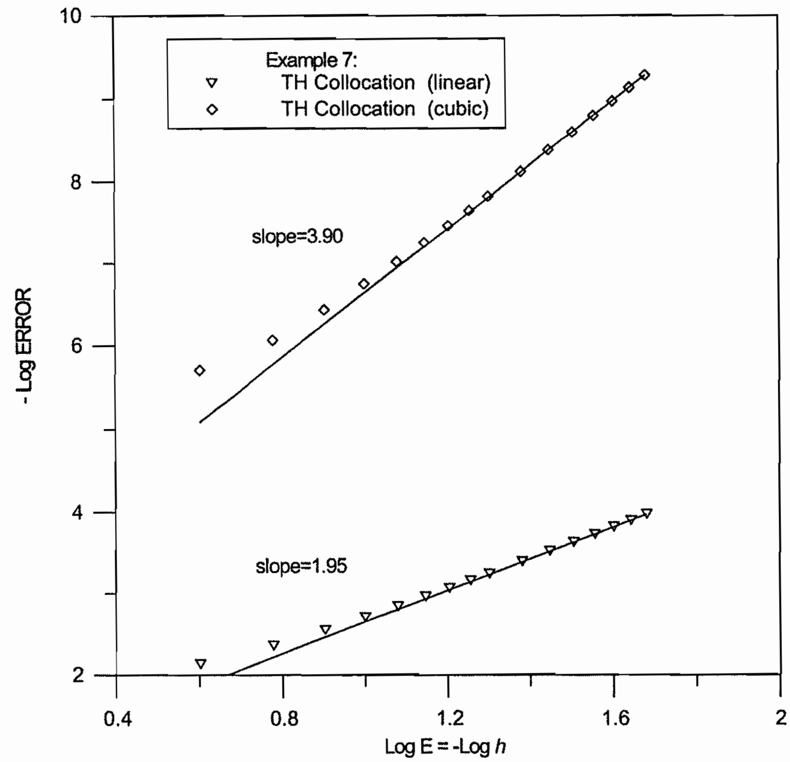

Fig. 10. Example 7: Convergence rate of Trefftz–Herrera collocation method using linear and cubic weighting functions

9. Conclusions

It has been shown that Trefftz Method, when it is formulated as in Herrera's unified theory, is quite broad and capable of incorporating and unifying many numerical methods for partial differential equations. It is a valuable tool both as a discretization procedure and as a domain decomposition method.

The unified theory of Trefftz Methods is quite systematic and possesses outstanding generality. It is applicable to any differential equation, or system of such equations, which is linear, independently of its type. This includes the case of discontinuous coefficients. The general problem treated is one in which, in addition to the conditions prescribed in the external domain boundary, jumps are prescribed in some internal boundaries. In this paper the basic concepts of this unified theory have been presented. Using this general framework two very wide classes of Trefftz Methods were identified: direct and indirect (or Trefftz–Herrera) methods. A brief description of each of these procedures has been presented. As an illustration, the indirect method was used to produce a non-standard method of collocation (Trefftz–Herrera Collocation) for elliptic problems, of second order, in several dimensions. TH-collocation exhibits several advantages with respect to standard collocation. Among them:

1. A dramatic reduction in the number of degrees of freedom associated with each node. In the standard method of collocation that number is two in one dimension (1-D); four in 2-D; and eight in 3-D. For some of the TH-collocation algorithms, this is one for all space dimensions.

2. In the standard method of collocation, using Hermite cubics, the global matrix is non-symmetric even when the differential operator is formally symmetric. In TH-collocation, the global matrix is symmetric and positive definite when the differential operator has these properties.

3. When Hermite cubics are used to approximate the local solutions, in the problems treated in this paper, the error is $O\left(h^4\right)$, if the test functions are piece-wise cubic on Σ, and it is $O\left(h^2\right)$ when

the test functions are only piece-wise linear. From theoretical considerations, an error $O\left(h^3\right)$ is expected for piece-wise-quadratic test functions.

4. The construction of the test functions and optimal interpolation are quite suitable for parallel computation.

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