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TH-collocation for the biharmonic equation

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Abstract

This paper is intended as a contribution to enhance orthogonal collocation methods. In this, a novel collocation method—*TH-collocation*—is applied to the biharmonic equation and the merits of such procedure are exhibited. TH-collocation relaxes the continuity requirements and, for the 2D problems here treated, leads to the development of algorithms for which the matrices are sparse (nine-diagonal), symmetric and positive definite. Due to these properties, the conjugate gradient method can be directly, and more effectively, applied to them. These features contrast with those of the standard orthogonal spline collocation on cubic Hermites, which yields matrices that are non-symmetric and non-positive. This paper is part of a line of research in which a general and unified theory of domain decomposition methods, proposed by Herrera, is being explored. Two kinds of contributions can be distinguished in this; some that are relevant for the parallel computation of continuous models and new discretization procedures for partial differential equations. The present paper belongs to this latter kind of contributions.

Keywords: Trefftz method; Collocation; Domain decomposition; Biharmonic equation; Discontinuous Galerkin

1. Introduction

This paper is part of a line of research in which a general and unified theory of domain decomposition methods (DDM), proposed by Herrera [1] and stemming from Trefftz method [2], is being explored. In it, the terms 'domain decomposition methods' are understood in broader sense than usual and they include many aspects of numerical methods for partial differential equations. As a matter of fact, Herrera's approach to partial differential equations constitutes a general and systematic formulation of discontinuous Galerkin methods [3], in which the use of 'fully' discontinuous functions is permitted. The investigations that are being carried out, in the line of research mentioned above, cover two different aspects. One is concerned with developing novel discretization procedures [4,5] and the other one deals with producing new ways of efficiently using parallel computing resources in the numerical simulation of continuous systems [3].

The main purpose of the present paper is to present an improved orthogonal-collocation treatment of the biharmonic equation. This is based on the application of a new general collocation method, 'TH-collocation', which was introduced in a pair of previous papers [4,5]. An interesting and attractive feature of TH-collocation is the relaxation of the continuity conditions, which allows using trial-spaces of functions that are globally only C^0. This, in turn, permits deriving algorithms with better-structured matrices. In particular, it produces symmetric and positive matrices when it is applied to differential systems with such properties, as is the case of Laplace's and the biharmonic operators. Also, the number of degrees of freedom associated with each node is reduced. For Poisson equation, TH-collocation yields an algorithm of fourth order precision whose global matrix, in addition to being symmetric and positive definite, is block nine-diagonal, with blocks of at most 3×3 [5]. This is to be compared with orthogonal spline collocation (OSC), which for the same order of accuracy yields a global matrix that is neither symmetric, nor positive, and whose blocks are 4×4. Furthermore, TH-collocation also yields another algorithm [5], of second order precision, whose global matrix is strictly nine-diagonal (i.e. with blocks 1×1). Such reduction is not

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possible when OSC is applied. Due to these important advantages, over standard collocation procedures, which TH-collocation possesses, domain decomposition methods (DDM) can be effectively applied to TH-collocation algorithms using the Conjugate Gradient Method (CGM) in a direct manner [3].

For the biharmonic equation semi-analytical discretization procedures, of the Trefftz–Jirousek type [6], have been developed by several authors [7–9] and a review of such methods can be found in [10]. As for non-analytical discretization methods, a recent paper by Lou et al. [11] presents a discussion, and a brief comparison, of several discretization methods that are available to deal with the biharmonic equation. According to them, some of the existing finite difference methods are very efficient and one due to Bjorstad is of optimal complexity. The order of accuracy of such methods is only second order. However, a fourth order collocation algorithm was introduced in [11].

When approaching the discretization of the biharmonic equation with non-analytical procedures, there are mainly two options. The first one consists in using a 13-point stencil [12,13] and in the second one, the 'splitting approach' [13], the biharmonic equation is rewritten as a system of two equations whose treatment requires solving two Poisson equations successively. When this latter procedure is applied, the effectiveness of the method and of its parallel computation depends essentially on those of the Poisson equations. The most popular collocation formulation for partial differential equations of second order, which the majority of the authors working in this field have used up to now, is OSC; i.e. the Hermite bi-cubic orthogonal spline collocation [14]. The OSC formulation is applied in a trial-space of functions which are globally C^1; this produces a global matrix, which in its usual form is neither symmetric nor positive definite, even when the differential operator has these properties.

In this paper, we tackle the biharmonic equation using the splitting approach and solve each one of the Poisson equations by means of TH-collocation, profiting from the advantageous features of the TH-collocation treatment of Poisson equation. Thus far, the order of accuracy of our algorithms has been only derived experimentally, as was done in [5] and in Section 7. However, an interesting characteristic of our method is that it actually produces the same solutions as those obtained by Lou et al. [11]. Using this fact, a rigorous theoretical proof of the fourth order accuracy of our algorithm can be constructed. However, such discussions will be presented elsewhere.

2. Notations

In our formulation the notations $\Omega \subset R^n$ and $\partial\Omega$ are used for a domain of the Euclidean space of dimension n and its boundary, respectively. Throughout this paper n is taken to be equal to 2. Let $\Pi \equiv \{\Omega_1,\ldots,\Omega_E\}$ be a partition of Ω. Given such a partition, the boundaries of the subdomains are $\partial\Omega_i$, $i=1,\ldots,E$. Clearly, $\partial\Omega \subset \cup_{i=1}^{E}\partial\Omega_i$ and the 'internal boundary', Σ, is defined to be the closed complement of $\partial\Omega$ relative to $\cup_{i=1}^{E}\partial\Omega_i$. Then, $\partial\Omega$ will be referred as 'external boundary'. In the external boundary, the unit normal vector is taken pointing outwards. As for the internal boundary, a positive side of Σ and a unit normal vector, also denoted by $\underline{n}$, are defined almost everywhere (a.e.) on it with the convention that $\underline{n}$ points toward the positive side.

It is assumed that for each $i=1,\ldots,E$, there is a linear space $D(\Omega_i)$, whose elements are functions defined in Ω_i. Then, trial and test functions are taken from the linear space D, defined by:

$$D \equiv D(\Omega) \equiv D(\Omega_1) \oplus \cdots \oplus D(\Omega_E) \tag{1}$$

Possible choices for $D(\Omega_i)$ are the Sobolev spaces $H^5(\Omega_i)$, $i=1,\ldots,E$. For the case of elliptic equations of second order that will be considered, it is convenient to take $s \geq 2$. In fact, when the space D is defined by Eq. (1), a function $u \in D$ is a finite sequence of functions $u \equiv (u_1,\ldots,u_E)$ such that $u_i \in D(\Omega_i)$, $i=1,\ldots,E$. It is assumed that the trace of every $u_i \in D(\Omega_i)$ is defined a.e. on $\partial\Omega_i$. Given any function $u \in D$, $u \equiv (u_1,\ldots,u_E)$, two traces are defined at every point of Σ, which are denoted by u_+ and u_-, respectively. Since generally, $u_+ \neq u_-$, it is useful to define the 'jump' and the 'average' of any function $u \in D$ by

$$[u] = u_+ - u_- \quad \text{and } \dot{u} = (u_+ + u_-)/2 \tag{2}$$

respectively. This notation will be applied not only for a function, but for its derivatives as well. Clearly, the definition of the jump of a function is dependent on the orientation of Σ; however, the expressions that will be handled in this paper are invariant with respect to such orientation.

In some previous works, for simplicity, we have written $\mathcal{L}u = f_\Omega$, in Ω, to mean:

$$\mathcal{L}u = f_\Omega, \text{ at each } \Omega_i, i = 1,\ldots,E \tag{3}$$

For greater clarity, in the present paper we will be more explicit and write directly, Eq. (3), since $w\mathcal{L}u$ is not, in general, defined on Σ when $u \in D$ and $w \in D$. Similarly, we also write $\sum_{i=1}^{E}\int_{\Omega_i} w\mathcal{L}u \,dx$ instead of $\int_\Omega w\mathcal{L}u \,dx$. Assume a tensor-valued function $\underline{\underline{a}}$ is defined in Ω and write $\underline{a}_n \equiv \underline{\underline{a}} \cdot \underline{n}$, then it can be shown that

$$\sum_{i=1}^{E} \int_{\partial\Omega_i} w\underline{a}_n \cdot \nabla u \,dx = \int_{\partial\Omega} w\underline{a}_n \cdot \nabla u \,dx - \int_{\Sigma} [w\underline{a}_n \cdot \nabla u] dx \tag{4}$$

3. Splitting formulation of the biharmonic equation

The formulation of well-posed problems in function spaces containing discontinuous functions require that some jump of the functions and their derivatives, across the internal boundary, be prescribed. A well-posed boundary value problem with prescribed jumps (BVPJ) for the biharmonic equation is considered in what follows. It is defined by the differential equation

$$\mathcal{L}u \equiv \Delta^2 u = f_\Omega, \quad \text{in } \Omega_i,\ i = 1, \ldots, E \tag{5}$$

the boundary conditions

$$u = g^0 \quad \text{and } \Delta u = g^1, \ \text{ on } \partial\Omega \tag{6}$$

and by the jump conditions:

$$[u] = j^0, \quad \left[\frac{\partial u}{\partial n}\right] = j^1, \quad [\Delta u] = j^2, \quad \left[\frac{\partial \Delta u}{\partial n}\right] = j^3, \ \text{ on } \Sigma \tag{7}$$

When $j^0 = j^1 = j^2 = j^3 = 0$ the solution of such problem is the same as that of a standard boundary value problem for the biharmonic equation, without jumps. Here, this problem is tackled using the splitting approach mentioned in Section 1. Then, the biharmonic equation is rewritten as a system of two Poisson equations [13]:

$$\begin{cases} -\Delta u = v, \\ -\Delta v = f_\Omega, \end{cases} \text{in } \Omega \tag{8}$$

Then the solution of BVPJ of Eqs. (5)–(7) can be obtained by solving sequentially two Poisson's equations, where each of them is subjected to non-homogeneous Dirichlet boundary conditions:

$$\begin{cases} -\Delta u = v, & \text{in } \Omega \\ u = g^0, & \text{on } \partial\Omega \\ [u] = j^0, \quad \left[\dfrac{\partial u}{\partial n}\right] = j^1, & \text{on } \Sigma \end{cases}$$
$$\text{and } \begin{cases} -\Delta v = f_\Omega, & \text{in } \Omega \\ v = g^1, & \text{on } \partial\Omega \\ [v] = j^2, \quad \left[\dfrac{\partial v}{\partial n}\right] = j^3, & \text{on } \Sigma \end{cases} \tag{9}$$

In this paper, the resulting coupled system of Eq. (9) will be solved by applying the indirect Trefftz–Herrera collocation procedures developed in Refs. [4,5].

A last comment, before finishing this section, is in order. The splitting approach to the biharmonic equation is applicable to some other combinations of boundary conditions. Boundary value problems in which the following pairs: (u, $\partial\Delta u/\partial n$), ($\partial u/\partial n$, Δu) or ($\partial u/\partial n$, $\partial\Delta u/\partial n$) are prescribed in the outer boundary can also be split into a system of two Poisson equations. However, other problems such as those in which the pairs (u, $\partial u/\partial n$) and (Δu, $\partial\Delta u/\partial n$) are prescribed in the outer boundary cannot be split in this manner. In such case, TH-collocation has to be applied directly to the original biharmonic equation using a procedure that is under investigation at present and that we intend to publish in the near future.

4. Solution of Poisson equation

TH-collocation, as presented in [5], will be here applied to each one of the Poisson's problems of Eq. (9). To this end, the results presented in [5] are specialized for the following boundary value problem with prescribed jumps

$$-\Delta u \equiv -\nabla\cdot\nabla u = f_\Omega, \ \text{ on } \Omega_i,\ i = 1, \ldots, E \tag{10}$$

together with:

$$u(x) = u_\partial \ \text{ on } \partial\Omega, \quad [u] = j^0_\Sigma \quad \text{and } \left[\frac{\partial u}{\partial n}\right] = j^1_\Sigma \ \text{ on } \Sigma \tag{11}$$

Our interest will focus in the case when $j^0_\Sigma = 0$ and $j^1_\Sigma = 0$ on Σ.

In Herrera's indirect approach, in which TH-collocation is based, a special class of test functions is used; they are taken from a linear subspace $N \subset D$ whose members, $w \in N$, fulfill the conditions [5]:

$$\begin{aligned} &\Delta w = 0, \ \text{ at } \Omega_i,\ i = 1, \ldots, E, \\ &[w] = 0, \ \text{ on } \Sigma \quad \text{and } w = 0 \ \text{ on } \partial\Omega \end{aligned} \tag{12}$$

Thus, such test functions are continuous in Ω, but their derivatives may have jump discontinuities across Σ. Let $u \in D$ be the solution of the Poisson BVPJ considered in this section and $\tilde{u} \in D$ be any function of D. Then, the weak formulation that will be applied (see [5]) states that $\dot{\hat{\tilde{u}}} \equiv \dot{\hat{u}}$, on Σ, if and only if:

$$\begin{aligned} -\int_\Sigma u\left[\frac{\partial w}{\partial n}\right]\mathrm{d}x = \sum_{i=1}^{E}\int_{\Omega_i} w f_\Omega \,\mathrm{d}x - \int_{\partial\Omega} u_\partial \frac{\partial w}{\partial n}\mathrm{d}x \\ - \int_\Sigma \left\{ w j^1_\Sigma - j^0_\Sigma \frac{\partial w}{\partial n}\right\}\mathrm{d}x, \quad \forall\, w \in N \end{aligned} \tag{13}$$

In passing, it is observed that when both v and $w \in N$ the following identity holds:

$$-\int_\Sigma v\left[\frac{\partial w}{\partial n}\right]\mathrm{d}x \equiv \int_\Omega \nabla v\cdot\nabla w \,\mathrm{d}x, \quad \forall\, v,\ w \in N \tag{14}$$

Therefore, the bilinear form $-\int_\Sigma v[\partial w/\partial n]\mathrm{d}x$ is symmetric and positive-definite on N.

At this point, it is convenient to introduce an auxiliary function, $u_0 \in D$, satisfying:

$$\begin{aligned} &u_0(x) = u_\partial(x), && \text{on } \partial\Omega \\ &[u_0] = [u_\Sigma] = j^0_\Sigma, && \text{on } x \in \Sigma \\ &\left[\frac{\partial u_0}{\partial n}\right] = \left[\frac{\partial u_\Sigma}{\partial n}\right] = j^1_\Sigma, && \text{on } x \in \Sigma \end{aligned} \tag{15}$$

Furthermore, define:

$$v(x) = u(x) - u_0(x), \ \text{ in } \Omega \tag{16}$$

Then

$$-\Delta v(x) = f_\Omega + \Delta u_0(x), \quad \text{for } x \in \Omega_i, \ i = 1, \ldots, E \tag{17}$$

$$\begin{aligned} &v(x) = 0, && \text{on } \partial\Omega \\ &[v]_\Sigma = 0, && \text{on } \Sigma \\ &\left[\frac{\partial v}{\partial n}\right]_\Sigma = 0, && \text{on } \Sigma \end{aligned} \tag{18}$$

and Eq. (13) reduces to:

$$-\int_\Sigma v\left[\frac{\partial w}{\partial n}\right] \mathrm{d}x = \sum_{j=1}^{E} \int_{\Omega_j} w(f_\Omega + \Delta u_0)\mathrm{d}x \quad \forall\, w \in N \tag{19}$$

In Eq. (19), the value of the function v on Σ has been used instead of the average $\dot{v}$, because $\dot{v} = v$, as v is continuous across Σ. Even more, when $f_\Omega \in H^r(\Omega)$ and $\Delta u_0(x) \in H^r(\Omega)$, with $r \geq 0$, then $v \in H^{r+2}(\Omega)$. In particular, the trace of v on Σ, belongs to $C^1(\Sigma)$. This fact permits simplifying the numerical implementation of the method, because the search for v can be carried out in a smaller space of functions.

The weak formulation of Eq. (19) is sufficient for obtaining the function v on the internal boundary Σ, exclusively, which in turn yields the average of the sought solution there, by means of Eq. (16); i.e.

$$\dot{u} = \dot{u}_0 + v, \quad \text{on } \Sigma \tag{20}$$

If desired, the solution u of the BVPJ can be obtained in the interior of the subdomains of the partition by 'optimal interpolation', which consists in solving well-posed Dirichlet problems in each of the subdomains of the partition. At a given subdomain, Ω_i, the boundary data for such a problem, at points belonging to $\Sigma \cap \partial\Omega_i$, are defined by the equations

$$u_+ \equiv \dot{u} + \frac{1}{2}[u_\Sigma] \ \text{ and } u_- \equiv \dot{u} - \frac{1}{2}[u_\Sigma] \tag{21}$$

and by the first of Eq. (11), at points belonging to $\partial\Omega \cap \partial\Omega_i$. Observe that the identity

$$-\int_\Sigma v\left[\frac{\partial w}{\partial n}\right] \mathrm{d}x \equiv \sum_{i=1}^{E} \int_{\partial\Omega_i} v \frac{\partial w}{\partial n} \mathrm{d}x \tag{22}$$

holds when $v \in D$ fulfills Eq. (18), and $w \in N$. The following chain of equalities

$$\begin{aligned} -\int_\Sigma v\left[\frac{\partial w}{\partial n}\right] \mathrm{d}x &= \int_{\partial\Omega} v \frac{\partial w}{\partial n} \mathrm{d}x - \int_\Sigma \left(v\left[\frac{\partial w}{\partial n}\right] + [v] \frac{\partial \dot{w}}{\partial n} \right) \mathrm{d}x \\ &= \int_{\partial\Omega} v \frac{\partial w}{\partial n} \mathrm{d}x - \int_\Sigma \left[v \frac{\partial w}{\partial n} \right] \mathrm{d}x = \sum_{i=1}^{E} \int_{\partial\Omega_i} v \frac{\partial w}{\partial n} \mathrm{d}x \end{aligned} \tag{23}$$

makes Eq. (22) clear.

5. TH-discretization

When TH-collocation is applied for solving Poisson equation, the construction of the global system of equations is based on Eq. (19) but the linear subspace $N \subset D$ of special test functions is replaced by a TH-complete system $\mathcal{E} \subset N$ (see [5]). TH-complete systems are infinite for problems in more than one independent variable and, in numerical applications of TH-collocation, it is necessary to approximate TH-complete systems by finite families, whose members belong to $N \subset D$. This, of course, implies a truncation error that is reflected in the accuracy of the approximate solutions so obtained.

Functions $w \in N$ are uniquely determined by their traces on the internal boundary Σ, because they fulfill Eq. (12). Thus, the subspace of test functions $\tilde{N} \subset N$ to be applied can be specified by taking a suitable finite-dimensional linear manifold of dimension m, of functions defined on Σ. This manifold, in turn, is determined when a basis that spans it is specified. We use the notation $\tilde{N} \subset N$ for the subspace spanned by $\mathcal{E} \equiv \{\tilde{w}^1, \ldots, \tilde{w}^m\} \subset N$. Also, the exact solution $v \in N$ will be approximated on Σ by

$$\tilde{v} \equiv \sum_{\alpha=1}^{m} c_\alpha \tilde{w}^\alpha, \text{ on } \Sigma \tag{24}$$

that fulfills Eq. (19), but only for all $\tilde{w} \in \tilde{N}$. Then

$$-\underline{\underline{S}} \cdot \underline{c} = \underline{b} \tag{25}$$

where $\underline{c} \equiv (c_1, \ldots, c_m)$ and $\underline{b} \equiv (b_1, \ldots, b_m)$, with:

$$b_\alpha \equiv \sum_{i=1}^{E} \int_{\Omega_i} \tilde{w}^\alpha (f_\Omega - \mathcal{L} u_0)\mathrm{d}x, \quad i = 1, \ldots, m \tag{26}$$

In addition, the elements of the matrix $-\underline{\underline{S}}$ are given by:

$$\begin{aligned} -S_{\alpha\beta} &\equiv -\int_\Sigma \tilde{w}^\beta \left[\frac{\partial \tilde{w}^\alpha}{\partial n}\right] \mathrm{d}x = \sum_{i=1}^{E} \int_{\partial\Omega_i} \tilde{w}^\beta \frac{\partial \tilde{w}^\alpha}{\partial n} \mathrm{d}x \\ &= \sum_{i=1}^{E} \int_{\Omega_i} \nabla \tilde{w}^\beta \cdot \nabla \tilde{w}^\alpha \, \mathrm{d}x \end{aligned} \tag{27}$$

The matrix $-\underline{\underline{S}}$ is symmetric and positive definite.

6. The test functions

As mentioned before, the test functions, $w \in N$, are uniquely determined by their traces on Σ. In a manner similar to what was done in [5], in the applications to Poisson equation that are considered in this paper, the linear subspace of test functions, $\tilde{N} \subset N$, is such that the traces on Σ of its members are continuous piecewise polynomials of a fixed degree G. Two algorithms will be constructed. For Algorithm 1, $G = 1$, and for Algorithm 2, $G = 3$. In addition,

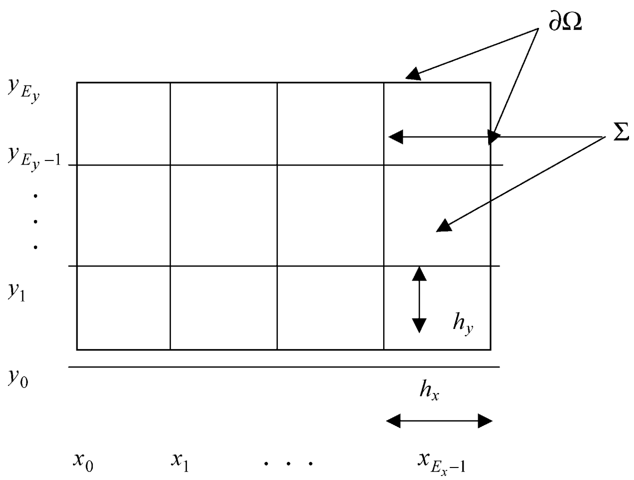

Fig. 1. Partition of the domain $\Omega=[x_{\min},x_{\max}]\times[y_{\min},y_{\max}]$ in rectangular $E_x\times E_y$ elements, where $h_x=x_i-x_{i-1}$; $i=1,\ldots,E_x$ and $h_y=y_j-y_{j-1}$; $j=1,\ldots,E_y$.

for Algorithm 2, the traces are required to be $C^1(\Sigma)$. Then, in each case the family of test functions $\mathcal{E}\subset\tilde{N}$, of Section 5, will be a basis of $\tilde{N}$, whose construction, for cases when the partition subdomains are either triangles or quadrilaterals, is relatively straightforward. The procedure is explained next for the case when the domain Ω of definition of the BVPJ is a rectangle and the subdomains of the partition are also quadrilaterals (Fig. 1). Then, associated with each internal node (x_i, y_j), we consider a cross which is made of two intervals limited, one of them, by the nodes (x_{i-1}, y_j) and (x_{i+1}, y_j), and, the other one, by the nodes (x_i, y_{j-1}) and (x_i, y_{j+1}), Fig. 2. They intersect at the 'center' of the cross (x_i, y_j). For both algorithms, there are systems $\mathcal{E}$ basis of $\tilde{N}$ whose functions have support contained in one of those crosses.

Algorithm 1. Given any node (x_i, y_j), there is one, and only one, piecewise linear function with support in the cross associated with it, such that it takes the value '1' at (x_i, y_j) and vanishes at the other end points of the cross. Then the system $\mathcal{E}\subset N$, basis of $\tilde{N}$, is the collection of functions of N whose restriction to Σ is one of such piecewise linear functions.

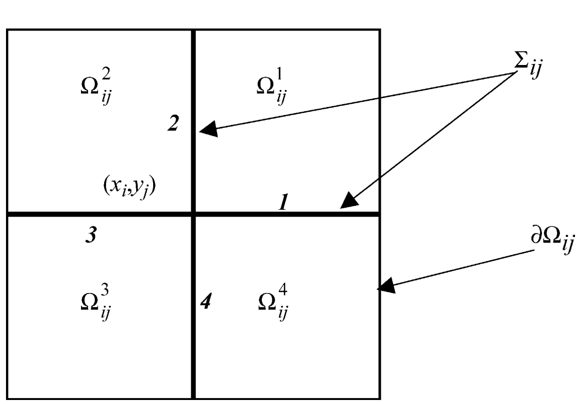

Fig. 2. Subregion Ω_{ij} associated with the node (x_i, y_j).

Algorithm 2. Here the following notations are used, $H_i^0(x)$ is the one dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}), which takes the value 1 at node x_i and zero at nodes x_{i-1} and x_{i+1}; and its first derivative is zero at all nodes x_{i-1}, x_i and x_{i+1}. Similarly, $H_i^1(x)$ is the one dimensional Hermite cubic polynomial with support in the interval (x_{i-1}, x_{i+1}), which takes the value zero at nodes x_{i-1}, x_i and x_{i+1}; and its first derivative takes the value 1 at node x_i and zero at the other nodes x_{i-1} and x_{i+1}. The use of the notations $H_j^0(y)$ and $H_j^1(y)$ is similar. Furthermore, let us order the intervals $(x_{i-1},y_j)-(x_{i+1},y_j)$, as 'first', and $(x_i,y_{j-1})-(x_i,y_{j+1})$, as 'second'. Then any function defined in the cross associated with the node (x_i, y_j), is made of an ordered pair of functions $\{p(x),q(y)\}$, where $p(x)$ is defined on (x_{i-1}, x_{i+1}), while $q(y)$ is defined on (y_{j-1}, y_{j+1}). For Algorithm 2, when (x_i, y_j) is an internal node, the trace on each one of the members the system of functions $\mathcal{E}\subset N$, is one of the following functions:

i. $\{H_i^0(x), H_j^0(y)\}$,
ii. $\{H_i^1(x), 0\}$ and
iii. $\{0, H_j^1(y)\}$.

If (x_i, y_j) lies on the external boundary $\partial\Omega$, then in order to fulfill the boundary condition of Eq. (12), only the functions of this set, which vanish on $\partial\Omega$ are taken. In particular, for boundary nodes that are not a corner, only one of the test function associated with each one of them is used. When (x_i, y_j) lies on a vertical boundary, such function is the restriction of $\{H_i^1(x), 0\}$ to the intersection of the cross with the domain Ω and, similarly, if (x_i, y_j) lies on a horizontal boundary, such function is the restriction of $\{0, H_j^1(y)\}$ to the intersection of the cross with the domain Ω. In addition, the set of functions associated with corner nodes is void, because none of the functions associated with such a node fulfills the homogeneous boundary condition of Eq. (12). Observe that the trace of each one of the members of the system $\mathcal{E}\subset N$ fulfills the requirement of belonging to $C^1(\Sigma)$. As mentioned in Section 4, this property is also enjoyed by the trace of v. Thus, the use of $\mathcal{E}\subset N$ defined in this manner, in the search for $v\in C^1(\Sigma)$, is consistent and reduces the number of degrees of freedom that are required. Indeed, it can be seen that if $v\notin C^1(\Sigma)$ then five functions, instead of three, at each internal node have to be introduced [5].

The above description defines uniquely a family of test functions $\mathcal{E}\subset\tilde{N}\subset N$, which is a basis of $\tilde{N}\subset N$. A simple counting—there are (E_x-1) (E_y-1) interior nodes, $2(E_x+E_y-2)$ boundary nodes plus 4 corners—shows that the dimension of $\tilde{N}$ is (E_x-1) (E_y-1), for Algorithm 1, while it is $3E_xE_y-(E_x+E_y)-1$ for Algorithm 2. These same numbers yield the number of degrees of freedom for each case, since test functions are also used as base functions on Σ. Here, E_x is the number of subintervals in the x-direction, while E_y is the number of subintervals in the y-direction.

Table 1
Analytic solutions for each one of the examples

Example	Exact solution
1	$x^2(x-1)^2y^2(y-1)^2$
2	$\sin^2(2\pi x)\sin^2(2\pi y)$
3	$(x^2-1)^2e^y+(y^2-1)^2e^x$
4	$\sin^2(2\pi x)\sin^2(2\pi y)+xy$
5	$\sin^2(2\pi x)\sin^2(2\pi y)+e^{xy}$

7. The numerical experiments

Two sets of numerical experiments were carried out. The first one consisted in applying Algorithm 1 of Section 6, for solving the Poisson's equations that occur when the *splitting method*, of Section 3, is used to treat several examples of the BVPJ of Eqs. (10) and (11). Exactly the same was done in the second set of numerical experiments but Algorithm 2 was used instead of Algorithm 1. The analytical solutions of each of these examples are given in Table 1. In all cases the domain of definition is the unit square $[0,1]\times[0,1]$. The imposed boundary values g^0 and g^1 and the right hand side term f_Ω are those implied by the analytical solutions.

Each one of the examples was solved in a uniform rectangular partition of the domain using Algorithm 1 and, subsequently, Algorithm 2, for which the weighting functions are piecewise linear and piecewise cubic, respectively, on Σ. In each of Figs. 3–12 the convergence rate of the error—measured in terms of the norm $\|\ \|_\infty$—is shown at nodes (x_i, y_j), on the internal boundaries (Σ_{ij}) and on the elements (Ω_{ij}).

Tables 2 and 3 summarize the numerical results for Algorithm 1 and Algorithm 2, respectively. For each

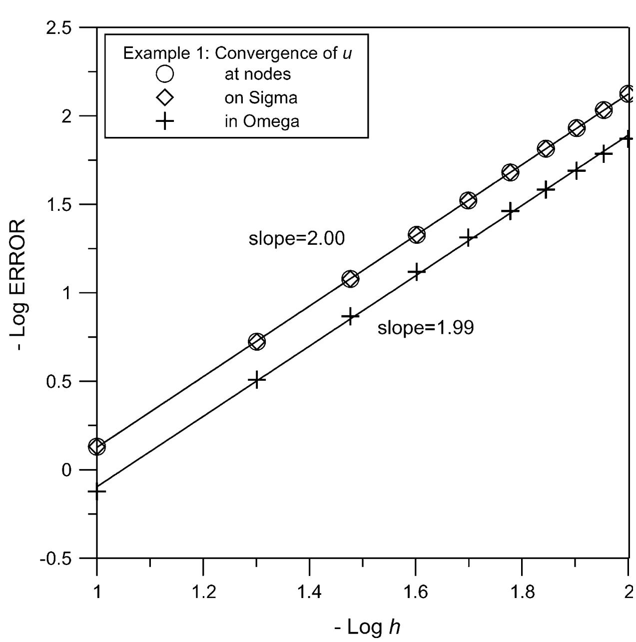

Fig. 3. Example 1: convergence rate of Trefftz–Herrera collocation method using linear weighting functions.

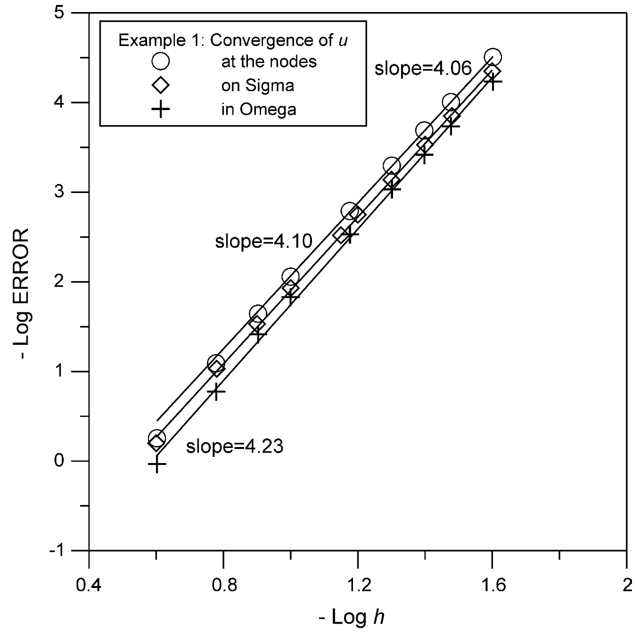

Fig. 4. Example 1: convergence rate of Trefftz–Herrera collocation method using cubic weighting functions.

example, the constant C and the power α of the estimated convergence error Ch^α based on least square log fits are reported.

8. Conclusions

The new method of collocation introduced in [4,5], TH-collocation, has been applied to the biharmonic equation

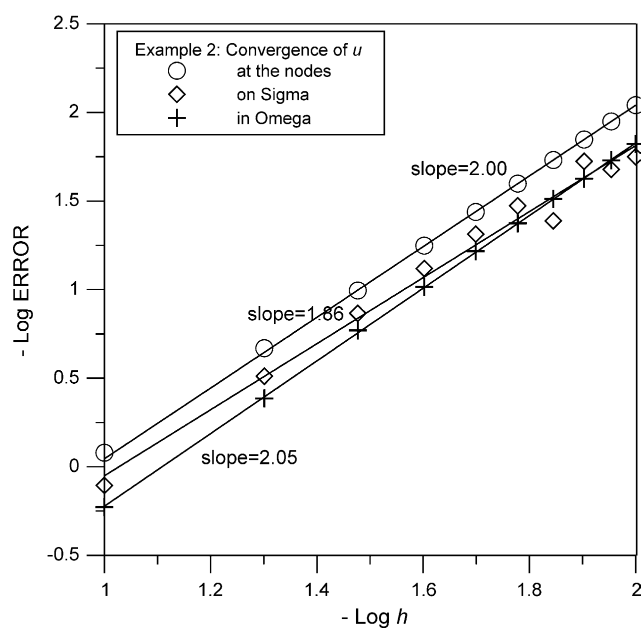

Fig. 5. Example 2: convergence rate of Trefftz–Herrera collocation method using linear weighting functions.

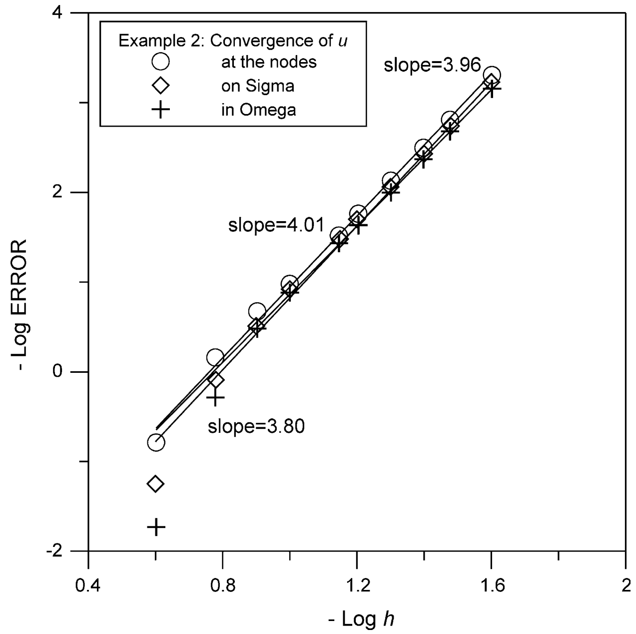

Fig. 6. Example 2: convergence rate of Trefftz–Herrera collocation method using cubic weighting functions.

subjected to one class of boundary conditions, using the splitting approach [13]. Two algorithms were developed in this manner that enjoy two general properties of discretization procedures, which are derived by means of Herrera's approach to domain decomposition; namely, the global matrices are symmetric and positive definite, when the original differential operators have these properties, and their constructions require solving local problems exclusively. These features are quite suitable for application of parallel computation.

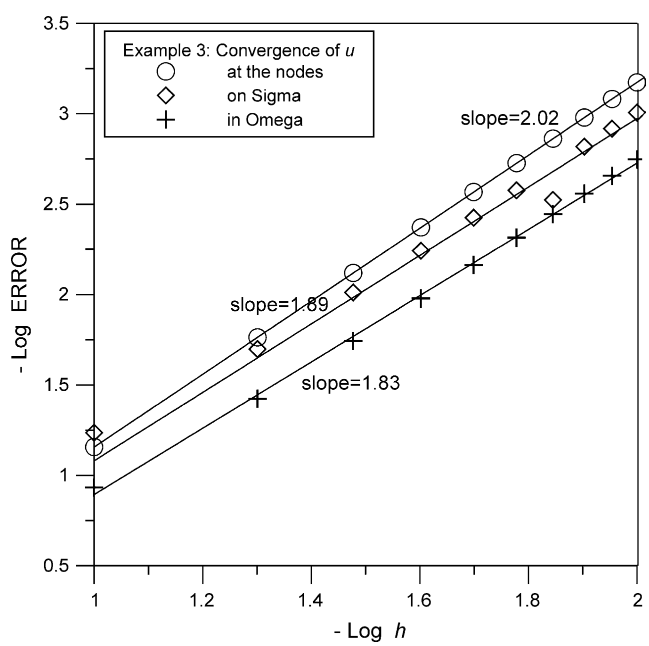

Fig. 7. Example 3: convergence rate of Trefftz–Herrera collocation method using linear weighting functions.

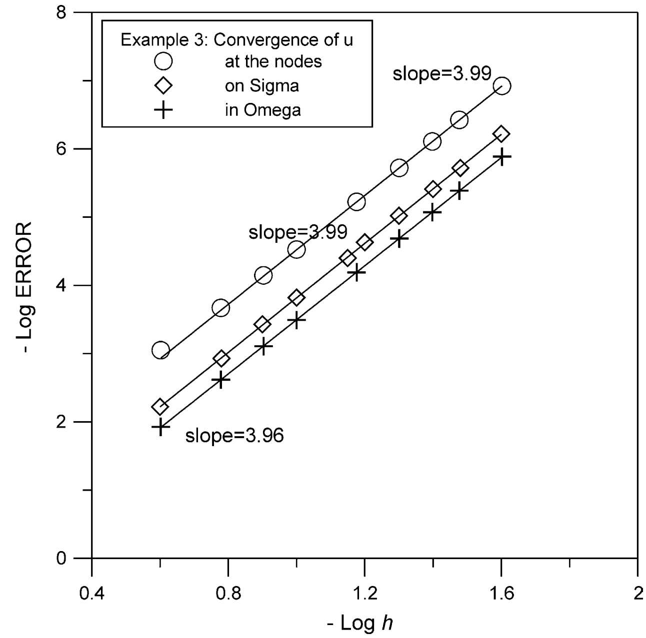

Fig. 8. Example 3: convergence rate of Trefftz–Herrera collocation method using cubic weighting functions.

One of these algorithms (Algorithm 1) uses piecewise linear interpolation in the internal boundary and the other one (Algorithm 2) uses piecewise cubic. The error behavior of Algorithm 1 is $O(h^2)$ and the corresponding properties for the global matrices follow:

I. They are strictly nine-diagonal (i.e. with blocks 1×1); and
II. The number of degrees of freedom associated with each internal node is one.

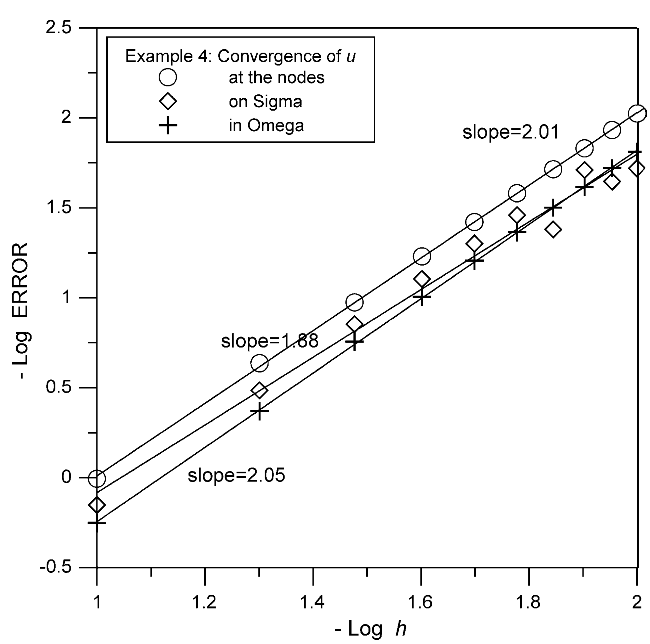

Fig. 9. Example 4: convergence rate of Trefftz–Herrera collocation method using linear weighting functions.

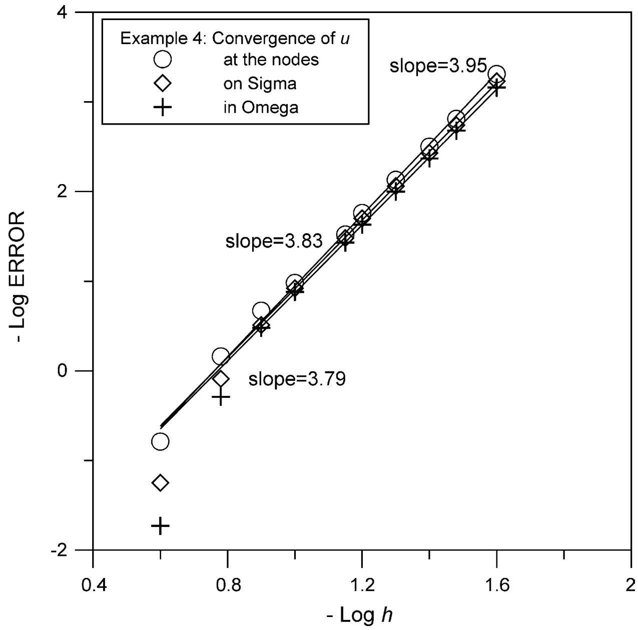

Fig. 10. Example 4: convergence rate of Trefftz–Herrera collocation method using cubic weighting functions.

As for Algorithm 2, the error behavior is O(h^4) and yields global matrices with the following properties:

i. They are block nine-diagonal, the blocks being at most 3×3; and
ii. The number of degrees of freedom associated with each internal node is three.

Because of the above features, both algorithms can be easily and effectively parallelized. Furthermore,

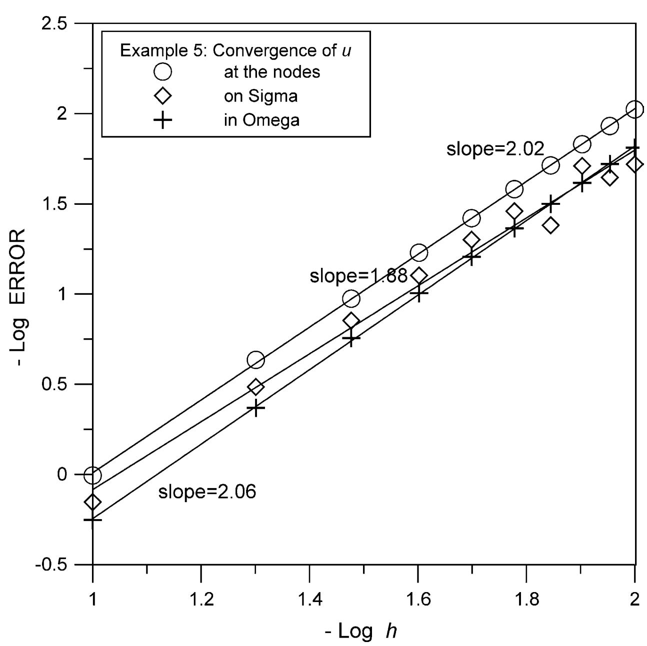

Fig. 11. Example 5: convergence rate of Trefftz–Herrera collocation method using linear weighting functions.

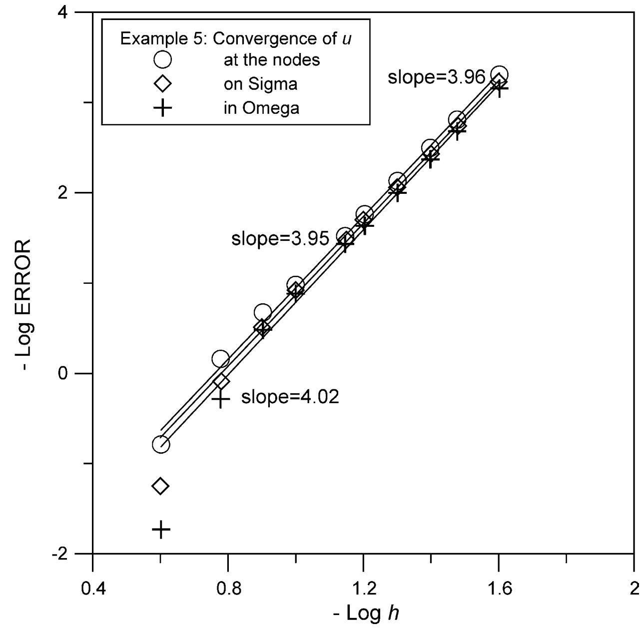

Fig. 12. Example 5: convergence rate of Trefftz–Herrera collocation method using cubic weighting functions.

the Conjugate Gradient Method is directly applicable, as has been done in [3].

It is natural to compare these algorithms with those derived by means of OSC (Hermite bi-cubic orthogonal spline collocation), which is the approach to collocation that has been more extensively used up to now. Regarding Algorithm 1, it must be mentioned that using OSC it is impossible to derive an algorithm that is strictly nine-diagonal. On the other hand, Algorithm 2, which is fourth order, and OSC have the same order of accuracy; so, they can be compared directly. When OSC is applied for solving the Poisson equations

Table 2
Convergence rates of Algorithm 1

Example	$\|u-\tilde{u}\|_\infty$		$\|u_\Sigma-\tilde{u}_\Sigma\|_\infty$		$\|u_\Omega-\tilde{u}_\Omega\|_\infty$	
	C	α	C	α	C	α
1	−1.87	2.00	−1.87	2.00	−2.08	1.99
2	−1.95	2.00	−1.91	1.86	−2.27	2.05
3	−0.86	2.02	−0.81	1.89	−0.94	1.83
4	−2.00	2.01	−1.96	1.88	−2.30	2.05
5	−2.01	2.02	−1.97	1.88	−2.31	2.06

Table 3
Convergence rates of Algorithm 2

Example	$\|u-\tilde{u}\|_\infty$		$\|u_\Sigma-\tilde{u}_\Sigma\|_\infty$		$\|u_\Omega-\tilde{u}_\Omega\|_\infty$	
	C	α	C	α	C	α
1	−2.00	4.06	−2.19	4.10	−2.48	4.23
2	−3.01	3.96	−3.18	4.01	−2.93	3.80
3	0.53	3.99	−0.17	3.99	−0.46	3.96
4	−3.00	3.95	−2.91	3.83	−2.92	3.79
5	−3.01	3.96	−3.08	3.95	−3.23	4.02

occurring in previous sections, the number of degrees of freedom associated with each internal node is four, and the global matrices are non-symmetric and non-positive. As mentioned before, when the TH-collocation Algorithm 2 is used instead, the number of degrees of freedom associated with each internal node is only three and the global matrices are symmetric and positive definite. Thus, a significant reduction of the number of degrees of freedom is achieved when TH-collocation is applied. Furthermore, because of the features explained above the algorithms derived using OSC cannot be easily parallelized and the Conjugate Gradient Method cannot be directly applied, while the opposite holds true when TH-collocation is used. This is probably the most important advantage of TH-collocation procedures. Even so, a method, which seems to be the most competitive one thus far developed for the biharmonic equation using OSC, deserves mention. It is due to Lou et al. [11], it is based on the splitting approach and combines OSC with a fast solver previously developed. However, because of the very special properties in which this is based, it can only be applied to Poisson equation in rectangular regions with rectangular meshes.

Here, for developing Algorithms 1 and 2, rectangular elements were used, but linear and cubic interpolation can also be applied in other kind of elements, such as triangles. When this is done, the global matrices associated with the resulting algorithms are sparse and enjoy the properties, common to all procedures that are based on Trefftz–Herrera domain decomposition approach, mentioned above: they are symmetric and positive definite, and their construction involves solving local problems exclusively. If rectangular elements are used, as has been done in this paper, the methods here presented can be applied in domains with shapes of considerable generality, since the problem domain does not need to be a rectangle, and many of the nice matrix properties are preserved. However, the geometrical diversity of the possible problem domains is enhanced if the method is applied using rectangular meshes of the type reported Barrera-Sanchez et al. (see [15,16]), or triangles.

As mentioned previously, it can be shown that Algorithm 2 yields exactly the same solution as the OSC. Therefore, our method derives the same solution as OSC but using much simpler matrices. In conclusion, the results of the present article exhibit the versatility of Trefftz–Herrera domain decomposition approach in one more instance. Other types of boundary conditions for the biharmonic equation can also be treated by means of the splitting approach, but not all of them. Observe that this limitation is shared by all procedures that are based on the splitting approach, including *Liu's* algorithm. However, more general boundary conditions can be dealt with applying TH-collocation directly to the original biharmonic equation, without splitting, but they need some special developments and the problem in its full generality is being investigated at present.

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