A General Effective Method for Combining Collocation and DDM: An Application of Discontinuous Galerkin Methods

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Domain decomposition methods (DDM) have received much attention in recent years. They constitute the most effective means of using parallel computing resources to model continuous systems. However, combining collocation procedures with domain decomposition methods presents complications that must be overcome in order to profit from the advantages of parallel computing. The present paper belongs to a line of research in which a theory that constitutes a general and systematic formulation of discontinuous Galerkin methods (dG) is being investigated. Based on it, a new method of collocation of general applicability, TH-collocation, was recently introduced. For a broad class of symmetric and positive continuous systems, TH-collocation yields symmetric and positive matrices. This clears the way for applying effectively DDM and parallel computing, in combination with collocation, to such systems. In this paper the general procedure is explained with some detail and then is applied to develop an effective method for processing elliptic equations of second order. This, by the way, overcomes the difficulties encountered in a previous Herrera and Pinder's article. © 2004 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 21: 672–700, 2005

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1. INTRODUCTION

This article belongs to a line of research on Herrera's theory of partial differential equations (PDEs) and its applications. This theory has implications in a variety of fields related with PDEs. Following approximately a chronological order of the most relevant publications, we may cite among them:

- Variational principles for boundary value problems [1],

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- Boundary methods; in particular, Trefftz method and its generalizations (Trefftz-Herrera method) [2–4],
- Complete systems of functions (including biorthogonal systems) [1, 5–7],
- Localized Adjoint Method (LAM) and its application to advection dominated transport (ELLAM) [8, 9],
- New collocation procedures for PDEs (TH-collocation) [10–15],
- Domain decomposition methods (DDM) [16–19] and
- Discontinuous Galerkin method (dG).

From the point of view of its numerical applications, Herrera's theory constitutes a very systematic and general formulation of discontinuous Galerkin methods (dG). In particular, *TH-collocation* [13, 14] is a new and very general procedure for applying orthogonal collocation to the discretization of boundary value problems with prescribed jumps (BVPJ) on the *internal boundary*, separating the subdomains of the domain partition from each other, and having possibly discontinuous operator coefficients. Systems of equations are also included in this theory. When elliptic differential equations of any order [15], or elliptic systems such as that of elasto-statics, that are symmetric and positive definite are discretized by TH-collocation, then one obtains global matrices that are also symmetric and positive definite, something that is not possible with other collocation methods.

On the other hand, domain decomposition methods (DDM) have received much attention in recent years [19], mainly because, at present, they constitute the most effective means for using parallel computing resources to model continuous systems. However, combining collocation procedures with DDM presents difficulties that must be overcome in order to profit from the advantages of parallel computing [20–23]. The main technical difficulties stem from the fact that, when boundary value problems are discretized by orthogonal spline collocation (OSC), the global matrices fail to be symmetric and positive, even when the differential operators involved enjoy this property. This failure precludes the direct use of the Conjugate Gradient Method (CGM) when DDM is applied, which in turn significantly reduces the effectiveness of DDM.

Herrera and Pinder first addressed this problem in 1994 [20]. Then, a DDM procedure was presented for which the interface conditions of the domain decomposition are expressed in terms of a positive definite transformation that can be used to apply CGM. However, their method in that form, which is based on the application of OSC in the interior of the subdomains of the *coarse partition*, is difficult to apply and introduces inaccuracies that deteriorate the final precision of the method. To a large extent, this is due to the fact that the system of equations obtained by application of the DDM that was introduced there, is not strictly equivalent to the global system of equations. More recently, in 1997 [22], Bialecki and Dryja derived an algorithm, which for a special domain shape and for a special kind of partition yields a nonsymmetric matrix, but with the property that its skew symmetric part is only $O(h)$. Furthermore, in 2003, they obtained [23] for Poisson's equation in an *L-shaped* domain a very efficient algorithm for combining collocation and DDM. Unfortunately, their methods are not easy to extend to more general situations because their procedures are based on very special properties of OSC, the equations considered, the domain shapes and the domain partitions.

As mentioned before, for very general classes of boundary value problems with prescribed jumps, TH-collocation yields global matrices that are symmetric and positive definite. Therefore, the introduction of TH-collocation overcomes, for a wide class of problems, the basic difficulty that had precluded combining collocation and DDM effectively. The main purpose of the present article is to present the general procedure for combining TH-collocation and DDM

in an efficient manner and illustrate it with the particular case in which the differential equation is symmetric and positive, of elliptic type and of second order.

A very important property of the procedure presented here is that the system of equations, after DDM is applied, is strictly equivalent to the global system of equations that is obtained by the TH-collocation method of discretization. Not only does this new procedure eliminate the inaccuracies, but it also resolves the difficulties that limit the applicability of the method proposed in [20]. This is achieved using, in addition to TH-collocation, a DDM that can be classified as a substructuring approach [24], but which also exhibits some novel attractive features, which make the implementation of the new procedure simpler. In our opinion this DDM should be developed further, but that will be the subject of other future publications.

It should be pointed out that in this article we have not tried to optimize the rate of convergence of the CGM applied because our goal here is not to develop software that could compete with the best available in the market. This would have taken us beyond the scope of the present article; as mentioned, our main purpose is to exhibit a new and very general method for combining orthogonal collocation with domain decomposition methods in a manner in which the CGM can be applied directly. In particular, we did not test different *preconditioners*, which of course will improve the rate of convergence of CGM when such preconditioners are incorporated.

Regarding TH-collocation, it must be mentioned that its origins can be traced back to a paper published in 1987 when, following a personal suggestion of Pinder, Herrera [12] applied orthogonal collocation in the framework of his theory, for the first time. One conspicuous feature of Herrera's discretization approaches is that in them *optimal test functions* are developed and applied (this is explained in Sections 4 and 5 of the present article). In numerical applications, this permits the complete separation of two processes: that of approximating the differential equation and that of approximating the sought solution in the *internal boundary*; i.e., the interelement boundary of the domain partition. At first, optimal test functions were developed using analytic approaches (see [25, 26]), which is very restrictive. However, in [12] they were constructed using orthogonal collocation, although only applications to ordinary differential equations were made. In spite of the fact that the feasibility of extending the method to several dimensions was mentioned from the very beginning, the subject remained dormant for some years. Recently, however, in 1999 [13], the previous method of [12] was reinterpreted as a nonstandard collocation method and then in [14], with Yates' collaboration, the first implementations of TH-collocation for problems in several dimensions were presented. In this latter article, TH-collocation was used in essentially the form in which it is applied in the present article.

The material that will be discussed here is organized as follows: In Section 2, some background material on Herrera's theory, which as mentioned constitutes a general framework for dG methods, is given. We felt this was necessary because this theory has remained, to a large extent, unnoticed and many readers are probably not familiar with it. In addition, a review paper is being prepared [27]. Section 3 is devoted to notations and preliminary notions that will be used in the sequel. In Section 4, the general indirect formulation of boundary value problems with jumps (BVPJ) of Herrera's theory is introduced. This is necessary because TH-collocation is an application of it. In Section 5, this formulation is applied to the general second-order elliptic equation in any number of dimensions. Section 6 is devoted to explain some general guidelines that are useful for the construction of families of optimal test functions that enjoy certain properties that are required by the method (*TH-complete families*). In Section 7, a detailed description of Herrera's general discretization procedures is given and then the application of orthogonal collocation for the construction of the *optimal test functions* and

optimal interpolation is explained in detail. A distinguishing, important and convenient feature of TH-collocation is that orthogonal collocation is always applied locally. More precisely, the optimal-test-function construction and the application of optimal interpolation require applying collocation in each one of the subdomains of the *fine* domain partitions, separately. Thus, these are simple operations in which the matrices involved are 4×4, for the 2-D problems treated in this article, and 1×1, for the *single-collocation-point methods* of [28]. Furthermore, it must be pointed out that 1×1 matrices are obtained in the *TH-single-collocation-point* methods, independently of the number of dimensions of the problem.[1] The DDM that is applied is explained in Section 8. As previously mentioned, this is a kind of substructuring procedure, but it has the conspicuous feature that the *Schur complement* is constructed in a simple and elegant manner. At present, the authors are researching its extension to more general problems. Sections 9 to 11 are devoted to explain and discuss the application of CGM, the parallel implementation of DDM and the numerical results obtained. We also include a final section of conclusions.

2. SOME BACKGROUND MATERIAL ON HERRERA'S THEORY

This theory has been developed over a long time span. Firstly, it was introduced as an algebraic theory of boundary value problems (BVP),[2] and in this form it was capable of supplying a very general framework, which accommodated practically all variational principles for BVP known at the time. It also encompassed boundary methods and biorthogonal systems of functions [6], as well as a criterion for completeness, presently known as TH-completeness and introduced by Herrera in 1980 with another name, that has been quite useful. In particular, Begehr and Gilbert in their book [30] that is devoted mainly to the construction of solutions to differential equations and/or systems using analytical methods, based their presentation of "complete (function) families for elliptic equations" (pp. 108–137) on this criterion. Furthermore, as a main conclusion of the corresponding chapter, one can read (p. 115):

> The function theoretic approach which was pioneered by Bergman [31] and Vekua [32] and then further developed by Colton [33–35], Gilbert [36, 37], Kracht-Kreyszig [38], Lanckau [39] and others, may now be effectively applied because of this result of the formulation by Herrera [1] as an effective means to solving boundary value problems.

A summary of the results obtained in that stage of the theory development is contained in Herrera [1], which was published in the Advanced Publishing Program of Pitman, London in 1984.

In 1985, a new kind of Green's formulas for "operators in discontinuous fields" was published in series of articles [10, 11] that appeared in the very first volume of *Numerical Methods for Partial Differential Equations*. These Green's formulas, which will be referred to as Green-Herrera formulas, constitute the backbone on which a very systematic and general formulation of *discontinuous Galerkin methods* is based. First, the boundary value problems are formulated in function spaces whose members are fully discontinuous (by 'fully' we mean that the functions themselves, and not only their derivatives, may be discontinuous across the *internal boundary*, which separates the subdomains of the domain partition). In order to have

[1] In passing, we mention that a different kind of single-collocation-point methods have been developed and applied to transport diffusion equations, by other authors [29], but their discussion is beyond the scope of the present article.

[2] By invitation, this theory was presented at a special session on Green's formulas and abstract adjoints at the 23rd Annual Meeting of the American Mathematical Society, at Denver, Colorado, in January, 1983.

well-posed problems in this setting, it is necessary to consider a generalized version of boundary value problems: boundary value problems with prescribed jumps (BVPJ). These are problems in which the usual boundary conditions that are prescribed at the external boundary are complemented with certain *jump conditions*, at the *internal boundary*. Generally, when the jumps are prescribed to be zero, the solution of such problem reduces to the solution of the standard BVP. For example, it has been shown [27, 40] that when considering elliptic equations of second order with continuous coefficients, a well-posed BVPJ is obtained if the jump of the function and its first normal derivative are prescribed across the internal boundary. More generally, for elliptic equations of order $2n$ the jumps of the normal derivatives up to order $2n - 1$ must be prescribed. In the particular case when the imposed jumps vanish, one recovers the solution of the corresponding standard boundary value problem (BVP).

A straightforward application of Green-Herrera formulas to such problems yields two equivalent weak formulations of the BVPJ: a weak formulation in terms of the data, which except for the fact that jumps are included, does not differ in an essential manner from other standard formulations, and a weak formulation in terms of the complementary information. This latter formulation, however, is nonstandard and plays an important role in the theory that is explained next.

By complementary information it is meant any information, about the solution of the BVPJ, that is not prescribed as data of the problem. Then, it must be observed that when the method of weighted residuals is applied, the complementary information that is contained in the approximate one is determined exclusively by the family of test functions that is used [10, 11, 41]. The usefulness of the weak formulation in terms of the complementary information stems from the fact that it constitutes a very effective tool for the analysis of the information supplied by different families of test functions. Based on these ideas, when Herrera's theory was in an earlier stage of its development, several numerical methods were introduced. In particular, the Localized Adjoint Method (LAM) [8] and its application to advection dominated transport (ELLAM) [9]; also, very general versions of the Trefftz Method, in whose development Herrera has played a leading role, widely recognized by that community (see, for example [42, 43]).

As for domain decomposition methods, in Herrera's theory they are interpreted as procedures for obtaining information about the sought solution at the internal boundary, exclusively. To this end, a target of information is defined there, which should be sufficient for defining well-posed local problems in each one of the subdomains of the partition, separately. Such a target is referred to as the sought information and in many applications getting the sought information is the final objective of the numerical procedure. This is similar to what one does in finite difference methods, where usually one is satisfied with obtaining the value of the solution at the nodes. However, once this sought information has been obtained, if the solution in the interior of the partition subdomains is desired, it can be obtained by solving the well-posed local problems that were previously mentioned. The process of solving these local problems is known as optimal interpolation. There are two general procedures that can be followed for gathering the sought information, which are referred to as direct (or Trefftz-Jirousek) and indirect (or Trefftz-Herrera) methods [4, 17].

The title Steklov-Poincare method could also be appropriate for the direct approach. To see this, observe that direct methods piece together the local solutions of the differential equations to build the global solution. However, in Herrera's theory the local solutions are used to establish compatibility conditions from which the required information on the internal boundary is derived. In fact, such methods can be based on a rather general formulation of Poincare-Steklov relations recently presented (see [17]).

As for the indirect approach to DDM, a systematic presentation of Herrera's indirect approach to domain decomposition was given in [14, 16]. The distinguishing feature of such methods is the use of families of specialized test functions, optimal test functions, that yield the sought information exclusively. Using the weak formulation in terms of the sought information as an effective tool for the analysis of the information supplied by different sets of test functions, one identifies the conditions that the test functions must satisfy in order to yield the target of information that was defined beforehand. In a sense, one designs the family of optimal test functions that must be applied in order to obtain the sought information.

In [13, 14], Herrera's indirect approach was applied to develop a new collocation method (TH-collocation) for elliptic differential equations, which yields symmetric positive definite matrices when the underlying differential operator is symmetric and positive. Essentially what was done in those articles was to apply the indirect approach, just as it has been explained, and then develop the local test functions by orthogonal collocation. In this article, our interest focuses in profiting from the enhanced properties exhibited by TH-collocation with respect to other collocation procedures, to obtain an effective method of combining collocation and domain decomposition. It must be noticed that using Herrera's indirect approach, other discretization methods can be derived, besides TH-collocation, if the approximate optimal test functions are constructed using other numerical procedures, instead of orthogonal collocation. Probably some of them are worth studying and researching, but that is beyond the scope of the present article.

3. PRELIMINARY NOTIONS AND NOTATIONS

In the general setting of Herrera's theory, trial and test functions are taken from different function-spaces. However, in this article, where only applications to symmetric operators will be considered, such generality is not necessary. Therefore, here, trial and test functions will be taken from the same linear space of functions, denoted by D. Functional-valued operators of the form $P : D \to D^*$, which are linear, will be considered, where D^* is the algebraic dual of D; i.e., D^* is the linear space whose elements are the real-valued linear functionals defined on D. Unless otherwise explicitly stated, no additional structure is assumed. Given $f \in D^*$, its value $f(v)$ at any $v \in D$ is denoted by $\langle f, v\rangle$, while $Pu \in D^*$ will be the value of $P : D \to D^*$ at any $u \in D$. Therefore, given $P : D \to D^*$, the expression $\langle Pu, v\rangle$ defines a unique bilinear functional on $D \times D$ and this establishes a one-to-one correspondence between the class of linear functional-valued operators considered and such bilinear functionals. In particular, given $P : D \to D^*$, the notation P^* is adopted for the operator $P^* : D \to D^*$ defined by

$$\langle P^*u, w\rangle \equiv \langle Pw, u\rangle, \qquad \forall (u, w) \in D \times D, \tag{3.1}$$

i.e., P^*, as a bilinear form, is the transpose of P. Given $f \in D^*$ and $u \in D$, the equation $Pu = f$ is an equality between linear functionals and, as such, it is tantamount to

$$\langle Pu, w\rangle = \langle f, w\rangle \qquad \forall w \in D. \tag{3.2}$$

The notations $\Omega \subset R^n$ and $\partial\Omega$ will be used for a domain of the Euclidean space of dimension n and its boundary, respectively. In the first part of our discussion, n can be any natural number, but starting with Section 4, n is taken to be equal to 2. Following Ciarlet [44], by a domain is

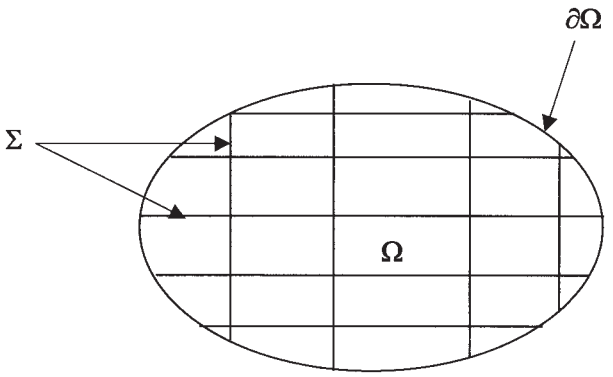

FIG. 1. Domain partition.

understood to be an open bounded, connected subset with a Lipschitz-continuous boundary. Let $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$ be a partition of Ω, where Ω_i, $i = 1, \ldots, E$, are subdomains (Fig. 1). Given such a partition, the boundaries of the subdomains are $\partial\Omega_i$, $i = 1, \ldots, E$. Clearly, $\partial\Omega \subset \bigcup_{i=1}^{E} \partial\Omega_i$ and the internal boundary, Σ, is defined to be the closed complement of $\partial\Omega$ relative to $\bigcup_{i=1}^{E} \partial\Omega_i$. It is assumed that for each $i = 1, \ldots, E$, there is a linear space $D(\Omega_i)$, whose elements are functions defined in Ω_i. Then the linear space D is defined to be

$$D \equiv D(\Omega) \equiv D(\Omega_1) \oplus \cdots \oplus D(\Omega_E). \tag{3.3}$$

In fact, when the space D is defined by Eq. (3.3), a function $u \in D$ is a finite sequence of functions $u \equiv (u_1, \ldots, u_E)$ such that $u_i \in D(\Omega_i)$, $i = 1, \ldots, E$. It is assumed that the trace of every $u_i \in D(\Omega_i)$ is defined at every point of $\partial\Omega_i$, except for, possibly, a set of measure zero. At any point of Σ, again with the possible exception of a set of measure zero, there is defined a unit normal vector $\underline{n}$ and the manifold Σ is oriented in this manner, taking as positive the side to which $\underline{n}$ points. Given a function $u \in D$, $u \equiv (u_1, \ldots, u_E)$, two traces are defined at every point of Σ, which are denoted by u_+ and u_-, respectively. Generally, $u_+ \neq u_-$ and the jump and the average of any function $u \in D$ are defined to be

$$[u] = u_+ - u_- \qquad \text{and} \qquad \dot{\bar{u}} = (u_+ + u_-)/2, \tag{3.4}$$

respectively.

Especially important for our discussions is a class of Sobolev spaces of the form introduced in Eq. (3.3), defined by

$$\hat{H}^s(\Omega) \equiv H^s(\Omega_1) \oplus \cdots \oplus H^s(\Omega_E). \tag{3.5}$$

When $\hat{H}^s(\Omega)$ is equipped with the norm

$$\|\hat{v}\|_{s,\Omega,\Pi} \equiv \left(\sum_{\alpha=1}^{E} \|v_\alpha\|_{s,\Omega_\alpha}^2 \right)^{1/2}, \tag{3.6}$$

defined for every $\hat{v} \equiv (v_1, \ldots, v_E) \in \hat{H}^s(\Omega)$, it becomes a Hilbert space. In Section 5, the most general elliptic operator of second order that is formally symmetric,

$$\mathcal{L}u \equiv -\nabla \cdot (\underline{\underline{a}} \cdot \nabla u) + cu \tag{3.7}$$

will be considered. It is assumed that $\underline{\underline{a}}$ is a symmetric tensor. When dealing with such elliptic equations of second order, it is convenient to take $s = 2$. In some previous works, for simplicity, we have written $\mathcal{L}u = f_\Omega$, in Ω, to mean

$$\mathcal{L}u = f_\Omega, \qquad \text{at each } \Omega_i,\ i = 1, \ldots, E. \tag{3.8}$$

However, for greater clarity, the more explicit notation of Eq. (3.8) will be used in what follows. Similarly, we also write $\sum_{i=1}^{E} \int_{\Omega_i} w\mathcal{L}u dx$ instead of $\int_\Omega w\mathcal{L}u dx$, since generally $w\mathcal{L}u$ is not defined on Σ when $u, w \in \hat{H}^2(\Omega)$. On the other hand, we adhere to the notation $\underline{a}_n \equiv \underline{\underline{a}} \cdot \underline{n}$. It can be shown that

$$\sum_{i=1}^{E} \int_{\partial\Omega_i} w\underline{a}_n \cdot \nabla u dx = \int_{\partial\Omega} w\underline{a}_n \cdot \nabla u dx - \int_{\Sigma} [w\underline{a}_n \cdot \nabla u] dx, \tag{3.9}$$

even if the coefficients of $\mathcal{L}$ have jump discontinuities across Σ. Above, the unit normal vector has been taken as explained before, on Σ, while it is taken pointing outwards from Ω and Ω_i, on $\partial\Omega$ and $\partial\Omega_i$, respectively.

4. INDIRECT FORMULATION OF THE BVPJ

The description of Herrera's indirect approach to the BVPJ, in purely algebraic terms, is extremely simple. Let $P : D \to D^*$, $Q : D \to D^*$, $B : D \to D^*$, $C : D \to D^*$, $J : D \to D^*$, and $K : D \to D^*$ be such that they fulfill

$$P - B - J = Q^* - C^* - K^*. \tag{4.1}$$

This equation will be referred to as the Green-Herrera formula. Furthermore, let f, g, and j be linear functionals; i.e., $f \in D^*$, $g \in D^*$, and $j \in D^*$. Explicit formulas for such linear and bilinear functionals in the case of the second-order elliptic equations are given in Section 5 of this article. Assume further that the equation

$$(P - B - J)u = f - g - j \tag{4.2}$$

is a weak formulation of the boundary value problem with prescribed jumps. Then, by virtue of Eq. (4.1), the equality

$$(Q^* - C^* - K^*)u = f - g - j \tag{4.3}$$

is an equivalent weak formulation of the BVPJ. In general, in applications such as those of Section 5, it is shown that if $u \in D$ is a solution of the BVPJ, then Q^*u, C^*u, and K^*u contain complementary information about the solution in the interior of the subdomains of the partition, the outer boundary and the internal boundary, respectively. This motivates that Eq. (4.3) be referred as the 'weak formulation in terms of the complementary information', while Eq. (4.2) is referred as the 'weak formulation in terms of the data'.

As mentioned in Section 2, in Herrera's indirect formulation one chooses a target of information, the sought information, which usually is required to be sufficient for defining well-posed local problems, and then one constructs optimal test functions, which have the property of yielding such information exclusively. Assume that the sought information is constituted by information about the solution on the internal boundary exclusively, as will be the case in the applications to be made in the present article. Then in order to achieve the proposed goal, one must choose test functions that eliminate the information Q^*u, in the interior of the partition subdomains and the information C^*u, in the outer boundary. That is, the optimal test functions $w \in D$ must have the property $\langle Q^*u, w\rangle = 0$ and $\langle C^*u, w\rangle = 0$. These conditions are satisfied independently of the unknown $u \in D$, if and only if $Qw = 0$ and $Cw = 0$, since $\langle Q^*u, w\rangle \equiv \langle Qw, u\rangle$ and $\langle C^*u, w\rangle \equiv \langle Cw, u\rangle$. When the weak formulation in terms of the complementary information of Eq. (4.3) is applied using test functions that fulfill these conditions, one obtains

$$-\langle K^*u, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in N_Q \cap N_C. \tag{4.4}$$

Here, we have written $N_Q \subset D$ and $N_C \subset D$ for the null subspaces of Q and C, respectively, and in what follows corresponding notations will be used for other operators.

Usually, the information contained in K^*u is more than just the sought information, and it is convenient to eliminate the excess of information. Assume the operator $S : D \to D^*$ is such that S^*u contains precisely the sought information. In Herrera's indirect formulation, in order to transform Eq. (4.4) into one that contains the sought information exclusively, one writes the operator $K^* : D \to D^*$ as a sum

$$K^* = S^* + R^*. \tag{4.5}$$

Here the definition $R \equiv K - S$ has been adopted. Then it is seen that Eq. (4.4) implies

$$-\langle S^*u, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in N_Q \cap N_C \cap N_R. \tag{4.6}$$

If $u \in D$ is the solution of the BVPJ, the linear functional R^*u, which has been eliminated, is referred to as the residual information and S^*u as the sought information. In addition, we say that any trial function $\hat{u} \in D$ such that $S^*\hat{u} = S^*u$ contains the sought information. Furthermore, in view of Eq. (4.6), functions $w \in N_Q \cap N_C \cap N_R$ are the optimal test functions we were seeking, since they possess the desired property of yielding the sought information exclusively.

Another useful concept is that of TH-completeness. As mentioned in Section 2, Herrera in 1980 [5], originally introduced it, with another name. Later this concept has been modified slightly, to a form better suited for the general theory formulation:

"A set $\mathscr{C} \subset D$ is said to be TH-complete for an operator $S^* : D \to D^*$, when for any $v \in D$, on has

$$\langle S^* v, w\rangle = 0, \qquad \forall w \in \mathscr{E} \Rightarrow S^* v = 0" \tag{4.7}$$

Using these concepts and Eq. (4.6), the following characterization of the sought information follows:
Assume a family of optimal test functions, $\mathscr{E} \subset D$, is TH-complete for S^*. Then a necessary and sufficient for any $\hat{u} \in D$ to contain the sought information is that

$$-\langle S^* \hat{u}, w\rangle = \langle f - g - j, w\rangle, \qquad \forall w \in \mathscr{E} \tag{4.8}$$

Proof. If $S^*\hat{u} = S^*u$, then Eq. (4.6) implies Eq. (4.8) since $\mathscr{E} \subset N_Q \cap N_C \cap N_R$. Conversely, assume Eq. (4.8). Then, subtracting Eqs. (4.6) and (4.8), it is obtained

$$\langle S^*(\hat{u} - u), w\rangle = 0, \qquad \forall w \in \mathscr{E}. \tag{4.9}$$

This implies $S^*\hat{u} = S^*u$, since $\mathscr{E} \subset N_Q \cap N_C \cap N_R$ is TH-complete for S^*.

It is important to make some comments. In the above discussions two characterizations have given: that of the sought information and that of the optimal test functions. They, together, constitute the basis of Herrera's indirect approach to domain decomposition. First, the fact that $w \in D$ is an optimal test function if and only $w \in N_Q \cap N_C \cap N_R$ constitutes the basic tool for designing them. In addition, the characterization of the sought information, given by Eq. (4.7), will be used to derive the global system of equations, from which the sought information will be obtained in the specific applications. By assumption, the sought information refers exclusively to features of the solution of the BVPJ on the internal boundary. However, once the sought information is known, if desired, one can apply optimal interpolation to obtaining the solution in the interior of the subdomains partition. Here, as it was mentioned before, optimal interpolation is the process of solving the well-posed local problems defined by the sought information and the data of the BVPJ. How to bring this down to earth and transform the theory into a useful numerical tool is explained in the following sections, in connection with second order elliptic problems.

5. INDIRECT FORMULATION OF SECOND-ORDER ELLIPTIC EQUATIONS

Let be $D = \hat{H}^2(\Omega) \equiv H^2(\Omega_1) \oplus \ldots \oplus H^2(\Omega_E)$. Define, for every $v \equiv (v_1, \ldots, v_E) \in D = \hat{H}(\Omega)$, the symmetric differential operator

$$\mathscr{L} v \equiv -\nabla \cdot (\underline{a} \cdot \nabla v_i) + c v_i, \qquad \text{in } \Omega_i,\ i = 1, \ldots, E, \tag{5.1}$$

and, for every $(v, w) \in D \times D$, the bilinear functionals

$$\langle P v, w\rangle \equiv \sum_{i=1}^{E} \int_{\Omega_i} w \mathscr{L} v dx; \qquad \langle Q^* v, w\rangle \equiv \sum_{i=1}^{E} \int_{\Omega_i} v \mathscr{L}^* w dx; \tag{5.2}$$

$$\langle B v, w\rangle \equiv \int_{\partial\Omega} \mathscr{B}(v, w) dx; \qquad \langle C^* v, w\rangle \equiv \int_{\partial\Omega} \mathscr{C}^*(v, w) dx; \tag{5.3}$$

$$\langle Jv, w\rangle \equiv \int_\Sigma \mathcal{J}(v, w)dx; \qquad \langle K^*v, w\rangle \equiv \int_\Sigma \mathcal{H}^*(v, w)dx. \tag{5.4}$$

Here, $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$; i.e., $\mathcal{L}^* = \mathcal{L}$, since $\mathcal{L}$ is a symmetric operator. In addition,

$$\mathcal{B}(v, w) \equiv v\underline{a}_n \cdot \nabla w \equiv \mathcal{C}(v, w) \qquad \text{and} \qquad \mathcal{J}(u, w) \equiv -[u]\widehat{\underline{a}_n \cdot \nabla w} + \dot{\hat{w}}[\underline{a}_n \cdot \nabla u] = \mathcal{H}(u, w). \tag{5.5}$$

With these definitions the Green-Herrera formula of Eq. (4.1) holds (see [14]). Furthermore, define

$$\mathcal{S}^*(u, w) \equiv \mathcal{S}(w, u) \equiv \dot{\hat{u}}[\underline{a}_n \cdot \nabla w] \qquad \text{and} \qquad \mathcal{R}^*(u, w) \equiv \mathcal{R}(w, u) \equiv -[w]\widehat{\underline{a}_n \cdot \nabla u} \tag{5.6}$$

together with

$$\langle S^*u, w\rangle \equiv \int_\Sigma \mathcal{S}^*(u, w)dx \qquad \text{and} \qquad \langle R^*u, w\rangle \equiv \int_\Sigma \mathcal{R}^*(u, w)dx. \tag{5.7}$$

Then Eq. (4.5), is fulfilled.

Let $f_\Omega \in H^0(\Omega)$, $g_\partial \in H^0(\partial\Omega)$, $j^0_\Sigma \in H^0(\Sigma)$ and $j^1_\Sigma \in H^0(\Sigma)$ be given functions. Then, define $g \in D^*$ and $j \equiv j^0 + j^1 \in D^*$ for every $w \in D$ by

$$\langle f, w\rangle \equiv \int_\Omega w f_\Omega dx, \qquad \langle g, w\rangle \equiv \int_{\partial\Omega} g_\partial(\underline{a}_n \cdot \nabla w)dx \tag{5.8}$$

$$\langle j^0, w\rangle \equiv -\int_\Sigma j^o_\Sigma(\widehat{\underline{a}_n \cdot \nabla w})dx; \qquad \langle j^1, w\rangle \equiv \int_\Sigma \dot{\hat{w}} j^1_\Sigma dx. \tag{5.9}$$

In this case, Eq. (4.2) is a weak formulation in terms of the data of the following BVPJ:

$$-\nabla \cdot (\underline{a} \cdot \nabla u) + cu = f_\Omega, \qquad \text{in } \Omega_i,\ i = 1, \ldots, E \tag{5.10}$$

$$u = g_\partial, \qquad \text{in } \partial\Omega \tag{5.11}$$

and

$$[u] = j^o_\Sigma \qquad \text{and} \qquad [\underline{a}_n \cdot \nabla u] = j^1_\Sigma, \qquad \text{on } \Sigma. \tag{5.12}$$

General conditions for the existence of a unique solution of the classical version of this problem ($j^o_\Sigma = j^1_\Sigma = 0$) are well known [45]. When such conditions are satisfied, then the same

is granted for the BVPJ of Eqs. (5.10) to (5.12), under the sole assumption that there exists a function $u_{\partial\Sigma} \in D$ fulfilling Eqs. (5.11) and (5.12). When such a function exists, the boundary and jump conditions are said to be compatible.

For the linear subspace of optimal test functions the notation $N \equiv N_Q \cap N_C \cap N_R \subset D \equiv \hat{H}^2(\Omega)$ is adopted. With the above definitions a function $w \in N \subset D \equiv \hat{H}^2(\Omega)$ is an optimal test function if and only if

$$-\nabla \cdot (\underline{a} \cdot \nabla w) + cw = 0, \qquad \text{in } \Omega_i,\ i = 1, \ldots, E;\ \text{i.e., } w \in N_Q \tag{5.13}$$

$$w = 0, \quad \text{on } \partial\Omega \qquad \text{and} \qquad [w] = 0, \quad \text{on } \Sigma; \qquad \text{i.e., } w \in N_C \cap N_R. \tag{5.14}$$

This last zero jump condition means that optimal test functions are continuous in Ω with possibly discontinuous normal derivatives across Σ.

Furthermore, a function $\hat{u} \in D$ contains the sought information (i.e., $S^*\hat{u} = S^*u$, where $u \in D$ is solution of the BVPJ), if and only if its average across Σ equals that of the solution of the BVPJ (i.e., $\dot{u} = \dot{\hat{u}}$, on Σ). Furthermore, the characterization of the sought information of Eq. (4.8) implies that, when $\hat{u} \in D$ and a family of optimal test functions $\mathscr{C} \subset N$ is TH-complete (for $S^* : D \to D^*$), the condition

$$-\langle S^*\hat{u}, w\rangle \equiv -\int_\Sigma \dot{\hat{u}}[\underline{a}_n \cdot \nabla w]dx = \langle f - g - j, w\rangle, \tag{5.15}$$

is fulfilled if and only if $\dot{u} = \dot{\hat{u}}$ on Σ. In this connection it is important to notice the relation

$$-\langle S^*v, w\rangle \equiv -\int_\Sigma v[\underline{a}_n \cdot \nabla w]dx = \sum_{i=1}^{E} \int_{\Omega_i} \{\nabla v \cdot \underline{\underline{a}} \cdot \nabla w + cvw\}dx, \qquad \forall (v, w) \in N \times N. \tag{5.16}$$

The last expression in the right-hand side of Eq. (5.16) has been used in many applications such as finite element methods (FEM) and DDM. Observe that, when $c \geq 0$, the bilinear form S^* is symmetric and positive definite on $N \times N$. This is a very important property. Due to it, TH-collocation yields algorithms for which the global matrices are symmetric and positive definite. When TH-collocation is used, this will clear the way for combining orthogonal collocation with DDM. Thus, this overcomes the complications encountered with other collocation formulations. However, in order to profit from such property of the bilinear functional S^*, some further elaboration is required and that will be done in the following sections. There, we explain procedures for, using Eqs. (5.15) or (5.16), deriving the average across Σ of the sought solution of the BVPJ. Once such an average has been obtained, one can reconstruct the solution everywhere in Ω by optimal interpolation, which consists in solving well-posed local problems exclusively in each one of the partition subdomains. In order to define such problems use is made of the identities:

$$u_+ = \hat{\bar{u}} + \tfrac{1}{2}[u] = \hat{\bar{u}} + \tfrac{1}{2}j^0_\Sigma \qquad \text{and} \qquad u_- = \hat{\bar{u}} - \tfrac{1}{2}[u] = \hat{\bar{u}} - \tfrac{1}{2}j^0_\Sigma, \text{ on } \Sigma. \tag{5.17}$$

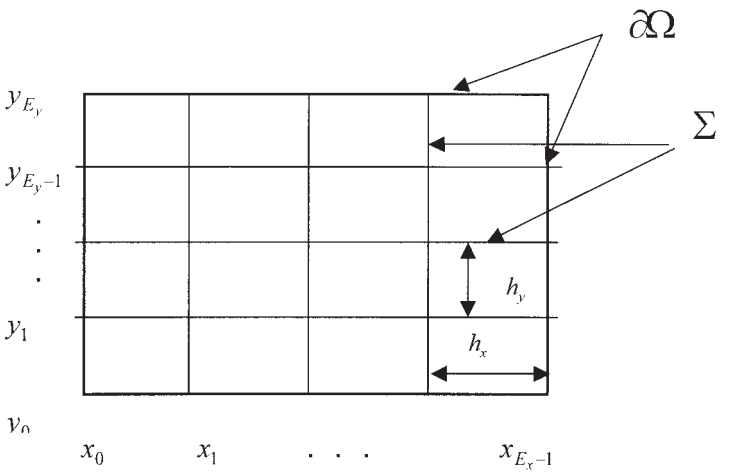

FIG. 2. Rectangular domain and its partition.

6. ON OPTIMAL TEST FUNCTIONS

For the elliptic equations that are being considered, a first observation is that optimal test functions $w \in N_Q \cap N_C \cap N_R \subset \hat{H}(\Omega)$ are uniquely determined by their traces on the internal boundary Σ. This is because Eqs. (5.13) to (5.14) together with the traces on the internal boundary, define well-posed problems in each one of the subdomains of the partition (Figs. 2 and 3). The situation is very similar to the case of optimal interpolation that was explained at the end of the last section. This establishes a one-to-one correspondence between optimal test functions and their traces. Using this bijection, it is advantageous to define the family of test functions to be applied by choosing a set of traces on Σ, as is done in what follows. Once the system of traces has been chosen, all that is required for constructing the optimal test functions is to solve the corresponding local boundary value problems in each one of the subdomains of the partition, separately. Furthermore, in the development of the theory it is more convenient to work, at first, with the exact optimal test functions. So, in the discussions of this section and the first part of the following one, it is assumed that the optimal test functions are the exact ones.

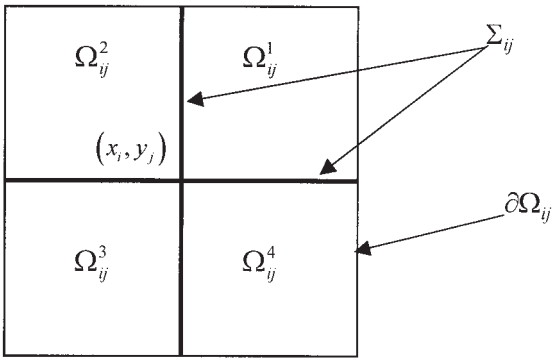

FIG. 3. Subdomains Ω_{ij}^{α}, $\alpha = 1, \ldots, 4$, associated with Ω_{ij}.

However, in the applications they generally have to be numerically approximated, and the corresponding discussion is given in the following section.

In one-dimensional problems the internal boundary consists of a finite set of points and the space $w \in N_Q \cap N_C \cap N_R \subset \hat{H}^2(\Omega)$ is finite dimensional [12, 13]. However, for problems in more than one independent variable the traces space dimension is infinite. Thus, for problems in several independent variables the space of optimal test functions, $N_Q \cap N_C \cap N_R \subset \hat{H}^2(\Omega)$, is infinite-dimensional and TH-complete systems of optimal test functions are also nonfinite sets. Then, a TH-complete system of optimal test functions is obtained by requiring that the set of traces on Σ be dense in $H^0(\Sigma)$. In previous applications, such a set has been chosen to be the set of piecewise polynomials, of all degrees, continuous on Σ, since optimal test functions have to be continuous, as has been previously shown.

In addition, when developing the TH-collocation method here explained, it is convenient to apply optimal test functions with local support; i.e., test functions whose support is contained in the union of a few subdomains of the partition. For example, for the rectangular domain with a rectangular partition illustrated in Fig. 3, one can develop a system of optimal test functions whose supports consist of the union of at most four rectangular partition subdomains. This feature of the optimal test functions yields an important property of the algorithms derived in this manner: the global matrix is sparse. In the case of 1-D problems the test functions supports consist of two subintervals of the partition and the global matrices are three-diagonal [12, 13].

In this article, as in [14], for two-dimensional problems, two algorithms are presented. For one of them (Algorithm 1) the traces are the linear manifold of $C^0(\Sigma)$ spanned by the piecewise linear polynomials and in the other one (Algorithm 2) they are the linear subspace of $C^1(\Sigma)$ spanned by the piecewise cubic polynomials. In these cases, the global matrices are block-nine-diagonal, with 1×1 and 3×3 blocks, respectively. Of course, the families of optimal test functions applied in each one of these algorithms are finite and the replacement of a TH-complete system of optimal test functions by a finite set implies a truncation error, even when the optimal test functions members of such set are the exact ones. In particular, when the *exact* optimal test functions are applied, Algorithm 1 is $O(h^2)$, while Algorithm 2 is $O(h^4)$.

As mentioned in the Introduction, this manner of tackling the problem permits separating thoroughly the process of approximating the differential equation in the interior of the partition subdomains, from that of approximating the solution in the internal boundary, Σ. When one constructs approximate optimal test functions by collocation, if desired, one can develop very accurate optimal test functions by increasing the number of collocation points in each partition subdomain, without modifying the degree of the polynomials used on Σ. Furthermore, one can decrease the accuracy of the optimal test function approximation by decreasing the number of collocation points used in its construction. However, the number of collocation points that can be applied cannot be reduced beyond a certain minimum, because the accuracy of the approximation of the optimal test functions must be consistent with the accuracy of the approximation implied by the traces used on Σ. In particular, going back to the algorithms that are applied in the present article, this minimum is 1, for Algorithm 1, and it is 4, for Algorithm 2. In the present article we used 4 collocation points for both of them. Also, for Algorithm 1, we used cubic polynomials in the interior of the partition subdomains, in spite of having used linear polynomials on the internal boundary, Σ.

7. TH-DISCRETIZATION AND TH-COLLOCATION

By TH-discretization we mean the discretization method that is derived from the indirect formulation of Section 4, when the exact optimal test functions are used and the exact solutions of the local problems are obtained when performing the optimal interpolation. Our interest in TH-discretization is mainly theoretical, since in most applications approximate optimal test functions are used and approximate optimal interpolation is performed. However, depending on the procedure used for carrying out the optimal interpolation and the optimal-test-functions approximation processes, from TH-discretization different numerical methods can be derived. TH-collocation is the discretization procedure that is obtained when the numerical approximation used is orthogonal collocation.

A. TH-Discretization

In this subsection, except for the fact that finite families will approximate TH-complete systems, all the developments will be exact. In particular, recalling Eqs. (5.13) and (5.14), the optimal test functions are assumed to exactly satisfy

$$-\nabla \cdot (\underline{a} \cdot \nabla w) + cw = 0, \qquad \text{in } \Omega_i,\ i = 1, \ldots, E \tag{7.1}$$

$$w = 0, \quad \text{on } \partial\Omega \qquad \text{and} \qquad [w] = 0 \quad \text{on } \Sigma. \tag{7.2}$$

In what follows, the developments are explained for the case of a rectangular domain Ω, and a rectangular partition (Figs. 2 and 3). Then one can associate with each internal node (x_i, y_j), four rectangles $\{\Omega_{ij}^1, \ldots, \Omega_{ij}^4\}$ and the notations Ω_{ij}, $\partial\Omega_{ij}$, and Σ_{ij} are adopted for the interior of the union of the four rectangle closures, the boundary of Ω_{ij}, and the intersection $\Sigma \cap \Omega_{ij}$, respectively.

The Optimal Test Functions of Algorithm 1 As mentioned previously, two TH-collocation algorithms are applied in this article. For Algorithm 1, the manifold of traces is the linear subspace of $C^0(\Sigma)$, whose members are piecewise linear and vanish on $\Sigma \cap \partial\Omega$. Then, we associate with each internal node (x_i, y_j), a unique optimal test-function, w_{ij}. This is uniquely defined by the following conditions: its support is contained in $\bar{\Omega}_{ij}$, the closure of Ω_{ij}, and its trace on Σ_{ij} is given by

$$w_{ij}(x_i, y_j) = \left|\frac{x - x_{i+1}}{x_i - x_{i+1}}\right| \left|\frac{y - y_{j+1}}{y_j - y_{j+1}}\right|, \qquad \text{on } \Sigma_{ij} \tag{7.3}$$

In view of Eqs. (7.1) and (7.2), w_{ij} is the unique function belonging to $\hat{H}^2(\Omega) \cap H^1(\Omega)$, with support contained in $\bar{\Omega}_{ij}$, whose trace on Σ_{ij} fulfills Eq. (7.3) and in addition satisfies the following four local differential equations

$$-\nabla \cdot (\underline{a} \cdot \nabla w_{ij}) + cw_{ij} = 0, \qquad \text{in } \Omega_{ij}^{\alpha}, \quad \alpha = 1, \ldots, 4. \tag{7.4}$$

For later use, we observe that the condition $w_{ij} \in \hat{H}^2(\Omega) \cap H^1(\Omega)$ implies that w_{ij} is continuous across Σ. Therefore,

$$w_{ij} = 0, \qquad \text{on } \partial\Omega_{ij} \tag{7.5}$$

since w_{ij} vanishes identically outside $\bar{\Omega}_{ij}$ and, also, it vanishes on $\partial\Omega$. Due to the fact that the local boundary value problems defined by Eqs. (7.3) to (7.5) have a unique solution, in this manner we have associated a unique optimal test function with each internal node. Furthermore, these four local boundary value problems, separately defined in each one of the partition subdomains Ω_{ij}^{α}, $\alpha = 1, \ldots, 4$, permit the actual construction of the optimal test functions, by solving local problems exclusively. Finally, observe that the traces of the family of functions defined in this manner span the linear manifold of $C^0(\Sigma)$ functions, which are piecewise linear and vanish on the outer boundary of Ω.

The Optimal Test Functions of Algorithm 2 The procedure used for defining the family of optimal test functions applied in Algorithm 2 is explained next. Three optimal test functions, $\{w_{ij}^0, w_{ij}^1, w_{ij}^2\}$, are associated with each internal node (x_i, y_j). Each one of them fulfills the same conditions as the function w_{ij}, previously introduced, except for Eq. (7.3). This equation is replaced by

$$w_{ij}^0(x, y) = H_i^0(x)H_j^0(y),\ w_{ij}^1(x, y) = H_i^1(x)H_j^0(y),\ w_{ij}^2(x, y) = H_i^0(x)H_j^1(y) \tag{7.6}$$

to be fulfilled on Σ_{ij}. Here, the notations $H_i^0(x)$ and $H_i^1(x)$ are used for Hermite cubic polynomials. More precisely, they are C^1 piecewise cubic polynomials, with support in the interval (x_{i-1}, x_{i+1}), such that $H_i^0(x_i) = 1$ and $(dH_i^0/dx)(x_i) = 0$ together with $H_i^1(x_i) = 0$ and $(dH_i^1/dx)(x_i) = 1$.

However, for Algorithm 2, associated with boundary nodes some additional test functions will be needed. To construct them uniformly some auxiliary rectangles surrounding the domain Ω are added, in a manner that after doing this each boundary node (x_i, y_j) is surrounded by four rectangular subdomains Ω_{ij}^{α}, $\alpha = 1, \ldots, 4$. Figure 3 and the previous definitions yield three functions, $\{w_{ij}^0, w_{ij}^1, w_{ij}^2\}$.

The following remarks are relevant. All the functions that have been defined, associated with internal nodes, that are obtained in this manner are optimal test functions. However, some of the functions that are associated with boundary nodes, do not, because they do not satisfy the outer boundary condition of Eq. (7.2). When this is the case, they are *not* optimal test functions (i.e., they do not belong to $N_Q \cap N_C \cap N_R$) and must be eliminated. If (x_i, y_j) lies in a vertical boundary, only w_{ij}^1 survives. If (x_i, y_j) lies in a horizontal boundary, only w_{ij}^2 survives. All the test functions associated with corner nodes have to be eliminated. Then, all the remaining functions are indeed optimal test functions, and they altogether constitute the system of test functions of Algorithm 2. Observe that the traces on Σ of each one of the members of this system are $C^1(\Sigma)$. Furthermore, the family of traces on Σ associated with this system spans the linear manifold of piecewise cubic polynomials that are in $C^1(\Sigma)$. On the other hand the optimal test functions themselves are continuous in Ω, but their normal derivatives are generally discontinuous across Σ.

The Global System of Equations Before deriving the global system of equations, it is convenient to introduce an auxiliary function $u_0 \in D$, satisfying

$$u_0 = u = g \qquad \text{on } \partial\Omega \qquad [u_0]_\Sigma = [u]_\Sigma = j^0 \qquad \text{on } \Sigma$$

$$[\underline{a}_n \cdot \nabla u_0]_\Sigma = [\underline{a}_n \cdot \nabla u]_\Sigma = j^1 \qquad \text{on } \Sigma. \tag{7.7}$$

The construction of such function is possible when the boundary and jump conditions are compatible, which has been assumed. In addition, its numerical construction is not difficult to carry out because it does not need to satisfy any differential equation. Making use of such $u_0 \in D$, we define

$$s = u - u_0, \text{ in } \Omega \tag{7.8}$$

Then

$$\mathcal{L}s = f_\Omega - \mathcal{L}u_0, \text{ for } x \in \Omega_i, i = 1, \ldots, E \tag{7.9}$$

$$s = 0, \text{ on } \partial\Omega \qquad [s] = 0, \text{ on } \Sigma; \qquad [\underline{a}_n \cdot \nabla s] = 0, \text{ on } \Sigma \tag{7.10}$$

When these conditions are fulfilled, $Ps = f - Pu_0$, $Bs = 0$ and $Js = 0$. Then, Green-Herrera formula of Eq. (4.1) implies

$$(Q^* - C^* - K^*)s = (P - B - J)s = f - Pu_0 \tag{7.11}$$

and, therefore,

$$-\langle S^*s, w\rangle = \langle f, w\rangle - \langle Pu_0, w\rangle \qquad \forall w \in N \equiv N_Q \cap N_C \cap N_R \tag{7.12}$$

Since $K^* = S^* + R^*$.

Observe that $f_\Omega \in H^0(\Omega)$ and $\mathcal{L}u_0(x) \in H^0(\Omega)$, so that $s \in H^2(\Omega)$. In particular, the trace of s on Σ belongs to $C^1(\Sigma)$. This fact is numerically advantageous. Indeed, for Algorithm 2, it permits carrying out the numerical search for the trace of s, on the $C^1(\Sigma)$ space only, instead of the whole $C^0(\Sigma)$ space. This justifies the use of a system of optimal test functions whose traces on Σ span the linear manifold of piecewise cubic polynomials that are $C^1(\Sigma)$, instead of a larger system for which the corresponding span would have been the linear manifold of piecewise cubic polynomials that would have been $C^0(\Sigma)$. Indeed, the additional effort would have been wasteful. In the latter case the number of degrees of freedom associated with each internal node would have been 5, instead of 3 that were in the former.

An important auxiliary result is the following: "There exists $v \in N \subset \hat{H}^2(\Omega)$, such that $S^*v = S^*s$." This is because each one of the local boundary value problems:

$$\mathcal{L}v = 0, \quad \text{in } \Omega_i \qquad \text{subjected to } v = s, \text{ on } \partial\Omega_i \tag{7.13}$$

is well posed for $i = 1, \ldots, E$. Using this fact we search for the trace on Σ of such a function $v \in N$. It is characterized by

$$-\langle S^*v, w\rangle \equiv -\int_\Sigma v[\underline{a}_n \cdot \nabla w]ds = \sum_{i=1}^{E} \int_{\Omega_i} \{\nabla v \cdot \underline{\underline{a}} \cdot \nabla w + cvw\}dx$$

$$= \sum_{j=1}^{E} \int_{\Omega_j} w(f_\Omega - \mathcal{L}u_0)dx \qquad \forall w \in N \tag{7.14}$$

Here Eq. (5.16) has been applied. Also, in Eq. (7.14) the value of the function $v \in N$ on Σ has been used instead of its average because it is continuous across Σ.

In what follows, a system of (exact) optimal test functions, $\mathscr{C} \equiv \{\tilde{w}^1, \ldots, \tilde{w}^m\} \subset N \equiv N_Q \cap N_C \cap N_R$, will be considered. For either Algorithm 1 or Algorithm 2, this is assumed to be the system introduced before, after adjusting the index notation and reordering. The linear subspace spanned by $\mathscr{C}$ will be $\tilde{N} \subset N \equiv N_Q \cap N_C \cap N_R$. To obtain the TH-discretization method the linear subspace $N \subset D$ above is replaced by the subspace $\tilde{N} \subset N \subset D$, which is spanned by $\mathscr{C} \subset N$. Thus, the trace on Σ of the function $v \in N$ will be approximated by the trace of a function $\tilde{v} \in \tilde{N}$, whose general expression is

$$\tilde{v} = \sum_{\alpha=1}^{m} c_\alpha \tilde{w}^\alpha, \qquad \text{on } \Sigma. \tag{7.15}$$

Here, the coefficients $\underline{c} \equiv (c_1, \ldots, c_m)$ are determined by imposing the condition that Eq. (7.14) is fulfilled for every $\tilde{w} \in \mathscr{C}$. This condition yields

$$\underline{\underline{S}} \cdot \underline{c} = \underline{b}. \tag{7.16}$$

Here the elements of the $(m \times m)$ $-$ *matrix*, $\underline{\underline{S}}$, are given by

$$S_{\alpha\beta} \equiv -\langle S\tilde{w}^\alpha, \tilde{w}^\beta \rangle \equiv \sum_{i=1}^{E} \int_{\Omega_i} \{\nabla \tilde{w}^\alpha \cdot \underline{\underline{a}} \cdot \nabla \tilde{w}^\beta + c\tilde{w}^\alpha \tilde{w}^\beta\} dx, \tag{7.17}$$

while the m-vector $\underline{b} \equiv (b_1, \ldots, b_m)$ is

$$b_\alpha \equiv \sum_{i=1}^{E} \int_{\Omega_i} \tilde{w}^\alpha (f_\Omega - \mathscr{L} u_0) dx, \; \alpha = 1, \ldots, m. \tag{7.18}$$

Clearly, when the matrix $\underline{\underline{S}}$ is defined in this manner, it is symmetric and positive definite. This is what motivated our having reversed the sign when going from $\langle S\tilde{w}^\alpha, \tilde{w}^\beta \rangle$ to $S_{\alpha\beta}$, in Eq. (7.17).

B. TH-Collocation

The procedure explained in Subsection A is what we call TH-discretization. It uses the exact optimal test functions. However, generally it is not possible to obtain such exact optimal test functions and it is necessary to resort to numerical approximate methods for constructing them. The method of TH-collocation results when the optimal test functions are constructed as solutions of the local boundary value problems, defined by Eqs. (7.4), (7.5) and (7.3) or (7.6), using orthogonal collocation in each one of the partition subdomains, separately. In general, the number of collocation points used in each one of them can be varied. In the present article, four Gaussian points are used in each partition rectangle Figure 3 for both Algorithm 1 and 2. However, theoretical considerations indicate that the single-collocation-point methods of [28] are compatible with the accuracy of Algorithm 1.

All that remains to be explained is the construction of the three optimal test-functions. In the subdomain Ω_{ij}^{λ}, $\lambda = 1, \ldots, 4$, for Algorithm 1 they are given by

$$w_{ij}(x, y) = \left| \frac{x - x_{i+1}}{x_i - x_{i+1}} \right| \left| \frac{y - y_{j+1}}{y_j - y_{j+1}} \right| + \sum_{\beta=0}^{1} \sum_{\alpha=0}^{1} c_{ij}^{\lambda\alpha\beta} H_{i+\alpha}^{1}(x) H_{j+\beta}^{1}(y), \tag{7.19}$$

while, for Algorithm 2, the functions $w_{ij}^{\mu}(x, y)$, $\mu = 0, 1, 2$, are

$$w_{ij}^{\mu}(x, y) = W_{ij}^{\mu}(x, y) + \sum_{\beta=0}^{1} \sum_{\alpha=0}^{1} c_{ij\mu}^{\lambda\alpha\beta} H_{i+\alpha}^{1}(x) H_{j+\beta}^{1}(y). \tag{7.20}$$

Here the coefficients $c_{ij}^{\lambda\alpha\beta}$ and $c_{ij\mu}^{\lambda\alpha\beta}$ are determined by collocation of the differential equation (7.4) in the four Gaussian points of Ω_{ij}^{λ}, $\lambda = 1, \ldots, 4$. In addition, we have used the functions:

$$W_{ij}^{0}(x, y) \equiv H_i^0(x) H_j^0(y),\ W_{ij}^{1}(x, y) \equiv H_i^0(x) H_j^1(y),\ W_{ij}^{2}(x, y) \equiv H_i^1(x) H_j^0(y). \tag{7.21}$$

8. DOMAIN DECOMPOSITION OF TH-COLLOCATION

The theory that was used in the last section to develop TH-collocation will be applied in this one for deriving an efficient procedure for combining TH-collocation with a domain decomposition method. The approaches that are used to formulate discretization and domain decomposition methods are quite similar—the main difference being that the number of degrees of freedom associated with each partition subdomain is small for the former, while it is much larger for the latter.

As in Section 7, let $\mathscr{C} \equiv \{\tilde{w}^1, \ldots, \tilde{w}^m\} \subset N$ be a finite family of optimal test functions and $\tilde{N} \subset N$ the linear subspace spanned by it. In this section such functions are assumed to be the exact, but in the numerical applications the approximate optimal test functions built in Section 7, will be applied. The partition, $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$, with internal boundary Σ, that has been considered in previous sections, will be referred as the fine partition of Ω. In addition, another coarse partition $\Pi_G \equiv \{\tilde{\Omega}_1, \ldots, \tilde{\Omega}_{E_G}\}$, with internal boundary Σ_G, will be introduced. The characteristic feature of the coarse partition is that its subdomains are unions of families of the fine partition subdomains. More precisely, the coarse and fine partitions are related by the fact that

$$\tilde{\Omega}_i = \bigcup_{k=1}^{k=E_i} \Omega_{\gamma_i(k)}. \tag{8.1}$$

Here, for each $i = 1, \ldots, E_G$, γ_i is a suitable natural-number-valued function. In particular, the family $\{\Omega_{\gamma_i(1)}, \ldots, \Omega_{\gamma_i(E_i)}\}$ is a partition of $\tilde{\Omega}_i$ and $E = \sum_{i=1}^{E_G} E_i$. Let $\tilde{N} \subset N$ be the subspace that was introduced in Section 7. Then, for each $i = 1, \ldots, E_G$, the linear subspaces $\tilde{N}_{0i} \subset \tilde{N}$ are defined by

$$\tilde{N}_{0i} \equiv \{\tilde{w} \in \tilde{N} | support(\tilde{w}) \subset closure\ of\ \tilde{\Omega}_i\} \tag{8.2}$$

The subspace $\tilde{N}_0 \subset \tilde{N}$ is taken to be

$$\tilde{N}_0 \equiv \sum_{i=1}^{E_G} \tilde{N}_{0i}. \tag{8.3}$$

Let m_{0i} be the dimension of $\tilde{N}_{0i}$ for each $i = 1, \ldots, E_G$. Then $m_0 \equiv \sum_{i=1}^{E_G} m_{0i}$ is the dimension of $\tilde{N}_0$. For each $i = 1, \ldots, E$, let $\mathscr{C}_{0i} = \{\tilde{w}_{0i}^{\alpha} : \alpha = 1, \ldots, m_{0i}\}$ be a basis of $\tilde{N}_{0i}$. Then $\mathscr{C}_0 = \bigcup_{i=1}^{E_G} \mathscr{C}_{0i}$ is a basis of $\tilde{N}_0$.

In addition, $\tilde{N}_G \subset \tilde{N}$ is taken to be a complementary linear subspace of $\tilde{N}_0$; i.e.,

$$\tilde{N} = \tilde{N}_0 + \tilde{N}_G \qquad \text{and} \qquad \tilde{N}_0 \cap \tilde{N}_G = \{0\} \tag{8.4}$$

The dimension of $\tilde{N}_G$ will be denoted by m_G and $\mathscr{C}_G \equiv \{\tilde{w}_G^{\gamma} : \gamma = 1, \ldots, m_G\}$ is taken as a basis of $\tilde{N}_G$. Thus, $m_0 + m_G = m$ and $\mathscr{C}_0 \cup \mathscr{C}_G = \mathscr{C}$ is a basis of $\tilde{N}$. The results of Section 7 will be applied using this basis of $\tilde{N}$.

Observe that the restriction to $N \times N$ of the bilinear functional S is an inner product. Then, $\bar{N}_G \subset \tilde{N} \subset N$ is defined to be the orthogonal complement of $\tilde{N}_0$, in $\tilde{N}$, with respect to this inner product. Furthermore, given any $\tilde{w}_G \in \tilde{N}_G$ its projection on $\bar{N}_G$ is denoted by $\bar{w}_G \in \bar{N}_G$. An explicit expression for such a projection is

$$\bar{w}_G \equiv \tilde{w}_G + \sum_{i=1}^{E_G} \sum_{\alpha=1}^{m_{0i}} d_{\alpha}^{i} \tilde{w}_{0i}^{\alpha}, \tag{8.5}$$

where $\underline{d} \equiv (\underline{d}_1, \ldots, \underline{d}_{E_G})$ is the unique solution of the system of equations

$$\underline{\underline{S}}_i^0 \cdot \underline{d}_i = \underline{e}_i, \qquad i = 1, \ldots, E_G. \tag{8.6}$$

Here, use has been made of Eq. (8.4). In addition, for each $i = 1, \ldots, E_G$, $\underline{\underline{S}}_i^0$ is an $m_{0i} \times m_{0i}$ matrix, $\underline{\underline{S}}_i^0 \equiv (S_{i\alpha\beta}^0)$, with

$$S_{i\alpha\beta}^0 \equiv -\langle S\tilde{w}_{0i}^{\alpha}, \tilde{w}_{0i}^{\beta} \rangle = -\int_{\Sigma} \tilde{w}_{0i}^{\alpha} [\underline{a}_n \tilde{w}_{0i}^{\beta}] dx \equiv \int_{\Omega} \{\nabla \tilde{w}_{0i}^{\alpha} \cdot \underline{\underline{a}} \cdot \nabla \tilde{w}_{0i}^{\beta} + c\tilde{w}_{0i}^{\alpha} \tilde{w}_{0i}^{\beta}\} dx,$$

$$\alpha, \beta = 1, \ldots, m_{0i} \tag{8.7}$$

while $\underline{e}_i \equiv (e_i^1, \ldots, e_i^{m_{0i}})$, is given by

$$e_{i\alpha} \equiv -\int_{\Sigma} \tilde{w}_{0i}^{\alpha} [\underline{a}_n \cdot \nabla \tilde{w}_G] dx \equiv \int_{\Omega} \{\nabla \tilde{w}_{0i}^{\alpha} \cdot \underline{\underline{a}} \cdot \nabla \tilde{w}_G + c\tilde{w}_{0i}^{\alpha} \tilde{w}_G\} dx, \qquad \alpha = 1, \ldots, m_{0i}. \tag{8.8}$$

The following properties must be noticed:

$$\text{i)} \quad \langle S\bar{w}_G, \tilde{w}_0 \rangle = 0, \qquad \forall \tilde{w}_0 \in \tilde{N}_0 \quad \text{and} \quad \forall \bar{w}_G \in \bar{N}_G \tag{8.9}$$

$$\text{ii)} \quad \tilde{N} = \tilde{N}_0 + \bar{N}_G \qquad \text{and} \qquad \tilde{N}_0 \cap \bar{N}_G = \{0\}. \tag{8.10}$$

The notation $\bar{w}_G^\lambda$ ($\lambda = 1, \ldots, m_G$) is adopted for the images under the projection mapping introduced above of each one of the elements of the basis $\mathscr{E}_G = \{\tilde{w}_G^1, \ldots, \tilde{w}_G^{m_G}\} \subset \tilde{N}_G$. Clearly,

$$\langle S\bar{w}^\alpha, \bar{w}^\beta \rangle = \langle S\tilde{w}^\alpha, \bar{w}^\beta \rangle, \tag{8.11}$$

because of the orthogonality between the subspaces $\tilde{N}_0$ and $\bar{N}_G$. Furthermore, the image of $\mathscr{E}_G$, which will be denoted by $\bar{\mathscr{E}}_G \equiv \{\bar{w}_G^1, \ldots, \bar{w}_G^{m_G}\} \subset \bar{N}_G$, is a basis of $\bar{N}_G$. Due to the second of the properties listed above, the family $\bar{\mathscr{E}} \equiv \mathscr{E}_0 \cup \bar{\mathscr{E}}_G$ is a basis of $\tilde{N}$ and the results of Section 7 can be applied, using $\bar{\mathscr{E}}$ instead of $\mathscr{E}$. When this is done, using i), it is seen that the matrix $\underline{\underline{S}}$ obtained by means of Eq. (7.17), has the following structure:

$$\underline{\underline{S}} = \begin{matrix} \underline{\underline{S}}^G & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}^0 \end{matrix}. \tag{8.12}$$

The components of $\underline{\underline{S}}^G$ are

$$S_{\alpha\beta}^G \equiv -\langle S\bar{w}_G^\alpha, \bar{w}_G^\beta \rangle = -\langle S\tilde{w}^\alpha, \bar{w}^\beta \rangle \equiv -\int_\Sigma \tilde{w}_G^\alpha [\underline{a}_n \cdot \nabla \bar{w}_G^\beta] dx = \sum_{j=1}^{E} \int_{\Omega_j} \{\nabla \tilde{w}_G^\alpha \cdot \underline{\underline{a}} \cdot \nabla \bar{w}_G^\beta + c\tilde{w}_G^\alpha \bar{w}_G^\beta\} dx, \; \alpha, \beta = 1, \ldots, m_G, \tag{8.13}$$

while

$$\underline{\underline{S}}^0 \equiv \begin{matrix} \underline{\underline{S}}_1^0 & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{S}}_2^0 & \cdots & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{S}}_{E_G-1}^0 & & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & & \underline{\underline{S}}_{E_G}^0 \end{matrix}. \tag{8.14}$$

Then the system of equations (7.16), can be split into

$$\underline{\underline{S}}^G \cdot \underline{c}^G = \underline{b}^G \tag{8.15}$$

and

$$\underline{\underline{S}}_i^0 \cdot \underline{c}_i^0 = \underline{b}_i^0, \qquad i = 1, \ldots, E_G, \tag{8.16}$$

where $\underline{c}^G \equiv (c_1^G, \ldots, c_{m_G}^G)$, $\underline{c}^0 \equiv (\underline{c}_1^0, \ldots, \underline{c}_{E_G}^0)$, and, in turn $\underline{c}_j^0 \equiv (c_1^j, \ldots, c_{m_{0j}}^j)$. The components of the vectors $\underline{b}^G$ and $\underline{b}_i^0$ are

$$b_\lambda^G \equiv \sum_{j=1}^{E} \int_{\Omega_j} \bar{w}_G^\lambda (f_\Omega - \mathscr{L}u_0) dx, \qquad \lambda = 1, \ldots, m_G$$

$$b_{i\alpha}^0 \equiv \sum_{j=1}^{E} \int_{\Omega_j} \tilde{w}_0^{i\alpha} (f_\Omega - \mathscr{L}u_0) dx, \qquad \alpha = 1, \ldots, m_{0i}, \; i = 1, \ldots, E_G. \tag{8.17}$$

Equation (7.15) can be rewritten as

$$\tilde{v} = \sum_{\alpha=1}^{m_G} c_\alpha^G \bar{w}_G^\alpha + \sum_{i=1}^{E_G} \sum_{\alpha=1}^{m_{0i}} c_\alpha^i \tilde{w}_{0i}^\alpha. \tag{8.18}$$

Observe that the system of equations (8.16) constitute a sequence of local problems corresponding to each one of the subdomains $\tilde{\Omega}_i$ of the coarse partition ($i = 1, \ldots, E_G$). The system of equations (8.15), will be referred as the global system, while that of Eq. (8.16) constitutes the intermediate problems. They are local in the coarse partition, but generally they are nonlocal in the fine partition. An additional interesting property is that the global system and the intermediate system are independent of each other. Also, each one of the equations that are obtained when $i = 1, \ldots, E_G$, in Eq. (8.16), is independent of the other ones.

Before finishing this Section, some remarks are in order. In the applications we have in mind, the region Ω where the BVPJ is defined, as well as the number of degrees of freedom associated with the discretized version of it, will be usually rather large. In this respect, one can distinguish three different scales: the small or fine scale, corresponding to the individual subdomains, Ω_i, of the fine partition Π; the intermediate scale, corresponding to the subdomains $\tilde{\Omega}_i$ of the coarse partition, Π_G; and the global or large scale of the whole domain Ω. The functions $\tilde{w}^i \in \mathscr{C} \subset N$ are local in the small scale; i.e., they have their support in the union of a few subdomains of the fine partition. So, usually their construction is computationally cheap because the number of degrees of freedom involved is rather small. The same can be said about functions $\tilde{w}_G^\alpha \in \tilde{N}_G$ and $\tilde{w}_0^i \in \tilde{N}_0$. However, the construction of the test functions of the set $\bar{\mathscr{C}}_G = \{\bar{w}_G^\lambda : \lambda = 1, \ldots, m_G\}$ requires solving the sequence of problems of Eq. (8.16), which are local in the intermediate scale and are considerably more expensive. However, in most domain decomposition methods such constructions are generally avoided, as it is done in the method introduced in the present article. Furthermore, in view of the fact that the functions $\tilde{w}_G^\alpha \in \tilde{N}_G$ have local support in the fine partition (a property that is not enjoyed by the functions $\bar{w}_G^\alpha \in \bar{N}_G$), Eq. (8.13) permits reducing the extension of the internal boundary, or the number of subdomains of the fine partition, on which the integrations have to be carried out. The integral actually does not extend beyond the union of a few subdomains of the fine partition, neighboring Σ_G. However, notice that such an integral generally is not restricted to Σ_G and care must be exercised for not overlooking this fact in actual computations. Furthermore, application of the first of Eqs. (8.17) would require previously obtaining the function $\bar{w}_G^\lambda$. To avoid this, it is better to apply the equations

$$b_\lambda^G \equiv \sum_{j=1}^{E} \int_{\Omega_j} \tilde{w}_G^\lambda (f_\Omega - \mathscr{L} u_0) dx - \sum_{j=1}^{E} \int_{\Omega_j} \{\nabla \tilde{v}_0 \cdot \underline{\underline{a}} \cdot \nabla \tilde{w}_G^\lambda + c \tilde{v}_0 \tilde{w}_G^\lambda\} dx. \tag{8.19}$$

This equation can be established defining

$$\bar{v}_G \equiv \sum_{\alpha=1}^{m_G} c_\alpha^G \bar{w}_G^\alpha \qquad \text{and} \qquad \tilde{v}_0 \equiv \sum_{i=1}^{E_G} \sum_{\alpha=1}^{m_{0i}} c_\alpha^i \tilde{w}_{0i}^\alpha, \tag{8.20}$$

which are the projections of $\tilde{v}$ on $\bar{N}_G$ and $\tilde{N}_0$, respectively. Then $\tilde{v} = \bar{v}_G + \tilde{v}_0$ and applying Eq. (8.13) one can derive Eq. (8.19).

9. APPLICATION OF THE CONJUGATE GRADIENT METHOD

For the applications that the method here presented intends to address, the global matrix, $\underline{\underline{S}}^G$, is quite large. Its direct solution, as well as its explicit computation, are usually expensive and should be avoided. Also, the number m_G is large and the derivation of each one of the members of $\bar{\mathscr{E}}_G = \{\bar{w}_G^\lambda : \lambda = 1, \ldots, m_G\}$ requires solving a problem of the form of Eq. (8.16), which is local in the intermediate scale so that its cost is significant. Thus, a healthy strategy requires avoiding the explicit construction of such functions, as it is done in most DDMs. In particular, the Conjugate Gradient Method is a very effective iterative solution procedure for solving the global system of equations (8.15) and its application to this system, is straightforward because the matrix $\underline{\underline{S}}^G$ is positive definite. However, the evaluation of the product $\underline{\underline{S}}^G \cdot \underline{p}$, for any given vector $\underline{p}$, without computing the matrix $\underline{\underline{S}}^G$, is more subtle. Such a procedure is explained next.

Consider the matrix $\underline{\underline{S}}^G \equiv (S^G_{\alpha\beta})$, as given by Eq. (8.13), and let $\underline{p} \equiv (p_1, \ldots, p_{m_G})$ be any m_G − *vector*. Defining $\bar{w}_G \equiv \sum_{\alpha=1}^{m_G} p_\alpha \bar{w}_G^\alpha$, it is seen that

$$(\underline{\underline{S}}^G \cdot \mathbf{p})_\beta = -\sum_{\alpha=1}^{m_G} p_\alpha \int_\Sigma \tilde{w}_G^\beta [a_n \cdot \nabla \bar{w}_G^\alpha] dx = -\int_\Sigma \tilde{w}_G^\beta [a_n \cdot \nabla \bar{w}_G] dx = \int_\Omega \{\nabla \tilde{w}_G^\beta \cdot \underline{\underline{a}} \cdot \nabla \bar{w}_G + c\tilde{w}_G^\beta \bar{w}_G\} dx, \; \beta = 1, \ldots, m_G. \quad (9.1)$$

Here, $\bar{w}_G \in \bar{N}_G \cap (\sum_{\alpha=1}^{m_G} p_\alpha \tilde{w}_G^\alpha + \tilde{N}_0)$; i.e.,

$$\bar{w}_G \equiv \sum_{\alpha=1}^{m_G} p_\alpha \tilde{w}_G^\alpha + \sum_{i=1}^{E_G} \sum_{\alpha=1}^{m_{0i}} d_\alpha^i \tilde{w}_{0i}^\alpha. \quad (9.2)$$

In addition, in view of the results of Section 8, the condition $\bar{w}_G \in \bar{N}_G$ implies that $\underline{d} \equiv (\underline{d}_1, \ldots, \underline{d}_{E_G})$, is the unique solution of the system of equations

$$\underline{\underline{S}}_i^0 \cdot \underline{d}_i = \underline{e}_i, \qquad i = 1, \ldots, E_G, \quad (9.3)$$

with

$$e_{i\beta} \equiv \sum_{\alpha=1}^{m_G} p_\alpha \int_\Sigma \tilde{w}_{0i}^\beta [\underline{a}_n \cdot \nabla \tilde{w}_G^\alpha] dx = -\sum_{\alpha=1}^{m_G} p_\alpha \int_\Omega \{\nabla \tilde{w}_{0i}^\beta \cdot \underline{\underline{a}} \cdot \nabla \tilde{w}_G^\alpha + c\tilde{w}_G^\beta \tilde{w}_G^\alpha\} dx. \quad (9.4)$$

Recall that the integrals in Eqs. (9.1) and (9.4) are local; i.e., they involve only a few subdomains of the fine partition, because the functions $\tilde{w}_G^\beta$ have local support.

10. PARALLEL IMPLEMENTATION OF DDM

When the equation system (7.16) is split into two systems—the global and the intermediate systems of Eqs. (8.15) and (8.16), respectively—they are solved independently of each other. Of course, the solution of the global system is more costly. While there is more than one way of approaching this global system and they involve different trade-offs and efficiencies, the

solution procedure implemented by the authors is presented as an illustration of the overall parallel procedure. A synopsis of this solution algorithm can be given in three main steps.

1. First, an efficient solution procedure must be set up for the intermediate system of Eq. (9.3). This involves the computations of the test functions $\tilde{w}_0^i \in \tilde{N}_0$ for each subdomain of the fine partition. As the original problem is assumed to be symmetric and positive-definite, then these local matrices are well formed, banded, and positive definite as well. Since the subdomains are independent, these processes can be achieved in parallel. In the authors' implementation *LU* inverses were constructed.
2. The right-hand side $\underline{b}^G$ of Eq. (8.15) is assembled. As the components of $\underline{b}^G$ involve integrals on the subdomains, this process can also be parallelized.
3. Equation (8.15) is now solved iteratively by use of the Conjugate Gradient Method (CGM). Direct application of CGM is feasible because the matrices are symmetric and positive definite. Then convergence is guaranteed.
4. Each iteration of the conjugate gradient procedure involves exactly one matrix multiplication of the form $\underline{\underline{S}}^0 \cdot \underline{p}$. This is done by the procedure explained in Section 9, which requires at each step solving once the equations (9.3), but does not require the explicit computation of matrix $\underline{\underline{S}}^0$. The solution of these local intermediate problems is obtained by a matrix operation using the LU inverses computed in step 1. Finally, this solution is employed in Eq. (9.1) through another local matrix multiplication to give the desired components $(\underline{\underline{S}}^G \cdot \underline{p})_\beta$, $\beta = 1, \ldots, m_G$. Again, these component calculations, which involve matrix multiplication in the independent subdomains, can be done in parallel.

11. NUMERICAL RESULTS

In [14], the authors presented a collection of numerical experiments, using two TH-collocation algorithms that, for symmetric and positive operators, reduce to those that have been presented here. It must be noticed that TH-collocation can be applied to any differential equation independently of whether it is symmetric or not. Indeed, a large proportion of the examples treated in [14] is nonsymmetric. In the present article, we have treated the same set of BVPJ, except that the nonsymmetric cases have been eliminated. Now, however, such a set of problems is tackled using a combination of TH-collocation and the method of domain decomposition introduced in previous sections.

The domain decomposition method here used may be classified as a substructuring method. It is based on the use Eqs. (8.15) and (8.16). In Section 8, this pair of equations has been shown to be strictly equivalent to the system of equations of Eq. (7.16), which is the original system yielded by the direct application of TH-domain-decomposition and was used in [14]. The numerical results obtained confirm experimentally the theoretical results of Section 8. Indeed, in [14], for all the examples of the set considered, the error behavior exhibited by the TH-collocation in the numerical experiments was $O(h^2)$ and $O(h^4)$, for Algorithm 1 and Algorithm 2, respectively. We recall that Algorithm 1 uses C^0 piecewise linear functions, on Σ, and Algorithm 2 uses piecewise C^1 cubic functions, on Σ, with four collocation points in the partition subdomains in both cases. It must be mentioned that a preliminary analysis indicates that the error order for Algorithm 1, $O(h^2)$, is also consistent with the use of only one collocation point, instead of four [28]. The authors implemented the algorithm detailed in Section 9, which applies the CGM.

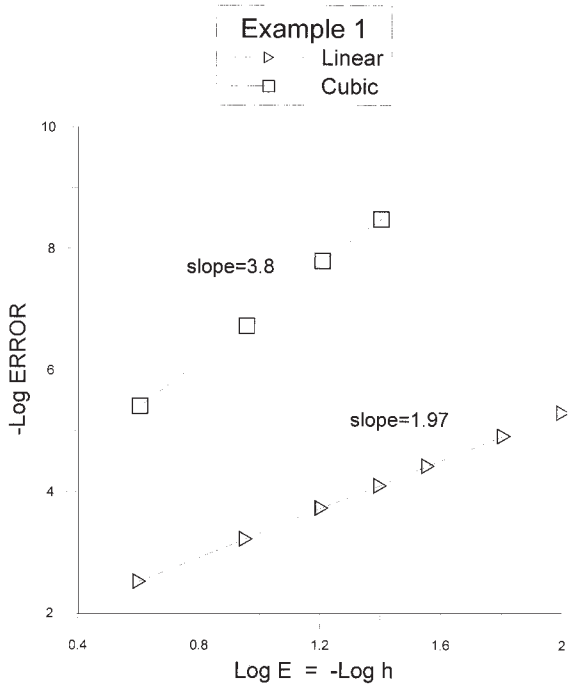

FIG. 4. Example 1. $-(\partial^2 u)/(\partial x^2) - (\partial^2 u)/(\partial y^2) + u = (1 - x^2 - y^2)e^{xy}$. $\Omega = [0, 1] \times [0, 1]$. $u(x, y) = e^{xy}$ on $\partial\Omega$. Solution: $u(x, y) = e^{xy}$.

In Figs. 4 and 5, the graph of only two of the BVPJ treated are illustrated, although more examples were carried out.

12. CONCLUSIONS

Domain decomposition methods (DDM) constitute, at present, the most effective means for applying parallel computing resources to continuous systems models. However, combining collocation procedures for discretizing differential equations, which are the basic models of such systems, with domain decomposition methods presents difficulties, as has been reported by several authors [20–23], that must be overcome in order to profit from parallel computing. The main technical difficulties stem from the fact that the system-matrices of models derived using Orthogonal Spline Collocation (OSC) fail to be symmetric and positive, even when the differential operators involved possess these properties.

Lately, an important change has occurred. Indeed, the recent development of TH-collocation [13, 14] supplies a broad basis for overcoming this problem in a wide class of applications to science and engineering. TH-collocation is a new approach to collocation methods of general applicability, which is an outcome of a general theory of discontinuous Galerkin methods, due to Herrera. It can be applied to any differential equation or system of such equations, regardless of whether they are symmetric or nonsymmetric, with possibly discontinuous coefficients. In addition, the general initial-boundary value problem treated by TH-collocation is one with prescribed jumps (BVPJ) in the internal boundaries of the domain partitions. When the

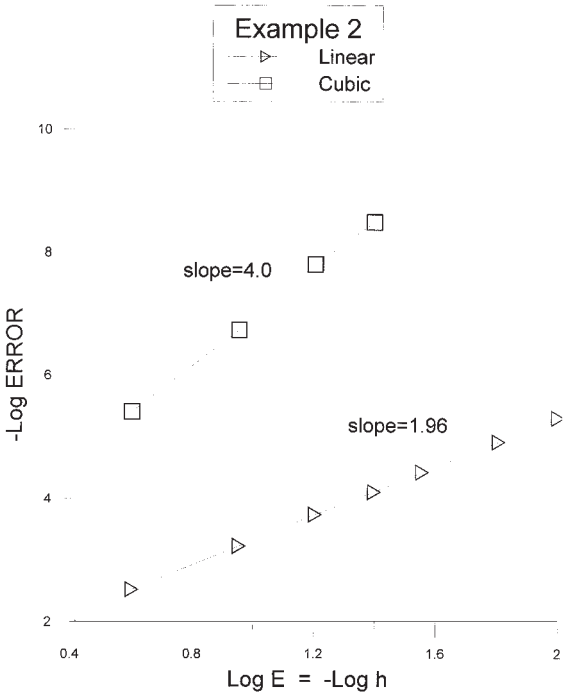

FIG. 5. Example 2. $-(\partial^2 u)/(\partial x^2) - (\partial^2 u)/(\partial y^2) + u = (2\pi^2 + 1)\sin \pi x \sin \pi y$. $\Omega = [0, 1] \times [0, 1]$. $u(x, y) = 0$ on $\partial\Omega$. Solution: $u(x, y) = \sin \pi x \cdot \sin \pi y$.

differential operators involved are symmetric and positive, the global matrices are also symmetric and positive. Thus, this overcomes the basic technical difficulty that has prevented the effective use of parallel computing in this class of problems and clears the way for combining orthogonal collocation with DDM in an effective manner, for a wide variety of problems. Higher order equations such as the biharmonic equation, for which TH-collocation has already been implemented [15,46], are included among them; also included are systems of equations, as those of elasto-statics, and many more.

As an illustration, because of its intrinsic importance, in this article an effective algorithm for combining TH-collocation with DDM has been implemented. It deals with the most general second-order equation for which the differential operator involved is elliptic, symmetric, and positive. A distinguishing feature is that when applying TH-collocation, the process of approximating the differential equation and that of approximating the solution in the internal boundary are carried out independently. This allows the use of different approximating procedures for them. In particular, in the present article the TH-collocation algorithms of [14] were applied. In one of them, of order h^2, the trial and test functions are, in the internal boundary, piecewise linear, but the differential equation was collocated in the four Gaussian points of the rectangular partition subdomains. This was done using cubic polynomials in the subdomain interiors. For the other algorithm that was applied, of order h^4, the trial and test functions are piecewise cubic in the internal boundary, but again the differential equation was collocated in the four Gaussian points of the rectangular partition subdomains and cubic polynomials were used in the subdomain interiors. It must be observed, however, that for algorithms of order h^2, this number can be reduced to 1, independently of the dimension of the space in which the problem is formulated.

These so-called single-collocation-point methods are being investigated at present [28] (using a different approach related algorithms were derived in [29]).

As for the DDM that is applied, it can be classified as a substructuring method [24], but it possesses some novel and attractive features, such as the systematic and simple manner in which the Schur-complement matrix is derived. In addition, an important property of the overall procedure is that the system of equations, after DDM is applied, is strictly equivalent to the global system that is obtained by the TH-collocation method of discretization. In particular, it eliminates the inaccuracies and overcomes the difficulties that were encountered with the method proposed in [20]. The fact that the conjugate gradient method (CGM) is applied directly, in its original form, is also very important. Of course, this is possible because of the symmetry and positive-definiteness of the global matrices, which are obtained when TH-collocation is applied.

Finally, this article contains a brief but systematic exposition of Herrera's theory of discontinuous Galerkin methods. Putting together, in a coherent manner, the results on which this theory is based was necessary for making this article more self-contained and readable, and for future reference.

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