Theory of Differential Equations in Discontinuous Piecewise-Defined Functions

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A truly general and systematic theory of finite element methods (FEM) should be formulated using, as trial and test functions, piecewise-defined functions that can be *fully* discontinuous across the internal boundary, which separates the elements from each other. Some of the most relevant work addressing such formulations is contained in the literature on discontinuous Galerkin (dG) methods and on Trefftz methods. However, the formulations of partial differential equations in discontinuous functions used in both of those fields are indirect approaches, which are based on the use of Lagrange multipliers and mixed methods, in the case of dG methods, and the frame, in the case of Trefftz method. This article addresses this problem from a different point of view and proposes a theory, formulated in discontinuous piecewise-defined functions, which is direct and systematic, and furthermore it avoids the use of Lagrange multipliers or a frame, while mixed methods are incorporated as particular cases of more general results implied by the theory. When boundary value problems are formulated in discontinuous functions, well-posed problems are boundary value problems with prescribed jumps (BVPJ), in which the boundary conditions are complemented by suitable jump conditions to be satisfied across the internal boundary of the domain-partition. One result that is presented in this article shows that for elliptic equations of order $2m$, with $m \geq 1$, the problem of establishing conditions for existence of solution for the BVPJ reduces to that of the "standard boundary value problem," without jumps, which has been extensively studied. Actually, this result is an illustration of a more general one that shows that the same happens for any differential equation, or system of such equations that is linear, independently of its type and with possibly discontinuous coefficients. This generality is achieved by means of an algebraic framework previously developed by the author and his collaborators. A fundamental ingredient of this algebraic formulation is a kind of Green's formulas that simplify many problems (some times referred to as Green-Herrera formulas). An important practical implication of our approach is worth mentioning: "avoiding the introduction of the Lagrange multipliers, or the 'frame' in the case of Trefftz-methods, significantly reduces the number of degrees of freedom to be dealt with." Numer Methods Partial Differential Eq 23: 000–000, 2007

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1. INTRODUCTION

A basic feature of finite element methods (FEM) and many other related numerical methods for partial differential equations is the use, after a partition of the problem-domain has been introduced, of trial and test functions that are piecewise-defined; i.e., they are defined separately in each one of the partition subdomains. Here, it is remarked that the most general class of piecewise-defined functions includes functions that are *fully* discontinuous (by this we mean that the function itself has a jump discontinuity) across the internal boundary (i.e., that which separates the partition subdomains from each other). This, because such functions are defined independently in each one of the partition subdomains and, on the common boundary of two subdomains, the limits from one and the other side need not coincide. Thus, a truly general and systematic theory of FEM should be formulated in function spaces in which trial and test functions can be fully discontinuous across the internal boundary. Such a theory would include discontinuous Galerkin (dG) methods and should permit moving smoothly, without interruption, from the standard FEM, based on continuous piecewise-defined functions, to the discontinuous Galerkin methods.

Most FEM methods that exist at present use piecewise-defined functions with certain degree of continuity; typically, for second-order elliptic equations the functions are taken from the Sobolev space $H^1(\Omega)$, in which the functions are continuous with possibly discontinuous first-order derivatives [1]. As for dG methods, they originally were introduced with the main purpose of treating hyperbolic systems of equations, albeit their scope has expanded considerably during the last few years (see [2]). Furthermore, it should be mentioned that the interior penalty methods that were applied in the 1970s to treat elliptic and parabolic equations [3–7] must be classified, using the present day terminology, as dG methods. Nevertheless, the recent broadening of the interest on dG methods was heralded by works such as that of Bassi and Rebay [8], on the numerical solution of the Navier-Stokes equations (see [9] for an updated review of dG methods for elliptic problems). Other important contributions are: the local discontinuous method (LDG) [10], the Galerkin/least-squares [11], stabilized methods (SUPG/SD [12] and USFEM [13]), residual free bubbles (RFB [14–17]), variational multiscale (VMS) [18], the partition of unity method (PUM) [19], and nearly optimal Petrov-Galerkin [20]. At present, some of the most popular procedures applied when formulating dG methods combine hybrid and mixed methods with Lagrange multipliers using a procedure explained by Brezzi and Fortin [21]. This yields several variants such as the three-field method [22] (see also [23]), as well as the hybrid variational formulation with weak continuity on which the discontinuous enrichment method (DEM, [24–28]) is based.

Formulating the basic problems in fully discontinuous functions and handling them systematically afterwards would be advantageous for many methods; among them: dG methods, Trefftz methods [29–33], domain decomposition methods (DDM) [23, 34–38], and collocation methods, as well as matrix condensation. In this respect, a general theory of partial differential equations in which fully discontinuous trial and test functions could be handled systematically would be very valuable. Several of the dG methods formulations of the forgoing paragraphs have yielded very relevant results. However, they are indirect approaches whose essential ingredients are Lagrange multipliers and mixed methods. In others, the discontinuous functions are introduced only after the dimensions of the function spaces have been reduced to a finite number (see, for example [9]). This article is intended as a contribution to the development of a theory of partial differential equations in which trial and test functions are piecewise-defined functions that are fully discontinuous. The formulation to be presented is direct and systematic and, furthermore, it avoids the use of Lagrange multipliers, while mixed methods are incorporated as particular cases of more general results implied by the theory (see Section VII of [39]).

Trefftz methods [29–33] also make extensive use of discontinuous functions to build the approximate solutions of partial differential equations. They need to do so because they apply locally analytical solutions that do not fulfill any matching conditions across the inter-element boundaries. At present, a convergence of interests of dG and Trefftz methods is taking place, since some approaches of the former also use, locally, analytical solutions. Indeed, for instance, Farhart and his collaborators have extensively treated Helmholtz problems using DEM that they apply using, as the enricher, a complete system of plane waves, which constitutes an analytical C-complete system [40] (also called T-complete or TH-complete) and that was first developed by the author and his collaborators in [41, p 480]; for other similar systems of enrichers (see [42, 43]). Thereby, it should be mentioned that Trefftz method, which is extensively used by a large community of practitioners [32], has deep connections with some dG methods, such as DEM; in particular, the frame that was introduced by Jirousek in Trefftz method [29, 30], plays essentially the same role as that of the Lagrange multipliers in DEM [33]. A frequent complaint is the increased number of degrees of freedom that the incorporation of Lagrange multipliers yields [28], and the same phenomenon occurs due to the frame in the case of Trefftz method [33]. In this connection, an important practical implication of our approach is worth mentioning: "The theory of partial differential equations formulated in function spaces in which trial and test functions can be fully discontinuous permits avoiding the introduction of Lagrange multipliers, in the case of dG methods, and the frame, in the case of Trefftz methods." Naturally, any implementation of our approach should not contain any such auxiliary functions, since the basic formulation does contain them either. Anyway, to be more clear and specific, the basic ideas of how to implement our procedures have been explained in [33], where also an illustration has been thoroughly discussed.

When partial differential equations are formulated in discontinuous piecewise-defined functions, the well-posed problems are boundary value problems with prescribed jumps (BVPJ), in which the boundary conditions are complemented by suitable jump conditions, to be satisfied across the internal boundary associated with the domain partition. One result that is presented in this article shows that for elliptic problems of order $2m$, with $m \geq 1$, the BVPJ satisfies existence if and only if the standard smooth boundary value problem does. Thus, this result essentially reduces the problem of establishing conditions for existence of solution for the boundary value problem with prescribed jumps to that of the standard boundary value problem, without jumps, which has been extensively studied. Actually, this result for elliptic problems is only a particular case of a more general one (Theorem 8.3 of this article), which establishes that the same is true for any linear differential equation, or system of such equations, independently of its type and with possibly discontinuous coefficients. Thereby, an attractive feature of the theory of partial differential equations in piecewise-defined functions here presented is exhibited: its ability for establishing statements of broad applicability.

This generality is achieved by means of an algebraic formulation that has been developed through a long time span ([40, 44–58]; see also [31, 33, 37–39]). It identifies and makes extensive use of some algebraic properties of boundary value problems. In the first stages of its development it was capable of supplying a general framework, which accommodated practically all variational principles for boundary value problems known at the time [52–55]. It also encompassed Trefftz methods [56, 57], biorthogonal systems of functions [58], and a criterion for completeness [40] (originally introduced as C-completeness, but later known as T-completeness, or TH-completeness). This theory also yields a suitable framework for the development of complete systems of solutions (see [59], Ch. II, where the exposition is based on

Herrera's T-completeness or TH-completeness, concept). Furthermore, according to Begehr and Gilbert the algebraic theory [44] supplies the basis for effectively applying to boundary value problems the function theoretic methods of partial differential equations. Indeed, in [59, p 115], they state: "The function theoretic approach which was pioneered by Bergman [60] and Vekua [61] and then further developed by Colton [62–64], Gilbert [65, 66], Kracht-Kreyszig [67], and Lanckau [68] and others, may now be effectively applied because of this result of the formulation by Herrera [44] as an effective means to solving boundary value problems."

On the other hand, the algebraic theory has also been useful for establishing the theoretical foundations of Trefftz methods. This time the citation comes from J. Jirousek, one of the most conspicuous representatives of Trefftz methods [29, p 324]: "the mathematical foundations of which [referring to Trefftz methods] have been laid mainly by Herrera and co-workers." In 1984, the Pitman's Advanced Publishing Program collected many of the results of the theory in a book [44]. It was immediately afterwards that the study of differential equations in discontinuous fields started with the introduction of a kind of Green's formulas [46–49], sometimes referred to as Green-Herrera formulas, which simplify many problems and have played a central role in later developments [31, 33, 37–39, 50, 51, 69–76]. This more recent work phase of the theory includes certain number of applications. Among them are the introduction of the Localized Adjoint Method (LAM) [49] that in turn supplied the theoretical basis of the Eulerian-Lagrangean LAM (ELLAM), a numerical method that has had considerable success in treating advection-dominated transport [39, 69, 70]; more advanced applications to Trefftz method [31, 33, 71] and studies of several aspects of domain decomposition methods [37, 38, 71]; and a general class of methods [72–78] that can be collectively denominated as "finite elements methods with optimal functions (FEM-OF)" [78]. This latter kind of methods is more general than LAM and has yielded some very effective procedures for applying orthogonal collocation [72–77].

The present article begins by recalling some elementary algebraic notions in Section 2 and progressively introduces some more focused concepts that supply the basic structure of the algebraic theory of boundary value problems with prescribed jumps. Section 3 is devoted to auxiliary concepts that will be used in the sequel. An abstract and general concept of boundary value problem is introduced in Section 4. The notions of Dirichlet and appropriate boundary operators that play an important role in the theory are given in Section 5, while Sections 6 and 7 are devoted to formal adjoints and Green's formulas. The abstract BVPJ, which is the main subject of the present article, is introduced and discussed in Section 8. It is in this Section where the general result mentioned before (Theorem 8.3) is derived. The class of piecewise-defined functions considered in the theory is made precise in Sections 9 and 10. In particular the concept of a Sobolev space of piecewise-defined functions is introduced in this latter Section, and the manner in which they are related to standard Sobolev spaces is discussed in Section 11. Piecewise-defined functions have been considered by other authors; however, the nomenclature and the point of view adopted here differs somewhat. The general BVPJ for elliptic operators in Sobolev spaces of piecewise-defined functions is introduced in Section 12 and Theorem 8.3 is applied to it. To do this, extensive use is made of results of the classical mathematical theory of partial differential equations [1, 79]. Finally, the conclusions of the article are summarized in Section 13.

Background material of the theory here presented has appeared in scattered publications; many of them have already been mentioned and are included in the list of references given at the end of the article. However, this is the first time that the question of developing a theory of partial differential equations in discontinuous piecewise-defined functions has been addressed in a systematic manner. To this end, a large number of new developments were required, while the material that was already available was thoroughly revised and reorganized. In particular, the nomenclature was improved, making it more systematic. Furthermore, to make it more readable, the present article is to a large extent self-contained.

2. PROBLEMS WITH LINEAR CONSTRAINTS

Many problems of partial differential equations may be formulated as problems with linear constraints ([55]; see also [44]). So, in this section D, I_1, and I_2 will be a linear space and two of its linear subspaces, respectively. Elements of D will be referred to as functions.

Definition 2.1. *The problem with linear constraints (PLC). Given a pair of functions $(u_\Omega, u_\partial) \in D \times D$, the "problem with linear constraints" consists in finding $u \in D$ such that*

$$u - u_\Omega \in I_1 \qquad \text{and} \qquad u - u_\partial \in I_2. \tag{2.1}$$

The pair $(u_\Omega, u_\partial) \in D \times D$ will be referred to as the data of this problem.

Definition 2.2. *The subspaces D^S and N. Define*

$$D^S \equiv I_1 + I_2 \subset D \qquad \text{and} \qquad N \equiv I_1 \cap I_2 \subset D. \tag{2.2}$$

Definition 2.3. *Consistency of the data. The data of the problem with linear constraints, $(u_\Omega, u_\partial) \in D \times D$, is said to be consistent when*

$$u_\Omega - u_\partial \in D^S. \tag{2.3}$$

Theorem 2.1. *The problem with linear constraints possesses a solution if and only if the data are consistent.*

Proof. We show first

$$(\exists u \in D \text{ solution of the problem with linear constraints}) \Rightarrow u_\Omega - u_\partial \in D^S. \tag{2.4}$$

Assume $\exists u \in D$ solution of the problem with linear constraints. Then,

$$u_\Omega - u_\partial = \{(u - u_\partial) - (u - u_\Omega)\} \in I_1 + I_2 \equiv D^S. \tag{2.5}$$

Conversely, next we show: $u_\Omega - u_\partial \in D^S \Rightarrow$ ($\exists u \in D$ solution of the problem with linear constraints). Assume $u_\Omega - u_\partial \in D^S$ and let $(u_\Omega - u_\partial)_i \in I_i$, $i = 1, 2$, be such that

$$u_\Omega - u_\partial = (u_\Omega - u_\partial)_1 + (u_\Omega - u_\partial)_2. \tag{2.6}$$

Then, it can be seen that the function $u \in D$, defined by

$$u \equiv u_\Omega - (u_\Omega - u_\partial)_1 \equiv u_\partial + (u_\Omega - u_\partial)_2, \tag{2.7}$$

is solution of the problem with linear constraints. Indeed

$$u - u_\Omega = -(u_\Omega - u_\partial)_1 \in I_1 \qquad \text{while } u - u_\partial = (u_\Omega - u_\partial) - (u_\Omega - u_\partial)_1 = (u_\Omega - u_\partial)_2 \in I_2. \tag{2.8}$$

This exhibits $u \in D$, as defined by Eq. (2.7), as a solution of the PLC. Thus, Eq. (2.3) implies the existence of solution.

Corollary 2.1. *The problem with linear constaints, with data* $(u_\Omega, 0) \in D \times D$, *possesses a solution, if and only if,* $u_\Omega \in D^S$.

Proof. Apply Theorem 2.1 with $u_\partial = 0$.

3. AUXILIARY CONCEPTS

In this section we introduce some algebraic concepts, which supply a suitable framework for a large class of weak formulations of partial differential equations. The notation D_1 and D_2 will be used for two linear spaces, to be referred to as the spaces of "trial and test functions," respectively. Sometimes the terms "base and weighting functions" will be used instead. Furthermore, $P : D_1 \times D_2 \to \mathbf{R}^1$, $B : D_1 \times D_2 \to \mathbf{R}^1, \ldots$, will be bilinear functionals, while $P^* : D_2 \times D_1 \to \mathbf{R}^1$, $B^* : D_2 \times D_1 \to \mathbf{R}^1, \ldots$, will stand for their transposes. The value of any bilinear functional $P : D_1 \times D_2 \to \mathbf{R}^1$, on $(u, w) \in D_1 \times D_2$, will be denoted by $\langle Pu, w\rangle \equiv \langle P^*w, u\rangle$. The linear spaces D_1^* and D_2^* will be the "algebraic duals" of D_1 and D_2, respectively (i.e., the elements of D_1^* and D_2^* are linear functionals, real-valued, defined on D_1 and D_2, respectively). Furthermore, with each bilinear functional of the type considered above, we associate in a unique manner, a linear "functional-valued operator." Indeed, given $P : D_1 \times D_2 \to \mathbf{R}^1$, we define $P : D_1 \to D_2^*$ for every $u \in D_1$ the linear functional $Pu \in D_2^*$ by

$$Pu(w) \equiv \langle Pu, w\rangle, \qquad \forall w \in D_2. \tag{3.1}$$

This establishes a one-to-one correspondence between bilinear functionals, $P : D_1 \times D_2 \to \mathbf{R}^1$, and functional-valued operators, $P : D_1 \to D_2^*$, that are linear. Thus, in what follows, we identify both.

Definition 3.1. *TH-completeness for operators. A set* $\mathbf{E} \subset D_2$ *is said to be "TH-complete" for* $P : D_1 \to D_2^*$ *when*

$$\langle Pu, w\rangle = 0, \qquad \forall w \in \mathbf{E} \Rightarrow Pu = 0. \tag{3.2}$$

The concepts that are introduced next are discussed more thoroughly in the Appendix.

Definition 3.2. *Conjugate subspaces and TH-completeness for subsets. An ordered pair of linear subspaces* $\{I_1, I_2\}$, $I_1 \subset D_1$ *and* $I_2 \subset D_2$, *is said to be "conjugate," with respect to* $R : D_1 \to D_2^*$, *when*

$$\langle Ru, w\rangle = 0, \qquad \forall (u, w) \in I_1 \times I_2. \tag{3.3}$$

i. Let $\{I_1, I_2\}$ be a conjugate pair, while $\mathbf{E}_1 \subset \mathrm{I}_1 \subset \mathrm{D}_1$ and $\mathbf{E}_2 \subset I_2 \subset D_2$. Then $\mathbf{E}_1$ is said to be TH-complete for I_2 when

$$\langle Ru, w\rangle = 0, \qquad \forall\, u \in \mathbf{E}_1 \Rightarrow w \in I_2. \tag{3.4}$$

Similarly, $\mathbf{E}_2$ is said to be TH-complete for I_1 when

$$\langle Ru, w\rangle = 0, \qquad \forall\, w \in \mathbf{E}_2 \Rightarrow u \in I_1. \tag{3.5}$$

ii. A conjugate pair is said to be "regular," when in addition $I_1 \supset N_R$ and $I_2 \supset N_{R^*}$.

iii. A conjugate pair, $\{I_1, I_2\}$, is said to be "completely regular" when I_1 is TH-complete for I_2 and, conversely, I_2 is TH-complete for I_1.

Remark 3.1. Notice the similarities and differences between the two concepts of TH-completeness: one for subsets that has been just been introduced and one for operators that was given in Definition 3.1. Observe also that a pair of linear subspaces $\{I_1, I_2\}$, $I_1 \subset D_1$ and $I_2 \subset D_2$, is completely regular if and only if

$$\langle Ru, w\rangle = 0, \qquad \forall w \in I_2 \Leftrightarrow u \in I_1 \tag{3.6}$$

and

$$\langle Ru, w\rangle = 0, \qquad \forall u \in I_1 \Leftrightarrow w \in I_2. \tag{3.7}$$

It can be seen every completely regular pair is necessarily a regular pair. In particular, $I_1 \supset N_R$ and $I_2 \supset N_{R^*}$ when $\{I_1, I_2\}$ is a completely regular pair.

Consider now a family of operators $\mathbf{F} \equiv \{R_i : i = 1, \ldots, n\}$; $R_i : D_1 \to D_2^*$. Given such a family we associate with it a collection of pairs of subspaces, $\{(I_{1i}, I_{2i}) : i = 1, \ldots, n\}$, defined by

$$I_{1i} = \bigcap_{i \neq j} N_{R_j} \qquad \text{and} \qquad I_{2i} \equiv \bigcap_{i \neq j} N_{R_j^*} \tag{3.8}$$

Definition 3.3. *Operator decomposition. Let $R : D_1 \to D_2^*$ be given. Then a family $\mathbf{F} \equiv \{R_i | i = 1, \ldots, n\}$ of operators, $R_i : D_1 \to D_2^*$, such that for each $i = 1, \ldots, n$ one has*

$$\left.\begin{array}{l} I_{1i} \text{ is TH-complete for } R_i^* \\ I_{2i} \text{ is TH-complete for } R_i \end{array}\right\} \tag{3.9}$$

is said to be an operator decomposition of R (or, simply, a decomposition of R), when

$$R = \sum_{i=1}^{n} R_i. \tag{3.10}$$

4. THE ABSTRACT BOUNDARY VALUE PROBLEM

Definition 4.1. *Boundary Operator. An operator $B : D_1 \to D_2^*$ is said to be a boundary operator for $P : D_1 \to D_2^*$ when $N_{B^*} \subset D_2$ is TH-complete for P.*

Lemma 4.1. *Assume B is a boundary operator for P. Then for every pair* $(u, \mathbf{v}) \in D_1 \times D_1$, *one has*

$$Pu - B\mathbf{v} = 0 \Leftrightarrow Pu = 0 \quad \text{and} \quad B\mathbf{v} = 0. \tag{4.1}$$

Furthermore,

$$N_{P-B} = N_P \cap N_B. \tag{4.2}$$

Proof. We prove

$$Pu - B\mathbf{v} = 0 \Leftrightarrow Pu = 0 \quad \text{and} \quad B\mathbf{v} = 0. \tag{4.3}$$

Clearly

$$Pu = 0 \quad \text{and} \quad B\mathbf{v} = 0 \Rightarrow Pu - B\mathbf{v} = 0. \tag{4.4}$$

To prove the converse, observe that the equation $Pu - B\mathbf{v} = 0$ means

$$\langle Pu - B\mathbf{v}, w\rangle = 0, \qquad \forall w \in D_2. \tag{4.5}$$

When this latter equation holds, we have in particular

$$\langle Pu, w\rangle = \langle Pu - B\mathbf{v}, w\rangle = 0, \qquad \forall w \in N_{B^*}. \tag{4.6}$$

Hence, by Definition 4.1, $Pu = 0$. When this latter equation holds, the equality $Pu - B\mathbf{v} = 0$ reduces to $B\mathbf{v} = 0$. Finally, Eq. (4.2) is equivalent to

$$(P - B)u = 0 \Leftrightarrow Pu = 0 \quad \text{and} \quad Bu = 0. \tag{4.7}$$

And this is Eq. (4.1) when $u = \mathbf{v}$.

Definition 4.2. *The boundary value problem* (*BVP*). *Let* $B : D_1 \to D_2^*$ *be a boundary operator for* $P : D_1 \to D_2^*$. *Then the* (*abstract*) *BVP consists in, given* $(f, g) \in P(D_1) \times B(D_1)$, *finding* $u \in D_1$ *such that*

$$Pu = f \quad \text{and} \quad Bu = g. \tag{4.8}$$

Frequently, the equations occurring in Eq. (4.8) *will be referred to as the* "*differential equation*" *and the* "*boundary conditions,*" *respectively. We will say that the BVP satisfies existence when it possesses at least one solution. Furthermore, when required for precision, the BVP just defined will be referred to as the BVP associated with the pair* (*P*, *B*).

Notice that the condition $(f, g) \in P(D_1) \times B(D_1)$ is tantamount to requiring the existence of a pair $(u_\Omega, u_\partial) \in (D_1 \times D_1)$ such that

$$f = Pu_\Omega \quad \text{and} \quad g = Bu_\partial. \tag{4.9}$$

Problems for which $(f, g) \notin P(D_1) \times B(D_1)$ lack interest, since they do not possess any solution, and Definition 4.2 excludes them.

Definition 4.3. *Weak formulation of the abstract BVP (ABVP). The equation*

$$\langle (P - B)u, w\rangle = \langle f - g, w\rangle, \qquad \forall w \in D_2 \tag{4.10}$$

or, equivalently,

$$(P - B)u = f - g \tag{4.11}$$

will be referred to as the weak formulation of the ABVP. Notice that Eqs. (4.10) *and* (4.11) *are indeed equivalent, because an equality between two linear functionals defined in D_2 is fulfilled if and only if their values at each member of D_2 are equal.*

Lemma 4.2. *A function $u \in D_1$ fulfills the weak formulation of the BVP, if and only if, u is solution of the BVP.*

Proof. Using the functions $(u_\Omega, u_\partial) \in D_1 \times D_1$, Eq. (4.11) is equivalent to

$$P(u - u_\Omega) - B(u - u_\partial) = 0. \tag{4.12}$$

By Lemma 4.1, this equation is equivalent to $Pu = Pu_\Omega \equiv f$ and $Bu = Bu_\partial = g$.

Introducing the pair of functions $(u_\Omega, u_\partial) \in (D_1 \times D_1)$, the BVP can be formulated as follows: "Find $u = D_1$ such that

$$u - u_\Omega \in N_P \qquad \text{and} \qquad u - u_\partial \in N_B. \tag{4.13}$$

Equation (4.13) reduces the BVP to the problem with linear constraints discussed in Section 2, if we define

$$I_1 \equiv N_P \qquad \text{and} \qquad I_2 \equiv N_B. \tag{4.14}$$

Then, the definitions and results presented in that section become available. In particular: the BVP possesses a solution if and only if $u_\Omega - u_\partial \in D_1^S$, where

$$D_1^S \equiv N_P + N_B \subset D_1. \tag{4.15}$$

Furthermore, the linear subspace D_1^S is characterized by the fact that $u_\Omega \in D_1^S$, if and only if, the BVP

$$Pu = Pu_\Omega \qquad \text{and} \qquad Bu = 0 \tag{4.16}$$

possesses a solution. The following Definition introduces some additional subspaces that will play significant roles in the theory.

Definition 4.4. *The subspaces* $D_1^R \subset D_1$, $N_2^{\otimes} \subset D_2$, $N_2^R \subset D_2$, *and* $N_2^S \subset D_2$. *We define the linear subspaces*:

$$N_2^{\otimes} \equiv N_{(P-B)^*} \equiv \{w \in D_2 | \langle (P-B)^* w, u\rangle = 0, \ \forall u \in D_1\} \subset D_2 \tag{4.17}$$

$$D_1^R \equiv \{u \in D_1 | \langle Pu, w\rangle = 0, \ \forall w \in N_2^{\otimes}\} \subset D_1 \tag{4.18}$$

$$N_2^R \equiv \{w \in D_2 | \langle (P-B)^* w, u\rangle = 0, \ \forall u \in D_1^R\}$$
$$N_2^S \equiv \{w \in D_2 | \langle (P-B)^* w, u\rangle = 0, \ \forall \ u \in D_1^S\}. \tag{4.19}$$

Remark 4.1. When $w \in N_2^{\otimes}$, for every $u \in D_1$ one has $\langle Pu, w\rangle = \langle Bu, w\rangle$, since $\langle (P - B)u, w\rangle = 0$. Therefore, Eq. (4.18) is equivalent to

$$D_1^R \equiv \{u \in D_1 | \langle Bu, w\rangle = 0, \ \forall w \in N_2^{\otimes}\} \subset D_1. \tag{4.20}$$

Thus the roles played by P and B in the definition of D_1^R are symmetrical. Furthermore, if $u \in N_P$, then $\langle Pu, w\rangle = 0$ for every $w \in N_2^{\otimes}$; and if $u \in N_B$, then $\langle Pu, w\rangle = \langle Bu, w\rangle = 0$ for every $w \in N_2^{\otimes}$. Hence, $N_P \subset D_1^R$, $N_B \subset D_1^R$, and $N_P + N_B \subset D_1^R$. So

$$D_1^S \subset D_1^R \subset D_1 \qquad \text{and} \qquad N_2^S \supset N_2^R \supset N_2^{\otimes}. \tag{4.21}$$

5. DIRICHLET AND APPROPRIATE BOUNDARY OPERATORS

To avoid excessive repetitions, throughout this section it is assumed that $B : D_1 \to D_2^*$ is a boundary operator for $P : D_1 \to D_2^*$ and the BVP associated with the pair (P, B) is the only one to be considered.

Lemma 5.1. *Write* $g \equiv Bu_\partial$ *and* $f \equiv Pu_\Omega$. *Then,* $u_\Omega - u_\partial \in D_1^R$ *if and only if*

$$\langle f - g, w\rangle = 0, \qquad \forall w \in N_2^{\otimes}. \tag{5.1}$$

Proof. Because

$$\langle f - g, w\rangle = \langle Pu_\Omega - Bu_\partial, w\rangle = \langle P(u_\Omega - u_\partial), w\rangle, \qquad \forall w \in N_2^{\otimes}. \tag{5.2}$$

Using Eq. (4.18) of Definition 4.4, Lemma 5.1 follows.

Theorem 5.1. *Equation* (5.1) *is a necessary condition for the existence of solution of the BVP.*

Proof. This theorem follows as a Corollary of Lemma 5.1, since $D_1^S \subset D_1^R$.

Recall that the subspaces N_2^S, N_2^R, and $N_2^{\otimes}$ are nested [Eq. (4.21)]. When N_2^S is as small as possible, one has $N_2^S = N_2^R = N_2^{\otimes}$.

Definition 5.1. *Let* $B : D_1 \to D_2^*$ *be a boundary operator for* $P : D_1 \to D_2^*$. *Then*

I. B is said to be a "Dirichlet operator for P" when

$$N_2^S = N_2^R = N_2^{\otimes}. \tag{5.3}$$

II. B is said to be an "appropriate operator for P" when

$$D_1^S = D_1^R. \tag{5.4}$$

Remark 5.1. Observe that, in view of Definitions 4.4 and 5.1, B is a Dirichlet operator for P, if and only if, $D_1^S \equiv N_P + N_B$ is TH-complete for $(P - B)^*$.

The next theorem supplies a property that can be used as an equivalent alternative definition of appropriate boundary operator.

Theorem 5.2. *B is an appropriate boundary operator for P if and only if, Eq.* (5.1) *is a necessary and sufficient condition for the existence of solution of the BVP.*

Proof. When $D_1^S = D_1^R$, then by virtue of Lemma 5.1, Eq. (5.1) is a necessary and sufficient condition for $u_\Omega - u_\partial \in D_1^S$. Conversely, when Eq. (5.1) is a sufficient condition for $u_\Omega - u_\partial \in D_1^S$, then $D_1^S \supset D_1^R$, which in turn implies $D_1^S = D_1^R$.

Theorem 5.3. *When $B : D_1 \to D_2^*$ is a boundary operator for $P : D_1 \to D_2^*$, one has*

i. If N_B is TH-complete for $(P - B)^*$, then B is a Dirichlet operator for P; and
ii. $P(N_B) \supset P(D_1^R)$, if and only if, B is an appropriate operator for P.

Proof. Recall $D_1^S \equiv N_P + N_B \supset N_B$. Thus, the statement N_B is TH-complete for $(P - B)^*$ implies D_1^S is TH-complete for $(P - B)^*$. Hence, Proposition i. Next, we tackle ii. First, we prove

$$P(N_B) \supset P(D_1^R) \Rightarrow D_1^S \supset D_1^R. \tag{5.5}$$

Now, given any $u \in D_1^R$, choose $u_B \in N_B$ such that $Pu_B = Pu$, which is possible when $P(N_B) \supset P(D_1^R)$. Then, $u = (u - u_B) + u_B$, where $(u - u_B) \in N_P$, while $u_B \in N_B$. Thus, $u \in N_P + N_B = D_1^S$. Hence, $u \in D_1^R \Rightarrow u \in D_1^S$; i.e., $D_1^S \supset D_1^R$. This establishes Eq. (5.5).

Second, we prove

$$N_P + N_B \equiv D_1^S \supset D_1^R \Rightarrow P(N_B) \supset P(D_1^R). \tag{5.6}$$

Let any $f \in P(D_1^R) \subset D_2^*$ be given, which can be written as $f \equiv Pu$ with $u \in D_1^R$. When, $N_P + N_B \supset D_1^R$ one can chose $u_P \in N_P$ and $u_B \in N_B$ such that $u = u_P + u_B$. Then, $P(u) = P(u_B) \in P(N_B)$. Thus, $u \in P(D_1^R)$ implies $u \in P(N_B)$, and the relation $P(N_B) \supset P(D_1^R)$ is clear. Hence, the Theorem.

6. FORMAL ADJOINTS AND GREEN'S FORMULAS

Definition 6.1. *Formal adjoints. Let $P : D_1 \to D_2^*$ and $Q : D_2 \to D_1^*$ be operators. Define $R \equiv P - Q^*$. Then, $P : D_1 \to D_2^*$ and $Q : D_2 \to D_1^*$, are said to be "formal adjoints" of each other, when R is a boundary operator for P, while R^* is a boundary operator for Q.*

Definition 6.2. *Green's formula. Let* $\{B_1, \ldots, B_n\}$ *and* $\{C_1, \ldots, C_n\}$ *be two families of operators, where* $B_i : D_1 \to D_2^*$ *and* $C_i : D_2 \to D_1^*$, $i = 1, \ldots, n$. *Assume* $P : D_1 \to D_2^*$ *and* $Q : D_2 \to D_1^*$ *are formal adjoints, while the family of operators is* $\{B_1, \ldots, B_n, -C_1^*, \ldots, -C_n^*\}$ *a decomposition of R. Then, the equation*

$$P - \sum_{i=1}^{n} B_i = Q^* - \sum_{i=1}^{n} C_i^* \tag{6.1}$$

is said to be a "Green's formula."

Only two possible values of n will be discussed in this article: $n = 1$ and $n = 2$. In the next section we deal with the case $n = 2$. Here, we take $B_1 \equiv B$ and $C_1 \equiv C$. Then, Eq. (6.1) yields

$$P - B = Q^* - C^*. \tag{6.2}$$

Remark 6.1. When Eq. (6.2) is satisfied and it is a Green's formula, then (see the Appendix):

$$\left.\begin{array}{r} B \text{ is a boundary operator for } C^* \\ C^* \text{ is a boundary operator for } B \\ C \text{ is a boundary operator for } B^* \\ B^* \text{ is a boundary operator for } C \end{array}\right\}. \tag{6.3}$$

This is equivalent to

$$\left.\begin{array}{r} N_B \text{ is TH-complete for } C \\ N_C \text{ is TH-complete for } B \\ N_{C^*} \text{ is TH-complete for } B^* \\ N_{B^*} \text{ is TH-complete for } C^* \end{array}\right\}. \tag{6.4}$$

Then

$$N_{P-Q^*} = N_B \cap N_{C^*} \qquad \text{and} \qquad N_{P^*-Q} = N_{B^*} \cap N_C. \tag{6.5}$$

Theorem 6.1. *When Eq.* (6.2) *is a Green's formulas, then*

1. B is a boundary operator for P;
2. C is a boundary operator for Q;
3. N_B is TH-complete for Q;
4. N_B is TH-complete for $(P - B)^* = Q - C$;
5. B is a Dirichlet (boundary) operator for P; and
6. $N_2^{\otimes} = N_Q \cap N_C$.

Proof. In view of Remark 6.1, one has

$$N_{B^*} \supset N_{B^*} \cap N_C = N_{R^*}. \tag{6.6}$$

This implies 1), since $R = P - Q^*$ is a boundary operator for P (Definition 6.1). A symmetrical argument yields 2). Part 3) follows from $N_B \supset N_B \cap N_{C^*} = N_R$ since R^* is a boundary operator for Q. To prove 4), assume

$$\langle (Q - C)w, u \rangle = 0, \qquad \forall u \in N_B. \tag{6.7}$$

Then

$$\langle (Q - C)w, u \rangle = \langle Qw, u \rangle = 0, \qquad \forall u \in N_R = N_B \cap N_{C^*} \subset N_B. \tag{6.8}$$

Equation (6.8) implies $Qw = 0$ and Eq. (6.7) reduces to

$$\langle Cw, u \rangle = 0, \qquad \forall u \in N_B. \tag{6.9}$$

This equation implies $Cw = 0$ since N_B is TH-complete for C (Remark 6.1), and the proof of 4) is complete. Part 5) follows from Part 4), since

$$\langle (P - B)^* w, u \rangle = \langle (Q - C)w, u \rangle = 0, \qquad \forall u \in N_B \subset D_1^S \Rightarrow w \in N_{Q-C} = N_{(P-B)^*}. \tag{6.10}$$

That is, $N_2^S \subset N_{(P-B)^*} = N_2^{\otimes}$, which implies $N_2^S = N_2^R = N_2^{\otimes}$, by virtue of Eq. (4.21). Finally, in view of Eq. (4.2) (Lemma 4.1) one has $N_{Q-C} = N_Q \cap N_C$; then Part 6) is clear since $N_2^S = N_{(P-B)^*} = N_{Q-C}$.

Corollary 6.1. *When Eq.* (6.2) *is a Green's formula and B is an appropriate operator for P, then the BVP posses a solution, if and only if*

$$\langle f - g, w \rangle = 0, \qquad \forall w \in N_{Q-C} = N_Q \cap N_C. \tag{6.11}$$

Proof. Because $N_2^{\otimes} \equiv N_{(P-B)^*} = N_Q \cap N_C$.

7. GREEN'S FORMULAS WITH TWO TERMS

Throughout this section it is assumed that $P : D_1 \to D_2^*$ and $Q : D_2 \to D_1^*$ are formal adjoints, while $R \equiv P - Q^*$ and Green's formulas of the form

$$P - B - J = Q^* - C^* - K^* \tag{7.1}$$

are discussed. Equation (6.1) reduces to Eq. (7.1) when $n = 2$, $B_1 \equiv B$, $B_2 \equiv J$, $C_1 \equiv C$, and $C_2 \equiv K$.

Lemma 7.1. *When Eq.* (7.1) *is a Green's formulas one has*

$$\text{I.} \quad \left. \begin{array}{l} N_K \text{ is TH-complete for } P, B, \text{ and } J \\ N_J \text{ is TH-complete for } Q, C, \text{ and } K \end{array} \right\}. \tag{7.2}$$

II. $N_R = N_{B+J-C^*-K^*} = N_B \cap N_J \cap N_{C^*} \cap N_{K^*}$. (7.3)

III. $N_{R^*} = N_{B^*+J^*-C-K} = N_{B^*} \cap N_{J^*} \cap N_C \cap N_K$. (7.4)

IV. $N_{P-B} = N_P \cap N_B$ and $N_{Q-C} = N_Q \cap N_C$. (7.5)

V. $N_{B+J} = N_B \cap N_J$ and $N_{C+K} = N_C \cap N_K$. (7.6)

VI. The system of operators $\{(B + J), -(C^* + K^*)\}$ decomposes R and

$$P - (B + J) = Q^* - (C + K)^* \tag{7.7}$$

is a Green's formula.

Proof. It is given in the Appendix. In Section 8, the results of Lemma 7.1 will be extensively used.

Corollary 7.1. *When Eq.* (7.1) *is a Green's formulas*, $(B + J)$ *is a boundary operator for* P, *while* $(C + K)$ *is a boundary operator for* Q.

Proof. This is implied by Part 1) of Theorem 6.1, since Eq. (7.7) is a Green's formula.

8. THE BOUNDARY VALUE PROBLEM WITH PRESCRIBED JUMPS

Throughout this and the next sections we consider linear spaces D_1, $\widehat{D}_1$, D_2, and $\widehat{D}_2$. It will be assumed that $D_1 \supset \widehat{D}_1$ and $D_2 \supset \widehat{D}_2$. Functions $u \in \widehat{D}_1$ and $w \in \widehat{D}_2$ will be said to be "smooth." The operators $P : D_1 \to D_2^*$, $B : D_1 \to D_2^*$, $J : D_1 \to D_2^*$, $Q : D_2 \to D_1^*$, $C : D_2 \to D_1^*$, and $K : D_2 \to D_1^*$ will be considered, together with $\widehat{P} : \widehat{D}_1 \to \widehat{D}_2^*$, $\widehat{P} : \widehat{D}_1 \to \widehat{D}_2^*$, $\widehat{Q} : \widehat{D}_2 \to \widehat{D}_2^*$, and $\widehat{C} : \widehat{D}_2 \to \widehat{D}_2^*$. It is assumed throughout that B is boundary operator for P. Furthermore, the following notations will be used:

$$\left.\begin{array}{l} N_2^{\otimes} \equiv N_{(P-B-J)^*} \subset D_2;\ \widehat{N}_2^{\otimes} \equiv N_{(\widehat{P}-\widehat{B})^*} \subset \widehat{D}_2 \\ D_1^R \equiv \{u \in D_1 | \langle Pu, w\rangle = 0,\ \forall w \in N_2^{\otimes}\} \subset D_1; \\ \widehat{D}_1^R \equiv \{u \in \widehat{D}_1 | \langle \widehat{P}u, w\rangle = 0,\ \forall w \in \widehat{N}_2^{\otimes}\} \subset D_1 \\ D_1^S \equiv N_P + N_{B+J}; \qquad \widehat{D}_1^S \equiv N_{\widehat{P}} + N_{\widehat{B}} \end{array}\right\}. \tag{8.1}$$

Definition 8.1. *The boundary value problem with prescribed jumps* (*BVPJ*). *Let* $f \in P(D_1)$, $g \in B(D_1)$, *and* $u_\Sigma \in D_1$ *be given. The BVPJ consists in finding* $u \in D_1$ *such that*

$$Pu = f,\ Bu = g \quad \text{and} \quad u - u_\Sigma \in \widehat{D}_1. \tag{8.2}$$

The condition $u - u_\Sigma \in \widehat{D}_1$ *will be referred to as the "jump conditions." Here, it is assumed that boundary and jump conditions are "compatible." By this we mean that there exists a function* $u_{\partial\Sigma} \in D_1$, *with the property that* $Bu_{\partial\Sigma} = g$ *and* $u_\Sigma - u_{\partial\Sigma} \in \widehat{D}_1$.

Definition 8.2. *Jump operator.* $J : D_1 \to D_2^*$ *is said to be a "jump operator" for the BVPJ when*

A. B+J is a boundary operator for P;

B. $N_{B+J} = N_B \cap N_J.$ (8.3)

C. $N_J = \widehat{D}_1.$ (8.4)

When J is a jump operator, we will write $j = Ju_\Sigma$ and Eq. (8.2), *in Definition* 8.1, *is equivalent to*

$$Pu = f, \qquad Bu = g, \qquad \text{and} \qquad Ju = j. \tag{8.5}$$

The assumption of compatibility of boundary and jump conditions is tantamount to the condition that there exists a function $u_{\partial\Sigma} \in D_1$ such that $g = Bu_{\partial\Sigma}$ and $j = Ju_{\partial\Sigma}$.

Three linear functionals $(f, g, j) \in P(D_1) \times B(D_1) \times J(D_1)$, which will be referred to as the "data of the BVPJ," are then sufficient for defining the BVPJ.

When a jump operator is available, the BVPJ can be transformed into a BVP as stated in the theorem that follows.

Theorem 8.1. *Assume that $J : D_1 \to D_2^*$ is a jump operator for the BVPJ. Consider the BVP associated with the pair $(P, B + J)$:*

$$Pu = f \qquad \text{and} \qquad (B + J)u = g + j. \tag{8.6}$$

Then, $u \in D_1$ is a solution of this BVP, if and only if, it is a solution of the BVPJ of Eq. (8.5).

Proof. Clearly Eq. (8.5) implies Eq. (8.6). To prove the converse, we write the second equality in Eq. (8.6) as $(B + J)u = (B + J)u_{\partial\Sigma}$. When J is a jump operator, this latter equation implies $Bu = Bu_{\partial\Sigma} = g$ and $Ju = Ju_{\partial\Sigma} = j$.

Corollary 8.1. *Under the assumptions of Theorem* 8.1, *one has*

i. The BVPJ has the weak formulation

$$(P - B - J)u = f - g - j. \tag{8.7}$$

ii. $$N_{P-B-J} = N_P \cap N_B \cap N_J. \tag{8.8}$$

Proof. Part i) is because Eq. (8.7) is the weak formulation of the BVP of Theorem 8.1 (See Section 3). As for Part ii), itself is a Corollary of Part i). Indeed,

$$(P - B - J)u = 0 \Leftrightarrow Pu = 0, Bu = 0, Ju = 0. \tag{8.9}$$

Definition 8.3. *Extension of a Green's formula to the BVPJ. Assume that each one of the equations*

$$\widehat{P} - \widehat{B} = \widehat{Q}^* - \widehat{C}^* \tag{8.10}$$

and

$$P - B - J = Q^* - C^* - K^* \tag{8.11}$$

is a Green's formula. Then, Eq. (8.11) *is said to be an 'extension of the Green's formula of Eq.* (8.10) *to the BVPJ' when*

i. $\widehat{P}$, $\widehat{B}$, $\widehat{Q}^*$, and $\widehat{C}^*$ are restrictions to $\widehat{D}_1 \times \widehat{D}_2$ *of* P, $B + J$, Q^* and $C^* + K^*$, respectively; and

ii. $$N_J = \widehat{D}_1 \qquad \text{and} \qquad N_K = \widehat{D}_2. \tag{8.12}$$

Furthermore, such an extension is said to be "range invariant," when

$$P(D_1) = P(\widehat{D}_1). \tag{8.13}$$

Lemma 8.1. *Assume that Eq.* (8.10) *is a Green's formula and Eq.* (8.11) *is an extension of it to the BVPJ. Then*

A. $$\left.\begin{array}{l}\widehat{D}_2 \text{ is TH-complete for } P, B, \text{and } J\\ \widehat{D}_1 \text{ is TH-complete for } Q, C, \text{and } K\end{array}\right\}. \tag{8.14}$$

B. $$\left.\begin{array}{l}N_P \supset N_{\widehat{P}} = N_P \cap \widehat{D}_1, N_B \supset N_{\widehat{B}} = N_B \cap \widehat{D}_1\\ N_Q \supset N_{\widehat{Q}} = N_Q \cap \widehat{D}_2, N_C \supset N_{\widehat{C}} = N_C \cap \widehat{D}_2\end{array}\right\}. \tag{8.15}$$

C. Given any subsets $\widehat{E} \subset \widehat{D}_1$ and $\widehat{F} \subset \widehat{D}_1$, the following equivalence relation holds:
$$P(\widehat{E}) \supset P(\widehat{F}) \Leftrightarrow \widehat{P}(\widehat{E}) \supset \widehat{P}(\widehat{F}). \tag{8.16}$$

D. $$N_2^{\otimes} = \widehat{N}_2^{\otimes} = N_{\widehat{Q}} \cap N_{\widehat{C}} \tag{8.17}$$

E. $$D_1^R \supset \widehat{D}_1^R \tag{8.18}$$

F. $$N_{B+J} = N_{\widehat{B}}. \tag{8.19}$$

G. $$\left.\begin{array}{l}B + J \text{ is a boundary operator for } P\\ C + K \text{ is a boundary operator for } Q\end{array}\right\}. \tag{8.20}$$

H. J is a jump operator for the BVPJ; and

I. $$\left.\begin{array}{l}N_{P-B-J} = N_P \cap N_B \cap N_J\\ N_{Q-C-K} = N_Q \cap N_C \cap N_K\end{array}\right\}. \tag{8.21}$$

Proof. Part A) of this Lemma follows from Theorem 7.1, since $\widehat{D}_2 = N_K$. Part B) follows from Part A), because when $u \in N_{\widehat{P}}$ one has

$$\langle Pu, w\rangle = \langle \widehat{P}u, w\rangle = 0, \qquad \forall w \in \widehat{D}_2 \Rightarrow Pu = 0. \tag{8.22}$$

Then, Eq. (8.15) can be established. The relation

$$u - \mathbf{v} \in N_{\widehat{P}} \Leftrightarrow u - \mathbf{v} \in N_P. \tag{8.23}$$

for every pair $(u, \mathbf{v}) \in \widehat{D}_1 \times \widehat{D}_1$, is then clear. In turn, Eq. (8.23) can be used to prove C). In view of Eqs. (8.10) and (8.11), one has $(P - B - J)^* = Q - C - K$ together with $(\widehat{P} - \widehat{B})^* = \widehat{Q} - \widehat{C}$. Hence,

$$N_2^{\otimes} \equiv N_{(P-B-J)^*} = N_{Q-C-K} = N_Q \cap N_C \cap N_K = (N_Q \cap \widehat{D}_2) \cap (N_C \cap \widehat{D}_2) = N_{\widehat{Q}} \cap N_{\widehat{C}} = \widehat{N}_2^{\otimes}. \tag{8.24}$$

This shows D). As for E), it follows from the fact that when $\widehat{u} \in \widehat{D}_1^R$, one has

$$\widehat{u} \in D_1 \qquad \text{and} \qquad \langle P\widehat{u}, w\rangle = 0, \qquad \forall w \in \widehat{N}_2^{\otimes} = N_2^{\otimes}. \tag{8.25}$$

This, in turn, applying the definition of Eq. (8.1) implies that $\widehat{u} \in D_1^R$. Hence, E) is established. Furthermore, F) follows from Eq. (8.15), since

$$N_{B+J} = N_B \cap N_J = N_B \cap \widehat{D}_1. \tag{8.26}$$

As for G), it is a restatement of Part VI), Lemma 7.1. Now, it is straight forward to see that the conditions of Definition 8.2 are fulfilled; thereby it is shown that J is a jump operator for the BVPJ. Then, the first relation in Eq. (8.21) follows from Corollary 8.1, Part ii), while the second one is obtained by means of symmetric arguments, interchanging P, B, and J with Q, C, and K.

The results of Lemma 8.1 will be used extensively in the proofs of the following theorems.

Theorem 8.2. *Assume the Eq.* (8.10) *is a Green's formula and Eq.* (8.11) *is an extension of it to the BVPJ. Then $B + J$ is a* (*boundary*) *Dirichlet operator for P.*

Proof. The equation

$$P - (B + J) = Q^* - (C + K)^* \tag{8.27}$$

is a Green's formula, by virtue of Lemma 7.1. Then, this theorem follows from Theorem 6.1, Part 5).

The following theorem permits deriving the existence of the BVPJ of Eq. (8.5), which is equivalent to the BVPJ of Eq. (8.2), from the existence of solution of the corresponding smooth BVP: Find $\widehat{u} \in \widehat{D}_1$, such that

$$\widehat{P}\widehat{u} = \widehat{f} \qquad \text{and} \qquad \widehat{B}\widehat{u} = \widehat{g}, \tag{8.28}$$

where $\widehat{f} \equiv \widehat{P}\widehat{u}_\Omega \in \widehat{P}(\widehat{D}_1)$ and $\widehat{g} \equiv \widehat{B}\widehat{u}_\partial \in \widehat{B}(\widehat{D}_1)$.

Theorem 8.3. *Assume the Eq.* (8.10) *is a Green's formula and Eq.* (8.11) *is a range invariant extension of it, to the BVPJ, then, the following statements are equivalent*:

1. The BVPJ of Eq. (8.5) possesses a solution, if and only if,
$$\langle f - g - j, w\rangle = 0, \qquad \forall\, w \in N_Q \cap N_C \cap N_K \subset D_2. \tag{8.29}$$

2. The BVP of Eq. (8.28) possesses a solution, if and only if,

$$\langle \widehat{f} - \widehat{g}, \widehat{w} \rangle = 0, \qquad \forall \widehat{w} \in N_{\widehat{Q}} \cap N_{\widehat{C}} \subset \widehat{D}_2. \tag{8.30}$$

Furthermore,

$$N_{\widehat{Q}} \cap N_{\widehat{C}} = N_Q \cap N_C \cap N_K. \tag{8.31}$$

Proof. First a lemma that will be used in the sequel is established.

Lemma 8.2. *Under the assumptions of Theorem* 8.3, *one has*

$$P(N_{B+J}) \supset P(D_1^R) \Leftrightarrow \widehat{P}(N_{\widehat{B}}) \supset \widehat{P}(\widehat{D}_1^R). \tag{8.32}$$

Proof of the Lemma. Observe $P(N_{\widehat{B}}) \supset P(\widehat{D}_1^R) \Leftrightarrow \widehat{P}(N_{\widehat{B}}) \supset \widehat{P}(\widehat{D}_1^R)$, by virtue of Eq. (8.16), since $N_{\widehat{B}} \subset \widehat{D}_1$ and $\widehat{D}_1^R \subset \widehat{D}_1$. Recall now Eq. (8.18) and (8.19), to see that $P(N_{\widehat{B}}) = P(N_{B+J})$ and $P(D_1^R) \supset P(\widehat{D}_1^R)$. Therefore, $P(D_{B+J}) \supset P(D_1^R)$ implies $P(N_{\widehat{B}}) \supset P(\widehat{D}_1^R)$, which is equivalent to $\widehat{P}(N_{\widehat{B}}) \supset \widehat{P}(\widehat{D}_1^R)$. This establishes the implication $\Rightarrow$ in Eq. (8.32). The relation

$$P(D_1^R) = P(\widehat{D}_1^R) \tag{8.33}$$

will be used in the sequel. We proceed to establish it, under the assumptions of Theorem 8.3. In view of Eq. (8.18), it is only necessary to show that $P(D_1^R) \subset P(\widehat{D}_1^R)$. Now if $l \in P(D_1^R) \subset P(D_1) = P(\widehat{D}_1)$, there exists $\widehat{u} \in \widehat{D}_1$ such that $P(\widehat{u}) = l$ and therefore $\widehat{u} \in \widehat{D}_1^R$, since $\widehat{u}$ satisfies the conditions of Eq. (8.1). Now we proceed to prove the reverse implication $\Leftarrow$ in Eq. (8.32). Clearly, the relation $P(N_{B+J}) \supset P(D_1^R)$ follows from $P(N_{\widehat{B}}) \supset P(\widehat{D}_1^R)$, since $P(N_{B+J}) = P(N_{\widehat{B}})$ and $P(D_1^R) = P(\widehat{D}_1^R)$.

Proof of the Theorem. Applying Theorem 5.3, it is seen that the condition $P(N_{B+J}) \supset P(D_1^R)$ is necessary and sufficient for $B + J$ being an appropriate operator for P, while the condition $\widehat{P}(N_{\widehat{B}}) \supset \widehat{P}(\widehat{D}_1^R)$ is necessary and sufficient for $\widehat{B}$ being an appropriate operator for $\widehat{P}$. Furthermore,

- $B + J$ is an appropriate operator for P, if and only if, Eq. (8.29) is a necessary and sufficient condition for the existence of solution of the BVP of Eq. (8.6), by Theorem 5.1, which in turn is equivalent to the BVPJ of Eq. (8.5), by Theorem 8.5 and Part H) of Lemma 8.1, while
- $\widehat{B}$ is an appropriate operator for $\widehat{P}$, if and only if, Eq. (8.30) is a necessary and sufficient condition for the existence of solution of the BVP of Eq. (8.28), by Theorem 5.1.

Finally, Eq. (8.31) is clear because

$$N_{\widehat{P}} \cap N_{\widehat{B}} = N_P \cap \widehat{D}_1 \cap N_B \cap \widehat{D}_1 = N_P \cap N_B \cap N_J. \tag{8.34}$$

9. PIECEWISE-DEFINED FUNCTIONS

In what follows, $\Omega \subset R^n$ will be a domain, in the sense of Ciarlet [1], and $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$ a domain partition of Ω; i.e., it is assumed that

i. Ω_α, for $\alpha = 1, \ldots, E$ is a subdomain of Ω.

$$\text{ii.} \qquad \Omega_\alpha \cap \Omega_\beta = \phi, \qquad \text{whenever } \alpha \neq \beta. \tag{9.1}$$

$$\text{iii.} \qquad \Omega \subset \bigcup_{\alpha=1}^{E} \bar{\Omega}_\alpha. \tag{9.2}$$

The notations $\partial\Omega$ and $\partial\Omega_\alpha$, $\alpha = 1, \ldots, E$, are adopted for the boundaries of Ω and Ω_α, respectively. Clearly, $\partial\Omega \subset \cup_{\alpha=1}^{E} \partial\Omega_\alpha$. In addition, $\Sigma \subset \cup_{\alpha=1}^{E} \partial\Omega_\alpha$ is defined to be the closed complement of $\partial\Omega$, with respect to $\cup_{\alpha=1}^{E} \partial\Omega_\alpha$, and will be called the internal boundary, while $\partial\Omega$ is referred to as the outer boundary. Observe that the internal boundary is also characterized by

$$\Sigma = \bigcup_{\alpha\neq\beta} \partial\Omega_\alpha \cap \partial\Omega_\beta. \tag{9.3}$$

The notation

$$\partial_\alpha\Omega \equiv (\partial\Omega_\alpha) \cap (\partial\Omega) \qquad \text{and} \qquad \Sigma_\alpha \equiv (\partial\Omega_\alpha) \cap \Sigma \tag{9.4}$$

will also be used. Notice that in general $\partial_\alpha\Omega$ and $\partial\Omega_\alpha$ are different. Also, it is assumed that except for a set of measure zero, every point $\underline{x} \in \Sigma$ belongs to the boundaries of two and only two subdomians, $\partial\Omega_\alpha$ and $\partial\Omega_\beta$ say, with $\alpha \neq \beta$. Fruthermore, it is also assumed that Σ has been oriented, so that the positive and negative sides of Σ have been defined, almost everywhere (a.e.), on Σ.

In what follows, two functions defined in Ω are identified when the set of points where they differ has Lebesgue measure zero. Given the partition $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$, by a piecewise-defined function we mean a sequence of functions $(w_1, \ldots, w_E)$, such that for each $\alpha = 1, \ldots, E$, the function w_α is defined a.e. in Ω_α; the functions w_α are said to be "locally defined." When a function w is defined a.e. in Ω, we can associate to it, uniquely, a piecewise-defined function $(w_1, \ldots, w_E)$. Indeed, to this end the function w_α, for every $\alpha = 1, \ldots, E$, is taken to be the restriction of w to Ω_α. The sequence $(w_1, \ldots, w_E)$ will be referred to as the piecewise representation of w. Conversely, given any piecewise-defined function $(w_1, \ldots, w_E)$, there is unique function, w, defined in Ω, such that $(w_1, \ldots, w_E)$ is the piecewise representation of w; indeed, such a function is defined in Ω by the condition

$$w \equiv w_\alpha; \qquad \text{a.e. in } \Omega_\alpha, \qquad \alpha = 1, \ldots, E. \tag{9.5}$$

Observe that Eq. (9.5) does not define the function w on Σ. However, the definition of w on Σ is immaterial because the Lebesgue measure of Σ is zero, so, w can be arbitrarily defined on Σ. This procedures establish a one-to-one correspondence between piecewise defined functions and functions defined a.e. in Ω, that will be referred to as the natural immersion of one of these spaces into the other. From now on we identify both, the function and the sequence.

When considering piecewise-defined functions, if the trace of w_α, is defined a.e. on $\partial\Omega_\alpha$, for every $\alpha = 1, \ldots, E$, then two functions (w_+, w_-) are defined a.e. on the positive and negative sides of Σ, respectively. This permits defining the jump and the average of w across Σ, by

$$[w] \equiv w_+ - w_- \qquad \text{and} \qquad \dot{w} \equiv \frac{1}{2}(w_+ + w_-) \tag{9.6}$$

respectively. Then, the following identities are fulfilled:

$$w_+ = \dot{w} + \frac{1}{2}[w] \qquad \text{and} \qquad w_- = \dot{w} - \frac{1}{2}[w]. \tag{9.7}$$

It must be mentioned that in many applications the functions w_α will be vector-valued; i.e., they may take values in R^m.

10. SOBOLEV SPACES OF PIECEWISE DEFINED FUNCTIONS

Given a family of linear spaces, $\{D(\Omega_1), \ldots, D(\Omega_E)\}$, such that $D(\Omega_\alpha)$, for every $\alpha = 1, \ldots, E$, is a linear space of functions defined a.e. in Ω_α, one can consider the space

$$D(\Omega) \equiv D(\Omega_1) \oplus \cdots \oplus D(\Omega_E). \tag{10.1}$$

Then, the elements of $D(\Omega)$ are piecewise defined functions, $(w_1, \ldots, w_E)$, with $w_\alpha \in D(\Omega_\alpha)$, $\alpha = 1, \ldots, E$. An example of such linear spaces is the Sobolev space of piecewise defined functions of order p, which is defined by

$$\hat{H}^p(\Omega) \equiv H^p(\Omega_1) \oplus \cdots \oplus H^p(\Omega_E), \qquad p = 0, 1, \ldots. \tag{10.2}$$

Here, $H^p(\Omega_\alpha)$ is the Sobolev space of order p, of functions defined in Ω_α. Only integer values $p \geq 0$ will be considered. Every function $w \in \hat{H}^p(\Omega)$ is a sequence, $w \equiv (w_1, \ldots, w_E)$, with $w_\alpha \in H^p(\Omega_\alpha)$, $\alpha = 1, \ldots, E$.

Observe that when $w \in H^p(\Omega)$, then the restriction, w_α, of w to Ω_α has the property that $w_\alpha \in H^p(\Omega_\alpha)$. Therefore

$$H^p(\Omega) \subset \hat{H}^p(\Omega). \tag{10.3}$$

For $p > 0$ this is a proper inclusion. However, $H^0(\Omega) \equiv \hat{H}^0(\Omega) \equiv L^2(\Omega)$. Furthermore,

$$H^0(\Omega) \equiv \hat{H}^0(\Omega) \supset \hat{H}^p(\Omega), \qquad \forall p \geq 0. \tag{10.4}$$

Here, the functions defined in Ω have been identified with their piecewise representations, as explained in Section 9. In view of Eq. (10.4), all the spaces $\hat{H}^p(\Omega)$, for $p = 0, 1, 2, \ldots$, are made of functions which belong to $H^0(\Omega) \equiv L^2(\Omega)$.

Furthermore, for each $p \geq 0$, a function $\hat{u} \equiv (u_1, \ldots, u_E) \in H^0(\Omega)$ belongs to $\hat{H}^p(\Omega)$ if and only if the norm

$$\|\hat{\mathbf{v}}\|_{p,\Omega,\Pi} \equiv \left(\sum_{\alpha=1}^{E} \|\mathbf{v}_\alpha\|^2_{p,\Omega_\alpha} \right)^{1/2}. \tag{10.5}$$

is well defined. Here, the subscripts Ω and Π have been included to emphasize the fact that such norm depends not only on the domain Ω considered, but on the partition Π, as well. When $\hat{H}^p(\Omega)$ is equipped with the norm of Eq. (10.5), it becomes a Hilbert space. The family of subspaces $\{\hat{H}^p(\Omega)|p = 0, 1, 2, \ldots\}$ is a nested family of Hilbert spaces in the sense that

$$H^0(\Omega) \supset \hat{H}^p(\Omega) \supset H^q(\Omega), \tag{10.6}$$

when $0 \leq p \leq q$.

For greater clarity some special features, which are concomitant of observations already made and are somewhat different of other authors'—such as Ciarlet [1]—notations, should be emphasized. Due to the inclusion of Eq. (10.4), all functions considered are members of $H^0(\Omega) = L^2(\Omega)$ and as such they are not defined on point sets of zero Lebesgue measure. In particular, it does not make any sense to talk about the value of a function $u \in H^0(\Omega)$ on the outer boundary $\partial\Omega$ or on the inner boundary Σ. That notwithstanding, we consider the set of functions $C^0(\Omega) \subset H^0(\Omega)$, which is defined to be the natural immersion of $\tilde{C}^0(\Omega)$ (the set of functions that are continuous in Ω). More precisely, a function $u \in C^0(\Omega)$ if and only if there exists a continuous function $\tilde{u} \in \tilde{C}^0(\Omega)$ such that $u = \tilde{u}$ a.e. in Ω. The set $C^0(\bar{\Omega}) \subset H^0(\Omega)$ is defined similarly, replacing Ω by $\bar{\Omega}$ above. On the other hand, if the trace of w_α on $\partial\Omega_\alpha$ exists and belongs to $H^0(\partial\Omega_\alpha)$, for every $\alpha = 1, \ldots, E$, then the functions $[w] \equiv (w_+ - w_-) \in H^0(\Sigma)$ and $\dot{w} \equiv \frac{1}{2}(w_+ + w_-) \in H^0(\Sigma)$. Similarly, in such a case, the trace of u, on $\partial\Omega$, belongs to $H^0(\partial\Omega)$.

We recall that when $u_\alpha, w_\alpha \in H^1(\Omega_\alpha)$ the following Green formula holds (see Ciarlet [1]):

$$\int_{\Omega_\alpha} u_\alpha \frac{\partial w_\alpha}{\partial x_i} dx = -\int_{\Omega_\alpha} w_\alpha \frac{\partial u_\alpha}{\partial x_i} dx + \int_{\partial\Omega_\alpha} u_\alpha w_\alpha n_i dx. \tag{10.7}$$

Therefore,

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u_\alpha \frac{\partial w_\alpha}{\partial x_i} dx = -\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} w_\alpha \frac{\partial u_\alpha}{\partial x_i} dx + \sum_{\alpha=1}^{E} \int_{\partial\Omega_\alpha} u_\alpha w_\alpha n_i dx \tag{10.8}$$

or, equivalently

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u_\alpha \frac{\partial w_\alpha}{\partial x_i} dx = -\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} w_\alpha \frac{\partial u_\alpha}{\partial x_i} dx + \int_{\partial\Omega} uwn_i dx - \int_{\Sigma} [uw] n_i dx. \tag{10.9}$$

Here, the identity

$$\sum_{\alpha=1}^{E} \int_{\partial\Omega_\alpha} u_\alpha w_\alpha n_i dx = \int_{\partial\Omega} uwn_i dx - \int_{\Sigma} [uw] n_i dx \tag{10.10}$$

has been used. Thereby, Eq. (10.10) illustrates our notation; the outer normal to Ω_α is being used in each one of the integrals $\int_{\partial\Omega_\alpha} u_\alpha w_\alpha n_i dx$, which is specified by the subscript of the integral symbol; on the other hand, to evaluate the integral $\int_\Sigma [uw]n_i dx$ the unit normal is chosen arbitrarily and at the same time such a choice defines the orientation of Σ. Then, of course, the value of that integral is independent of that choice. Finally, as specified by the subscript $\partial\Omega$, the outer normal vector to Ω is used when evaluating $\int_{\partial\Omega} uwn_i dx$. Finally, it should be noticed that Eq. (10.9) can also be written as

$$\sum_{\alpha=1}^{E}\int_{\Omega_\alpha} u_\alpha \frac{\partial w_\alpha}{\partial x_i} dx + \sum_{\alpha=1}^{E}\int_{\Omega_\alpha} w_\alpha \frac{\partial u_\alpha}{\partial x_i} dx = \int_{\partial\Omega} uwn_i dx - \int_\Sigma (\dot{u}[w] + \dot{w}[u])n_i dx, \quad (10.11)$$

and, in particular, when $w \in \mathcal{D}(\Omega)$ Eq. (10.11) yields

$$\sum_{\alpha=1}^{E}\int_{\Omega_\alpha} w \frac{\partial u_\alpha}{\partial x_i} dx + \sum_{\alpha=1}^{E}\int_{\Omega_\alpha} u_\alpha \frac{\partial w}{\partial x_i} dx = -\int_\Sigma w[u]n_i dx. \quad (10.12)$$

Here, as in what follows, $\mathcal{D}$ (Ω) stands for the space whose members are functions of $\mathscr{C}^\infty(\Omega)$ with compact support contained in Ω.

11. RELATION BETWEEN DIFFERENT KINDS OF SOBOLEV SPACES

In the last section it was seen that $\hat{H}^p(\Omega) \supset H^p(\Omega)$, for every $p \geq 0$. A basic problem that is addressed in this section is how to characterize the space $H^p(\Omega)$ as a subset of $\hat{H}^p(\Omega)$. In particular, necessary and sufficient conditions for members $u \equiv (u_1, \ldots, u_E) \in \hat{H}^p(\Omega)$ to be a members of $H^p(\Omega)$, will be given.

When $m \geq 1$ and $u \equiv (u_1, \ldots, u_E) \in \hat{H}^m(\Omega)$, the traces of u_α, on $\partial\Omega_\alpha$, for $\alpha = 1, \ldots, E$, belong to $L^2(\partial\Omega_\alpha)$ [1]. Then, as explained in Section 9, after having oriented the internal boundary Σ the functions $u_+ \in L^2(\Sigma)$ and $u_- \in L^2(\Sigma)$, as well as the jump and the average across Σ, are well defined.

Lemma 11.1. *Let $u \in \hat{H}^1(\Omega)$, then*

i. For every $\phi \in \mathcal{D}(\Omega)$, one has

$$\sum_{\alpha=1}^{E}\int_{\Omega_\alpha} \phi \frac{\partial u_\alpha}{\partial x_i} dx + \int_\Omega u \frac{\partial \phi}{\partial x_i} dx = -\int_\Sigma \phi[u]n_i dx. \quad (11.1)$$

ii. $u \in H^1(\Omega)$, if and only if, $[u] = 0$ on Σ.

Proof. Let $u \equiv (u_1, \ldots, u_E) \in \hat{H}^1(\Omega) \subset H^0(\Omega)$. Then, by the definition of the distributional derivative, $u \in H^1(\Omega)$ if and only if, for every $i = 1, \ldots, n$, there is a function $w \in H^0(\Omega)$ with the following property:

$$\int_\Omega w\phi dx = -\int_\Omega u \frac{\partial\phi}{\partial x_i} dx, \qquad \forall \phi \in \mathcal{D}(\Omega). \quad (11.2)$$

When Eq. (11.2) is fulfilled, $w \in H^0(\Omega)$ is the distributional derivative of $u \in H^1(\Omega)$. Observe that

$$\int_\Omega u \frac{\partial \phi}{\partial x_i} dx = \sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u_\alpha \frac{\partial \phi}{\partial x_i} dx. \tag{11.3}$$

But, in view of the fact that $u_\alpha \in H^1(\Omega_\alpha)$ and $\phi|_{\Omega_\alpha} \in \mathscr{C}^\infty(\bar{\Omega}_\alpha) \subset H^1(\Omega_\alpha)$, for each $\alpha = 1, \ldots, E$, the following Green formula holds [1]:

$$\int_{\Omega_\alpha} u_\alpha \frac{\partial \phi}{\partial x_i} dx = -\int_{\Omega_\alpha} \phi \frac{\partial u_\alpha}{\partial x_i} dx + \int_{\partial\Omega_\alpha} u_\alpha \phi n_i dx, \tag{11.4}$$

with $(\partial u_\alpha / \partial x_i) \in H^0(\Omega_\alpha)$. Therefore,

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u_\alpha \frac{\partial \phi}{\partial x_i} dx = -\int_{\Omega} \phi \frac{\partial u}{\partial x_i} dx + \sum_{\alpha=1}^{E} \int_{\partial\Omega_\alpha} u_\alpha \phi n_i dx \tag{11.5}$$

or

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u_\alpha \frac{\partial \phi}{\partial x_i} dx = -\int_{\Omega} \phi \frac{\partial u}{\partial x_i} dx - \int_{\Sigma} \phi [u] n_i dx. \tag{11.6}$$

This equation implies Eq. (11.1). Furthermore, in view of Eq. (11.6), the function

$$w \equiv \left(\frac{\partial u_1}{\partial x_i}, \ldots, \frac{\partial u_E}{\partial x_i} \right) \in H^0(\Omega)$$

fulfills

$$\int_\Omega w \phi dx + \int_\Sigma \phi [u] n_i dx = -\int_\Omega u \frac{\partial \phi}{\partial x_i} dx. \tag{11.7}$$

Comparing Eq. (11.2) with Eq. (11.7), it is seen that Eq. (11.2) holds for every $\phi \in \mathscr{D}(\Omega)$ and every $i = 1, \ldots, n$, if and only if the function $[u] \in L^2(\Sigma)$ is the zero function. In particular, if for some $i = 1, \ldots, n$, $[u] n_i \neq 0$, the functional associated with the term $\int_\Sigma \phi [u] n_i dx$ is not continuous with respect to the norm $\| \, \|_{0,\Omega}$.

Lemma 11.2. *Let* $u \in \hat{H}^2(\Omega) \cap H^1(\Omega)$. *Then*

$$\left[\frac{\partial u}{\partial x_i} \right] = \left[\frac{\partial u}{\partial n} \right] n_i, \qquad \text{a.e. on } \Sigma. \tag{11.8}$$

Proof. When $u \equiv (u_1, \ldots, u_E) \in \hat{H}^2(\Omega) \cap H^1(\Omega)$, one has

$$\int_\Omega u \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx = -\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_j} dx = -\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \frac{\partial u_\alpha}{\partial x_i}\frac{\partial \phi}{\partial x_j} dx = \sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j} dx + \int_\Sigma \phi \left[\frac{\partial u}{\partial x_i}\right] n_j dx. \tag{11.9}$$

Interchanging i and j and subtracting the resulting equation, one gets

$$\int_\Sigma \phi \left\{ \left[\frac{\partial u}{\partial x_i}\right] n_j - \left[\frac{\partial u}{\partial x_j}\right] n_i \right\} dx = 0, \qquad \forall \phi \in \mathcal{D}(\Omega). \tag{11.10}$$

This implies

$$\left[\frac{\partial u}{\partial x_i}\right] n_j = \left[\frac{\partial u}{\partial x_j}\right] n_i, \qquad \text{a.e. on } \Sigma. \tag{11.11}$$

Multiplying by n_j and adding the resulting equations, for $j = 1, \ldots, n$, one gets

$$\left[\frac{\partial u}{\partial x_i}\right] = \left[\frac{\partial u}{\partial n}\right] n_i, \qquad \text{on } \Omega, i = 1, \ldots, n. \tag{11.12}$$

This is Eq. (11.8).

Lemma 11.3. *Let $u \in \hat{H}^2(\Omega) \cap H^1(\Omega)$, then*

i. For every $\phi \in \mathcal{D}(\Omega)$, one has

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial^2 u_\alpha}{\partial x_i \partial x_j} dx - \int_\Omega u \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx = -\int_\Sigma \phi \left[\frac{\partial u}{\partial v}\right] v_i v_j dx. \tag{11.13}$$

ii. $u \in H^2(\Omega)$ if and only if

$$\left[\frac{\partial u}{\partial n}\right] = 0. \tag{11.14}$$

Proof. Eqs. (11.9) and (11.11) together, yield Eq. (11.13). Once this has been shown, ii) is clear.

Observe that

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = \left(\frac{\partial^2 u_1}{\partial x_i \partial x_j}, \ldots, \frac{\partial^2 u_E}{\partial x_i \partial x_j}\right) \in H^0(\Omega),$$

when the jump of the normal derivative vanishes. In words, when $u \in \hat{H}^2(\Omega) \cap H^1(\Omega)$ and jump of the normal derivative across Σ vanishes, the second derivatives of u are obtained differentiating at each Ω_α ($\alpha = 1, \ldots, E$), separately.

In the following lemma the multi-index notation, which is essentially the same as that used by Ciarlet ([1]; see also Lions and Magenes [54]) is adopted: Given a multi-index $\underline{\lambda} \equiv (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$, the norm of $\underline{\lambda}$ is defined by $|\underline{\lambda}| \equiv \sum_{i=1}^{n} \lambda_i$.

Theorem 11.1. *Let $u \in \hat{H}^m(\Omega) \cap H^{m-1}(\Omega)$ and $\underline{\lambda} \equiv (\lambda_1, \dots, \lambda_n) \in \mathbf{N}^n$, with $|\underline{\lambda}| = m$. Then*

i. For every $\phi \in \mathcal{D}(\Omega)$, one has

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial^m u_\alpha}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx - (-1)^m \int_\Omega u \frac{\partial^m \phi}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx = -\int_\Sigma \phi \left[\frac{\partial^{m-1} u}{\partial n^{m-1}}\right] n_1^{\lambda_1} \cdots n_n^{\lambda_n} dx. \tag{11.15}$$

ii. $u \in H^m(\Omega)$ if and only if

$$\left[\frac{\partial^{m-1} u}{\partial n^{m-1}}\right] = 0. \tag{11.16}$$

Proof. We prove this Theorem by induction over m. For $m = 1$ and $m = 2$, it reduces to Lemmas 11.1 and 11.3, respectively. Assume it holds for $|\underline{u}| < m$, and let $\underline{\lambda}' \equiv (\lambda_1', \dots, \lambda_n') \in \mathbf{N}^n$ be such that $|\underline{\lambda}'| = m - 1$, with $\lambda_j' = \lambda_j$, except for $j = i$ in which case $\lambda_i' = \lambda_i - 1$. Here, i is any integer such that $1 \le i \le n$. Then

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial^m u_\alpha}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx = \sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial}{\partial x_i} \left(\frac{\partial^{m-1} u_\alpha}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right) dx = -\int_\Omega \frac{\partial^{m-1} u}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}} \frac{\partial \phi}{\partial x_i} dx$$
$$+ \int_\Sigma \phi \left[\frac{\partial^{m-1} u}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right] n_i dx. \tag{11.17}$$

Applying Lemma 11.2, one has

$$\left[\frac{\partial^{m-1} u}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right] = \left[\frac{\partial}{\partial x_j} \left(\frac{\partial^{m-2} u}{\partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right)\right] = \left[\frac{\partial}{\partial n} \left(\frac{\partial^{m-2} u}{\partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right)\right] n_j. \tag{11.18}$$

Then

$$\left[\frac{\partial^{m-1} u}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}}\right] n_i = \left[\frac{\partial^{m-1} u}{\partial^{m-1} n}\right] n_1^{\lambda_1} \cdots n_n^{\lambda_n} \tag{11.19}$$

by virtue of the induction assumption. Also

$$\int_\Omega \frac{\partial^{m-1} u}{\partial x_j \partial x_1^{\lambda_1'} \cdots \partial x_n^{\lambda_n'}} \frac{\partial \phi}{\partial x_i} dx = (-1)^{m-1} \int_\Omega u \frac{\partial^m \phi}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx, \tag{11.20}$$

since $u \in H^{m-1}(\Omega)$. Replacing these expressions into Eq. (11.17), it is obtained

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \phi \frac{\partial^m u_\alpha}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx = (-1)^m \int_{\Omega} u \frac{\partial^m \phi}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} dx - \int_{\Sigma} \phi \left[\frac{\partial^{m-1} u}{\partial^{m-1} n} \right] n_1^{\lambda_1} \cdots n_n^{\lambda_n} dx. \quad (11.21)$$

Rearranging this equality, Eq. (11.15) is obtained. Proposition ii) now follows as a corollary of Proposition i). Even more, observe that when $u \in H^m(\Omega)$ its derivatives are equal (a.e.) to those of $u_\alpha \in H^m(\Omega_\alpha)$, at each one of the subdomains of the partition.

Theorem 11.2. *Let $m \geq 1$ and assume $u \in \hat{H}^m(\Omega)$. Then, $u \in H^m(\Omega)$ if and only if*

$$\left[\frac{\partial^{\gamma-1} u}{\partial n^{\gamma-1}} \right] - 0, \qquad \text{for } \gamma = 1, \ldots, m. \quad (11.22)$$

Proof. This is a corollary of Theorem 11.1. It can be derived from this theorem by induction over m, starting with $m = 1$.

12. ELLIPTIC DIFFERENTIAL EQUATIONS WITH PRESCRIBED JUMPS

In this section, we mainly use Lions and Magenes' notations [79]:

i. The domain of definition of the problem $\Omega \subset \mathbf{R}^n$, as well as the partition subdomains, will be domains in the sense that they are assumed to be open, bounded, connected subsets with a Lipschitz-continuous boundary $\partial\Omega$ [1, 80]. In the next paragraphs the notations are explained for the whole domain Ω, but corresponding notations will used for each one of the patition subdomains: Ω_i, $i = 1, \ldots, E$.

ii. $\mathcal{D}(\Omega)$ will be the set of indefinitely differentiable functions in Ω, with compact support in Ω; while $\mathcal{D}(\bar{\Omega})$ is the set of infinitely differentiable functions in the closure of Ω;

iii. The elliptic differential operator $\mathbf{L}$ of order $2m$, with $m \geq 1$, and its formal adjoint $\mathbf{L}^*$, are defined by

$$\mathbf{L}u \equiv \sum_{|p|,|q| \leq m} (-1)^{|p|} D^p (a_{pq} D^q u) \quad \text{and} \quad \mathbf{L}^* w \equiv \sum_{|p|,|q| \leq m} (-1)^{|p|} D^p (a_{qp} D^q w), \quad (12.1)$$

with $a_{pq} \in D(\bar{\Omega})$ and it is properly elliptic in $\bar{\Omega}$. Here

$$D^\alpha \equiv \frac{\partial^{\alpha_1 + \ldots + \alpha_n}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad \alpha = \{\alpha_1, \ldots, \alpha_n\}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n. \quad (12.2)$$

iv. The "differential boundary operators" B_j^{LM}, of order m_j, are defined by

$$B_j^{LM} u \equiv \sum_{|h| \leq m_j} b_{jh} D^h u, \quad (12.3)$$

with $b_{jh} \in \mathcal{D}(\partial\Omega)$, $0 \leq m_j \leq 2m - 1$, the system $\{B_j^{LM}\}_{j=0}^{m-1}$ being normal on $\partial\Omega$ and covering $\mathbf{L}$ on $\partial\Omega$.

v. The systems of boundary operators $\{S_j^{LM}\}_{j=0}^{m-1}$, $\{C_j^{LM}\}_{j=0}^{m-1}$, and $\{T_j^{LM}\}_{j=0}^{m-1}$ are also normal on $\partial\Omega$, having infinitely differentiable coefficients and being of orders $0 \leq \mu_j \leq 2m - 1$, $2m - 1 - \mu_j$, and $2m - 1 - m_j$, respectively.

vi. Furthermore, each one of the systems $\{B_0^{LM}, \ldots, B_{m-1}^{LM}, S_0^{LM}, \ldots, S_{m-1}^{LM}\}$ and $\{C_0^{LM}, \ldots, C_{m-1}^{LM}, T_0^{LM}, \ldots, T_{m-1}^{LM}\}$ is a "differential Dirichlet system of order $2m$" on $\partial\Omega$ (see [54]). This assumption implies that the sets $\{m_0, \ldots, m_{m-1}\} \cup \{\mu_0, \ldots, \mu_{m-1}\}$ and $\{0, \ldots, 2\text{m} - 1\}$ are equal.

Modifying slightly the notation used in [79], let us define the "local" *trace* operator

$$\overline{\mathcal{P}}_{BLM} : u \rightarrow \overline{\mathcal{P}}_{BLM}u \equiv \{B_0^{LM}u, \ldots, B_{m-1}^{LM}u\}. \tag{12.4}$$

Here, for each $i = 0, \ldots, m - 1$ and each $\alpha = 1, \ldots, E$, $B_i^{LM}u$ stands for the trace on $\partial\Omega_\alpha$, so that $\overline{\mathcal{P}}_{BLM}$ is a transformation of functions belonging to $H^{2m}(\Omega_\alpha)$, and so defined in Ω_α, into functions defined on $\partial\Omega_\alpha$. In [79] it is shown that when the boundary $\partial\Omega_\alpha$ is an $n - 1$ dimensional infinitely differentiable variety, the image under this mapping of $H^{2m}(\Omega_\alpha)$ is given by

$$\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\} = \prod_{j=0}^{m-1} H^{2m-m_j-(1/2)}(\partial\Omega_\alpha). \tag{12.5}$$

In the developments that follow, where we only are assuming that each one of the partition subdomains $\Omega_\alpha \subset R^n$ is a domain with Lipschitz-continuous boundary $\partial\Omega_\alpha$, the *weaker* condition will be used:

$$\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\} \subset \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}}, \tag{12.6}$$

and

$$\text{span}(\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\}) = \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}}. \tag{12.7}$$

Here, the closure operation is taken with respect to the metric of the product space

$$\overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}}.$$

When the relation $\partial\Omega_\alpha \subset \cup_{\beta=1}^{E} \partial\Omega_\beta = \Sigma \cup \partial\Omega$ is taken into account and the natural embedding of $H^0(\partial\Omega_\alpha)$ into $H^0(\Sigma \cup \partial\Omega)$ is applied, it is clear that $H^0(\partial\Omega_\alpha) \subset H^0(\Sigma \cup \partial\Omega)$. Then

$$\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\} \subset \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}} \subset \overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}}. \quad (12.8)$$

In what follows, we will write $\text{span}(\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\})$ for the closure of $\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\}$, where the closure operation taken with respect to the metric of the product space

$$\overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}}.$$

Then, Eq. (12.8) implies

$$\text{span}(\overline{\mathcal{P}}_{BLM}\{H^{2m}(\Omega_\alpha)\}) \subset \overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}}. \quad (12.9)$$

In an analogous manner to the definition of Eq. (12.4), we define the local mappings

$$\begin{aligned} \overline{\mathcal{P}}_{SLM} : u &\rightarrow \overline{\mathcal{P}}_{SLM} u \equiv \{S_0^{LM} u, \ldots, S_{m-1}^{LM} u\} \\ \overline{\mathcal{Q}}_{TLM} : w &\rightarrow \overline{\mathcal{Q}}_{TLM} w \equiv \{T_0^{LM} w, \ldots, T_{m-1}^{LM} w\} \\ \overline{\mathcal{Q}}_{CLM} : w &\rightarrow \overline{\mathcal{Q}}_{CLM} w \equiv \{C_0^{LM} w, \ldots, C_{m-1}^{LM} w\}. \end{aligned} \quad (12.10)$$

Then, arguments that parallel those that led to Eqs. (12.7) and (12.9) permit writing

$$\left.\begin{aligned} \text{span}(\overline{\mathcal{P}}_{SLM}\{H^{2m}(\Omega_\alpha)\}) &= \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}} \\ \text{span}(\overline{\mathcal{Q}}_{TLM}\{H^{2m}(\Omega_\alpha)\}) &= \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}} \\ \text{span}(\overline{\mathcal{Q}}_{CLM}\{H^{2m}(\Omega_\alpha)\}) &= \overbrace{H^0(\partial\Omega_\alpha) \times \cdots \times H^0(\partial\Omega_\alpha)}^{m \text{ times}} \end{aligned}\right\} \quad (12.11)$$

and to

$$\left.\begin{aligned} \text{span}(\overline{\mathcal{P}}_{SLM}\{H^{2m}(\Omega_\alpha)\}) &\subset \overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}} \\ \text{span}(\overline{\mathcal{Q}}_{TLM}\{H^{2m}(\Omega_\alpha)\}) &\subset \overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}} \\ \text{span}(\overline{\mathcal{Q}}_{CLM}\{H^{2m}(\Omega_\alpha)\}) &\subset \overbrace{H^0(\Sigma \cup \partial\Omega) \times \cdots \times H^0(\Sigma \cup \partial\Omega)}^{m \text{ times}} \end{aligned}\right\}. \quad (12.12)$$

At this point, we define the spaces of trial and test functions to be

$$D_1 \equiv D_2 \equiv D \equiv \hat{H}^{2m}(\Omega), \quad (12.13)$$

while the subspaces of smooth functions are taken to be

$$\widehat{D}_1 \equiv \widehat{D}_2 \equiv \widehat{D} \equiv H^{2m}(\Omega). \tag{12.14}$$

As explained in Section 9, the internal boundary has been oriented, taking the unit normal vector $\underline{n}$ pointing toward the positive side. In the outer boundary $\partial\Omega$, the unit normal vector is always taken pointing outward. Then, the differential expressions $B_i^{LM}u$, $S_i^{LM}u$, $T_i^{LM}w$, and $C_i^{LM}w$ are computed using such normal vectors, and we introduce the following "global" trace mappings:

$$\left.\begin{array}{lll} \mathcal{P}^{\partial\Omega}_{BLM}u \equiv \{B_0^{LM}u, \ldots, B_{m-1}^{LM}u\} & \text{and} & \mathcal{P}^{\partial\Omega}_{SLM}u \equiv \{S_0^{LM}u, \ldots, S_{m-1}^{LM}u\} \\ \mathcal{Q}^{\partial\Omega}_{TLM}u \equiv \{T_0^{LM}w, \ldots, T_{m-1}^{LM}w\} & \text{and} & \mathcal{Q}^{\partial\Omega}_{CLM}u \equiv \{C_0^{LM}w, \ldots, C_{m-1}^{LM}w\} \end{array}\right\} \quad \text{on } \partial\Omega. \tag{12.15}$$

Together with

$$\left.\begin{array}{l} \mathcal{P}_J u \equiv \{[\![B_0^{LM}u]\!], \ldots, [\![B_{m-1}^{LM}u]\!], \quad [\![S_0^{LM}u]\!], \ldots, [\![S_{m-1}^{LM}u]\!]\} \\ \mathcal{P}_K u \equiv \{\{\{B_0^{LM}u\}\}, \ldots, \{\{B_{m-1}^{LM}u\}\}, \quad \{\{S_0^{LM}u\}\}, \ldots, \{\{S_{m-1}^{LM}u\}\}\} \end{array}\right\} \quad \text{on } \Sigma, \tag{12.16}$$

and

$$\left.\begin{array}{l} \mathcal{Q}_J w \equiv \{\{\{T_0^{LM}w\}\}, \ldots, \{\{T_{m-1}^{LM}w\}\}, \quad \{\{C_0^{LM}w\}\}, \ldots, \{\{C_{m-1}^{LM}w\}\}\} \\ \mathcal{Q}_K w \equiv \{[\![T_0^{LM}w]\!], \ldots, [\![T_{m-1}^{LM}w]\!], [\![C_0^{LM}w]\!], \ldots, [\![C_{m-1}^{LM}w]\!]\} \end{array}\right\} \quad \text{on } \Sigma. \tag{12.17}$$

Here, $[\![\ \ ]\!]$ and $\{\{\ \}\}$ stand for the "jump" and the "average," across Σ, of the functions that are within. Thus, for example,

$$[\![S_j^{LM}u]\!] \equiv (S_j^{LM}u)_+ - (S_j^{LM}u)_-$$

$$\{\{C_j^{LM}w\}\} \equiv \frac{1}{2}\{(C_j^{LM}w)_+ + (C_j^{LM}w)_-\}. \tag{12.18}$$

Then, Eqs. (12.7) and (12.11) imply

$$\left.\begin{array}{l} \operatorname{span}(\mathcal{P}^{\partial\Omega}_{BLM}\{\hat{H}^{2m}(\Omega)\}) = \overbrace{H^0(\partial\Omega) \times \cdots \times H^0(\partial\Omega)}^{m \text{ times}} \\ \operatorname{span}(\mathcal{P}^{\partial\Omega}_{SLM}\{\hat{H}^{2m}(\Omega)\}) = \overbrace{H^0(\partial\Omega) \times \cdots \times H^0(\partial\Omega)}^{m \text{ times}} \\ \operatorname{span}(\mathcal{Q}^{\partial\Omega}_{TLM}\{\hat{H}^{2m}(\Omega)\}) = \overbrace{H^0(\partial\Omega) \times \cdots \times H^0(\partial\Omega)}^{m \text{ times}} \\ \operatorname{span}(\mathcal{Q}^{\partial\Omega}_{CLM}\{\hat{H}^{2m}(\Omega)\}) = \overbrace{H^0(\partial\Omega) \times \cdots \times H^0(\partial\Omega)}^{m \text{ times}} \end{array}\right\}, \tag{12.19}$$

and

$$\left.\begin{aligned}
\text{span}(\mathcal{P}_J\{\hat{H}^{2m}(\Omega)\}) &= \overbrace{H^0(\Sigma)\times\cdots\times H^0(\Sigma)}^{2m\text{ times}}\\
\text{span}(\mathcal{P}_K\{\hat{H}^{2m}(\Omega)\}) &= \overbrace{H^0(\Sigma)\times\cdots\times H^0(\Sigma)}^{2m\text{ times}}\\
\text{span}(\mathcal{Q}_J\{\hat{H}^{2m}(\Omega)\}) &= \overbrace{H^0(\Sigma)\times\cdots\times H^0(\Sigma)}^{2m\text{ times}}\\
\text{span}(\mathcal{Q}_K\{\hat{H}^{2m}(\Omega)\}) &= \overbrace{H^0(\Sigma)\times\cdots\times H^0(\Sigma)}^{2m\text{ times}}
\end{aligned}\right\}. \tag{12.20}$$

Separately, in each one of the partition subdomains Ω_α, for every pair of functions $(u, w) \in H^{2m}(\Omega_\alpha) \times H^{2m}(\Omega_\alpha)$, the equation:

$$\int_{\Omega_\alpha} w\mathcal{L}u dx + \sum_{j=0}^{m-1}\int_{\partial\Omega_\alpha} B_j^{LM} u T_j^{LM} w dx = \int_{\Omega_\alpha} u\mathcal{L}^* w dx + \sum_{j=0}^{m-1}\int_{\partial\Omega_\alpha} S_j^{LM} u C_j^{LM} w dx \tag{12.21}$$

holds. Equation (12.21) can be written as

$$\int_{\Omega_\alpha} w\mathcal{L}u dx - \int_{\partial\Omega_\alpha} \mathcal{B}(u, w) dx = \int_{\Omega_\alpha} u\mathcal{L}^* w dx - \int_{\partial\Omega_\alpha} \mathcal{C}(w, u) dx, \tag{12.22}$$

if the definitions

$$\mathcal{B}(u, w) = -\sum_{j=0}^{m-1} B_j^{LM} u T_j^{LM} w \quad \text{and} \quad \mathcal{C}(w, u) \equiv -\sum_{j=0}^{m-1} S_j^{LM} u C_j^{LM} w \tag{12.23}$$

are adopted. Adding up the Eqs. (12.21) corresponding to each one of the partition subdomains, it is obtained

$$\sum_{\alpha=1}^{E}\int_{\Omega_\alpha} w\mathcal{L}u dx - \sum_{\alpha=1}^{E}\int_{\partial\Omega_\alpha} \mathcal{B}(u, w) dx = \sum_{\alpha=1}^{E}\int_{\Omega_\alpha} u\mathcal{L}^* w dx - \sum_{\alpha=1}^{E}\int_{\partial\Omega_\alpha} \mathcal{C}(w, u) dx. \tag{12.24}$$

Furthermore,

$$\left.\begin{aligned}
\sum_{\alpha=1}^{E}\int_{\partial\Omega_\alpha} \{\mathcal{B}(u, w) - \mathcal{C}(w, u)\} dx = \int_{\partial\Omega} \{\mathcal{B}(u, w) - \mathcal{C}(w, u)\} dx +\\
\int_{\Sigma} \{\mathcal{J}(u, w) - \mathcal{K}(w, u)\} dx
\end{aligned}\right\}, \tag{12.25}$$

where

$$\mathcal{J}(u,w) \equiv \mathcal{C}(\{\{w\}\}, [\![u]\!]) - \mathcal{B}([\![u]\!], \{\{w\}\})$$
$$\mathcal{K}(w,u) \equiv \mathcal{B}(\{\{u\}\}, [\![w]\!]) - \mathcal{C}(\{\{u\}\}, [\![w]\!]). \tag{12.26}$$

For later use, we observe that Eqs. (12.23) and (12.26) together imply that

$$\left.\begin{aligned} \mathcal{J}(u,w) &= \sum_{j=0}^{m-1} [\![S_j^{LM}u]\!]\{\{C_j^{LM}w\}\} - \sum_{j=0}^{m-1} [\![B_j^{LM}u]\!]\{\{T_j^{LM}w\}\} \\ \mathcal{K}(w,u) &= \sum_{j=0}^{m-1} \{\{S_j^{LM}u\}\}[\![C_j^{LM}w]\!] - \sum_{j=0}^{m-1} \{\{B_j^{LM}u\}\}[\![T_j^{LM}w]\!] \end{aligned}\right\} \quad \text{on } \Sigma. \tag{12.27}$$

for every $(u, w) \in D \times D$.

Using these results, Eq. (12.24) yields

$$\left.\begin{aligned} &\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} w\mathcal{L}u dx - \int_{\partial\Omega} \mathcal{B}(u,w)dx - \int_{\Sigma} \mathcal{J}(u,w)dx = \\ &\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u\mathcal{L}^* w dx - \int_{\partial\Omega_\alpha} \mathcal{C}(w,u)dx - \int_{\Sigma} \mathcal{K}(w,u)dx \end{aligned}\right\} \quad \forall (u,w) \in D \times D. \tag{12.28}$$

Furthermore, it can be seen that

$$\sum_{\alpha=1}^{E} \int_{\Omega_\alpha} w\mathcal{L}u dx - \int_{\partial\Omega} \mathcal{B}(u,w)dx = \sum_{\alpha=1}^{E} \int_{\Omega_\alpha} u\mathcal{L}^* w dx - \int_{\partial\Omega_\alpha} \mathcal{C}(w,u)dx, \qquad \forall (u,w) \in \widehat{D} \times \widehat{D}. \tag{12.29}$$

The operators $P : D \to D^*$, $Q : D \to D^*$, $B : D \to D^*$, $C : D \to D^*$, $J : D \to D^*$, and $K : D \to D^*$ are now defined, for every $(u, w) \in \hat{H}^{2m}(\Omega) \times \hat{H}^{2m}(\Omega)$, to be given by

$$\left.\begin{aligned} &\langle Pu, w\rangle \equiv \sum_{i=1}^{E} \int_{\Omega_i} w\mathcal{L}u dx \quad \text{and} \quad \langle Qw, u\rangle \equiv \sum_{i=1}^{E} \int_{\Omega} u\mathcal{L}^* w dx \\ &\langle Bu, w\rangle \equiv \int_{\partial\Omega} \mathcal{B}(u,w)dx \quad \text{and} \quad \langle Cw, u\rangle \equiv \int_{\partial\Omega} \mathcal{C}(w,u)dx \\ &\langle Ju, w\rangle \equiv \int_{E} \mathcal{J}(u,w)dx \quad \text{and} \quad \langle Kw, u\rangle \equiv \int_{\Sigma} \mathcal{K}(w,u)dx \end{aligned}\right\}. \tag{12.30}$$

Then Eq. (12.28) is

$$\langle Pu, w\rangle - \langle Bu, w\rangle - \langle Ju, w\rangle = \langle Qw, u\rangle - \langle Cw, u\rangle - \langle Kw, u\rangle, \qquad (12.31)$$

which holds for every $(u, w) \in \hat{H}^{2m}(\Omega) \times \hat{H}^{2m}(\Omega)$. Equation (12.31) can also be written as an identity between bilinear functionals:

$$P - B - J = Q^* - C^* - K^*. \qquad (12.32)$$

Recalling that $\widehat{D}_1 \equiv \widehat{D}_2 \equiv \widehat{D} \equiv H^{2m}(\Omega)$, we define $\widehat{P} : \widehat{D} \to \widehat{D}^*$, $\widehat{Q} : \widehat{D} \to \widehat{D}^*$, $\widehat{B} : \widehat{D} \to \widehat{D}^*$, and $\widehat{C} : \widehat{D} \to \widehat{D}^*$ as the restrictions to $\widehat{D} \times \widehat{D}$ of P, Q, B, and C, respectively. Then, Eq. (12.29) is

$$\widehat{P} - \widehat{B} = \widehat{Q}^* - \widehat{C}^*. \qquad (12.33)$$

Theorem 12.1. *Equation* (12.33) *is a Green's formula and Eq.* (12.32) *is a range invariant extension of it to the BVPJ.*

Proof. In view of Definition 8.3, we need to prove that each one of the Eqs. (12.32) and (12.33) is a Green's formula, together with

$$N_J = H^{2m}(\Omega) \qquad \text{and} \qquad N_K = H^{2m}(\Omega) \qquad (12.34)$$

and

$$\mathcal{L}\{\hat{H}^{2m}(\Omega)\} = \mathcal{L}\{H^{2m}(\Omega)\}. \qquad (12.35)$$

Equation (12.35) is clear because

$$\mathcal{L}\{\hat{H}^{2m}(\Omega)\} = H^0(\Omega) = \mathcal{L}\{H^{2m}(\Omega)\}. \qquad (12.36)$$

Let us define the family of operators by

$$\{R_1, R_2, R_3, R_4\} \equiv \{B, J, -C^*, -K^*\}. \qquad (12.37)$$

Then

$$\left.\begin{array}{l} I_{11} = \mathcal{P}_{BLM}\{\hat{H}^{2m}(\Omega)\};\ I_{12} = \mathcal{P}_J\{\hat{H}^{2m}(\Omega)\} \\ I_{13} = \mathcal{P}_{SLM}\{\hat{H}^{2m}(\Omega)\};\ I_{14} = \mathcal{P}_K\{\hat{H}^{2m}(\Omega)\} \end{array}\right\}, \qquad (12.38)$$

together with

$$\left.\begin{array}{l} I_{21} = \mathcal{Q}_{TLM}\{\hat{H}^{2m}(\Omega)\};\ I_{22} = \mathcal{Q}_J\{\hat{H}^{2m}(\Omega)\} \\ I_{23} = \mathcal{Q}_{CLM}\{\hat{H}^{2m}(\Omega)\};\ I_{24} = \mathcal{Q}_K\{\hat{H}^{2m}(\Omega)\} \end{array}\right\}. \qquad (12.39)$$

Then it is easy to verify, for every $i = 1, \ldots, n$, that

$$\langle R_i u, w\rangle = 0, \qquad \forall w \in I_{2i} \Rightarrow \langle R_i u, w\rangle = 0, \qquad \forall w \in D$$

$$\langle R_i u, w\rangle = 0, \qquad \forall\, u \in I_{1i} \Rightarrow \langle R_i u, w\rangle = 0, \qquad \forall\, u \in D. \tag{12.40}$$

This shows that the system of operators $\{R_1, R_2, R_3, R_4\}$, as defined above, decomposes $R \equiv P - Q^*$. Now, it can be seen that

$$N_J = \{u \in \hat{H}^2(\Omega) | \mathcal{P}_J u = 0\} \qquad \text{and} \qquad N_k = \{w \in \hat{H}^2(\Omega) | \mathcal{Q}_K w = 0\}. \tag{12.41}$$

Furthermore, the condition $\mathcal{P}_J u = 0$ is tantamount to

$$[\![B_i^{LM} u]\!] = [\![S_i^{LM} u]\!] = 0, \qquad i = 0, \ldots, m-1, \tag{12.42}$$

which in turn is fulfilled, if and only if,

$$\left[\frac{\partial^\gamma u}{\partial n^\gamma}\right] = 0, \qquad \text{for } \gamma = 0, \ldots, 2m-1. \tag{12.43}$$

Similarly, a function $w \in N_k$ if and only if,

$$\left[\frac{\partial^\gamma w}{\partial n^\gamma}\right] = 0, \qquad \text{for } \gamma = 0, \ldots, 2m-1. \tag{12.44}$$

When the functions u and w belong to $D \equiv \hat{H}^2(\Omega)$, Eqs. (12.43) and (12.44) are fulfilled if and only if they belong to $\widehat{D} \equiv H^2(\Omega)$.

In what follows the "standard smooth elliptic boundary value problem" will be as follows: "Given $f_\Omega \in H^0(\Omega)$ and g_∂^i, on $\partial\Omega$, $i = 0, \ldots, m-1$, find $u \in H^{2m}(\Omega)$ such that

$$\begin{aligned} \mathcal{L}u &= f_\Omega; \qquad \text{in } \Omega \\ B_i^{LM} u &= g_\partial^i; \qquad \text{on } \partial\Omega,\ i = 0, \ldots, m-1. \end{aligned} \tag{12.45}$$

Here, it is assumed that the boundary data are such that there exists a function $u_\partial \in H^{2m}(\Omega)$ such that $g_\partial^i = B_i^{LM} u_\partial$ on $\partial\Omega$, $i = 0, \ldots, m-1$."

On the other hand, the elliptic BVPJ will be:

"Given $f_\Omega \in H^0(\Omega)$, g_∂^i, on $\partial\Omega$, $i = 0, \ldots, m-1$, and j_Σ^i, on $\partial\Omega$, $j = 0, \ldots, 2m-1$, find a $u \in \hat{H}^{2m}(\Omega)$ such that

$$\begin{aligned} \mathcal{L}u &= f_\Omega; \qquad \text{in } \Omega \\ B_i^{LM} u &= g_\partial^i; \text{ on } \partial\Omega, \quad \text{for } i = 0, \ldots, m-1 \\ \left[\frac{\partial^j u}{\partial n^j}\right] &= j_\Sigma^j; \qquad \text{on } \Sigma, \quad \text{for } j = 0, \ldots, 2m-1. \end{aligned} \tag{12.46}$$

Here, it is assumed that the boundary and jump conditions are compatible; i.e., there exists a function $u_{\partial\Sigma} \in \hat{H}^{2m}(\Omega)$ such that $g_\partial^i = B_i^{LM} u_{\partial\Sigma}$, on $\partial\Omega$, and $i = 0, \ldots, m-1$ and

$$\left[\frac{\partial^j u_{\partial\Sigma}}{\partial n^j}\right] = j^j_\Sigma; \qquad \text{on } \Sigma, \quad \text{for } j = 0, \ldots, 2m-1.\text{''} \tag{12.47}$$

When considering the standard smooth elliptic boundary value problem, for each $w \in H^{2m}(\Omega)$, we define on $\partial\Omega$ the function

$$g_\partial(w) \equiv \mathcal{B}(u_\partial, w) \equiv -\sum_{j=0}^{m-1} B_j^{LM} u_\partial T_j^{LM} w, \qquad \text{on } \partial\Omega. \tag{12.48}$$

Correspondingly, when considering the elliptic BVPJ, for each $w \in \widehat{H}^{2m}(\Omega)$ we define

$$g_\partial(w) \equiv \mathcal{B}(u_{\partial\Sigma}, w), \quad \text{on } \partial\Omega, \qquad \text{and} \qquad j_\Sigma(w) \equiv \mathcal{J}(u_{\partial\Sigma}, w), \quad \text{on } \Sigma. \tag{12.49}$$

Here, $\mathcal{J}(u_{\partial\Sigma}, w)$ is defined by Eq. (12.26). It can be verified that both $g_\partial(w)$ and $j_\Sigma(w)$ are uniquely determined by the functions g^i_∂ and j^j_Σ occurring on Eqs. (12.45) and (12.46) (i.e., they are well defined). Furthermore, the functions $w \in H^{2m}(\Omega) \subset \hat{H}^{2m}(\Omega)$ that fulfill

$$\mathcal{L}^* w = 0; \qquad \text{in } \Omega$$
$$C_i^{LM} w = 0; \qquad \text{on } \partial\Omega, i = 0, \ldots, m-1 \tag{12.50}$$

constitute a linear subspace contained in $H^{2m}(\Omega) \subset \hat{H}^{2m}(\Omega)$, which will be denoted by $N^{\otimes}$.

Theorem 12.2. *The following statements are equivalent.*

1. The standard smooth elliptic boundary value problem possesses at least one solution, if and only if,

$$\int_\Omega w f_\Omega dx - \int_{\partial\Omega} g_\partial(w) dx = 0, \qquad \forall\, w \in N^{\otimes}. \tag{12.51}$$

2. The elliptic BVPJ possesses at least one solution, if and only if,

$$\int_\Omega w f_\Omega dx - \int_{\partial\Omega} g_\partial(w) dx - \int_\Sigma j_\Sigma(w) dx = 0, \qquad \forall\, w \in N^{\otimes}. \tag{12.52}$$

Proof. In view of Theorem 12.1, one can apply Theorem 8.3.

13. CONCLUSIONS

In this article a theory of partial differential equations in discontinuous piecewise-defined functions spaces has been presented, which is applicable to any differential equation or system of such equations that is linear, independently of its type and with possibly discontinuous coefficients. When finite element methods are formulated in this theory's setting trial and test functions can be fully discontinuous across the internal boundary and, so, dG methods are included. Such a formulation permits moving smoothly, without interruption, from the standard finite element method, based on continuous piecewise-defined functions, to the discontinuous Galerkin methods.

The theory is direct and systematic and, furthermore, it avoids the use of Lagrange multipliers or a frame, while mixed methods are incorporated as particular cases of more general results implied by the theory [39]. Some of the advantages of systematically handling discontinuous piecewise-defined functions have been illustrated for methods such as discontinuous Galerkin, Trefftz, and domain decomposition and collocation; among them, more efficient collocation procedures have been exhibited [50, 72–77], as well as the elimination of Lagrange multipliers and the frame, with the concomitant reduction of the number of degrees of freedom. This latter feature has been illustrated in [33], where details of its implementation have been provided.

Another important motivation for writing the present article was to complete the theoretical foundations of a line of research developed by the author and his coworkers, through a long time span [31, 33, 37–58, 71–78]. In this respect, many of the results obtained in previous work have been incorporated for the first time, in the framework of a systematic and rigorous theory of partial differential equations in Sobolev spaces of discontinuous piecewise-defined functions. To this end, a large number of new developments were required, while the material that was already available was thoroughly revised and reorganized. In particular, the nomenclature was improved making it more systematic.

APPENDIX

A1. Operators Decompositions

In this section a fixed operator $R : D_1 \to D_2^*$ is considered; then, we are concerned with families of operators $\mathbf{F} \equiv \{R_i : i = 1, \dots, n\}$, where $R_1 : D_1 \to D_2^*$. When any such a family is given, we associate with it a collection of pairs of subspaces, $\{(I_{1i}, I_{2i}) : i = 1, \dots, n\}$, defined by

$$I_{1i} \equiv \bigcap_{i \neq j} N_{R_j} \qquad \text{and} \qquad I_{2i} \equiv \bigcap_{i \neq j} N_{R_j^*} \tag{A1.1}$$

Furthermore, we define

$$\Sigma_{1i} \equiv \sum_{j \neq i} I_{1j} \qquad \text{and} \qquad \Sigma_{2i} \equiv \sum_{j \neq i} I_{2j} \tag{A1.2}$$

and

$$\hat{D}_1 \equiv \sum_{j=1}^{n} I_{1j} \qquad \text{and} \qquad \hat{D}_2 \equiv \sum_{j \neq i} I_{2j}. \tag{A1.3}$$

The following properties should be noticed, because they will be used in the sequel.

I. For any functions $u \in D_1$ and $\mathbf{v} \in D_2$, one has

$$u \in I_{1i} \Leftrightarrow R_j u = 0, \qquad \forall j \neq i \tag{A1.4}$$

and

$$\mathbf{v} \in I_{2i} \Leftrightarrow R_j^* \mathbf{v} = 0, \qquad \forall j \neq i. \tag{A1.5}$$

II. Write $\mathbf{N} \equiv \{1, \ldots, n\}$, and let $\mathbf{N}' \subset \mathbf{N}$, $\mathbf{N}'' \subset \mathbf{N}$ be such that

$$\mathbf{N}' \cup \mathbf{N}'' = \mathbf{N} \quad \text{and} \quad \mathbf{N}' \cap \mathbf{N}'' = \varnothing. \tag{A1.6}$$

Then,

$$\left.\begin{array}{lll} \bigcap_{i\in\mathbf{N}'} \Sigma_{1i} \equiv \sum_{i\in\mathbf{N}''} I_{1i} & \text{and} & \bigcap_{i\in\mathbf{N}'} \Sigma_{2i} \equiv \sum_{i\in\mathbf{N}''} I_{2i} \\ \bigcap_{i\in\mathbf{N}''} \Sigma_{1i} \equiv \sum_{i\in\mathbf{N}'} I_{1i} & \text{and} & \bigcap_{i\in\mathbf{N}''} \Sigma_{2i} \equiv \sum_{i\in\mathbf{N}'} I_{2i} \end{array}\right\}. \tag{A1.7}$$

Definition A1.1. *Operator decomposition. Let a family of operators,* $\mathbf{F} \in \{R_i | i = 1, \ldots, n\}$, *where* $R_i : D_1 \rightarrow D_2^*$, *be given. Assume each one of the pairs* $(I_{1i}, N_{R_i^*})$ *and* (N_{R_i}, I_{2i}), $i = 1, \ldots, n$ *is completely regular (with respect to* R*). Then,* $\mathbf{F}$ *is said to be an operator decomposition of* R *(or, simply, a decomposition of* R*), when*

$$R = \sum_{i=1}^{n} R_i. \tag{A1.8}$$

Remark A1.1. When the family of operators $\mathbf{F}$ is an operator decomposition of R, one has

$$I_{1i} \supset N_R; \qquad \Sigma_{1i} \supset N_R; \qquad I_{2i} \supset N_{R^*}; \qquad \Sigma_{2i} \supset N_{R^*}; \quad i = 1, \ldots, n. \tag{A1.9}$$

The case when the family of operators $\mathbf{F}$ is a pair has special interest. In such a case Definition A1.1, reads as follows.

Definition A1.2. *Let* $R : D_1 \rightarrow D_2^*$ *be given. Then a pair of operators* $\mathbf{F} \equiv \{R_1, R_2\}$ *of operators, is said to be a decomposition of* R, *when*

$$R = R_1 + R_2 \tag{A1.10}$$

and each one of the pairs of linear subspaces: $(N_{R_2}, N_{R_1^*})$ *and* $(N_{R_1}, N_{R_2^*})$ *are completely regular.*

Remark A1.2. When the pair of operators $\mathbf{F} \equiv \{R_1, R_2\}$ decomposes R, then

$$\left.\begin{array}{l} R_1 \text{ is a boundary operator for } R_2 \\ R_2 \text{ is a boundary operator for } R_1 \\ R_1^* \text{ is a boundary operator for } R_2^* \\ R_2^* \text{ is a boundary operator for } R_1^* \end{array}\right\}. \tag{A1.11}$$

This is equivalent to

$$\left.\begin{array}{l} N_{R_1} \text{ is TH-complete for } R_2^* \\ N_{R_2} \text{ is TH-complete for } R_1^* \\ N_{R_1^*} \text{ is TH-complete for } R_2 \\ N_{R_2^*} \text{ is TH-complete for } R_1 \end{array}\right\}. \tag{A1.12}$$

Furthermore,

$$N_R = N_{R_1} \cap N_{R_2} \qquad \text{and} \qquad N_{R*} = N_{R_1^*} \cap N_{R_2^*} \tag{A1.13}$$

As can be seen using the fact that $N_{R_1} + N_{R_2} \subset D_1$ and $N_{R_1^*} + N_{R_2^*} \subset D_2$.

Lemma A1.1. *When* $\mathbf{F} \equiv \{R_i : i = 1, \ldots, n\}$ *is a decomposition of* R, *the following properties hold.*

1.
$$\left.\begin{aligned} N_R &= \bigcap_{k=1}^{n} N_{R_k} = \bigcap_{k=1}^{n} \Sigma_{1k} = \bigcap_{k=1}^{n} I_{1k} = I_{1i} \cap I_{1j} \\ N_{R^*} &= \bigcap_{k=1}^{n} N_{R_k^*} = \bigcap_{k=1}^{n} \Sigma_{2k} = \bigcap_{k=1}^{n} I_{2k} = I_{2i} \cap I_{2j} \end{aligned}\right\} \quad \forall i \neq j \tag{A1.14}$$

2.
$$N_{R_i} \supset \Sigma_{1i} \qquad \text{and} \qquad N_{R_i^*} \supset \Sigma_{2i}. \tag{A1.15}$$

3. Each one of the pairs, (I_{1i}, Σ_{2i}) and (Σ_{1i}, I_{2i}), is a regular conjugate pair (with respect to R).

Proof. When the conditions of Definition A1.1 are fulfilled, and $i \neq j$, one has

$$N_R \supset \bigcap_{k=1}^{n} N_{R_k} = I_{1i} \cap I_{1j} = \bigcap_{k=1}^{n} I_{1k} = \bigcap_{k=1}^{n} \Sigma_{1k} \supset N_R. \tag{A1.16}$$

So, the first part of Eq. (A1.14) is clear. To see property 2), observe that $N_{R_i} \supset I_{1j} \equiv \cap_{k \neq j} N_{R_k}$ whenever $i \neq j$. This, in turn implies that $N_{R_i} \supset \Sigma_{1i}$. The second part of Eq. (A1.15) can be shown in a similar manner, to complete the proof of Property 2). As for Property 3), the pair (I_{1i}, Σ_{2i}) is conjugate because the pair $(I_{1i}, N_{R_i^*})$ is comletely regular and $N_{R_i^*} \supset \Sigma_{2i}$. Furthermore, $\Sigma_{2i} \supset N_{R^*}$ and $I_{1i} \supset N_R$, as indicated in Remark A1.1.

In the following theorem, we consider two subfamilies $\mathbf{F}' \subset \mathbf{F}$ and $\mathbf{F}'' \subset \mathbf{F}$ of a decomposition $\mathbf{F} \equiv \{R_i | i = 1, \ldots, n\}$ of R. It is assumed that

$$\mathbf{F}' \cup \mathbf{F}'' = \mathbf{F} \qquad \text{and} \qquad \mathbf{F}' \cap \mathbf{F}'' = \varnothing. \tag{A1.17}$$

Then, the notations $\mathbf{N}' \subset \mathbf{N}$ and $\mathbf{N}'' \subset \mathbf{N}$ will be used for the sets of subindices spanned by the families $\mathbf{F}'$ and $\mathbf{F}''$, respectively. It can be seen that Eq. (A1.17), implies Eq. (A1.6).

Theorem A1.1. *Assume* $\mathbf{F}$ *is a decomposition of* R *and let* $\mathbf{F}' \subset \mathbf{F}$ *and* $\mathbf{F}'' \subset \mathbf{F}$ *fulfill Eq.* (*A*1.17). *Define*

$$R' \equiv \sum_{i \in \mathbf{N}'} R_i \qquad \text{and} \qquad R'' \equiv \sum_{i \in \mathbf{N}''} R_i. \tag{A1.18}$$

Then

$$\text{I.}\quad \left.\begin{array}{lll} N_{R'} = \bigcap_{i\in\mathbf{N}'} N_{R_i} \supset \sum_{i\in\mathbf{N}''} I_{1i} & \text{and} & N_{R''} = \bigcap_{i\in\mathbf{N}''} N_{R_i} \supset \sum_{i\in\mathbf{N}'} I_{1i} \\ N_{(R')*} = \bigcap_{i\in\mathbf{N}'} N_{(R_i)*} \supset \sum_{i\in\mathbf{N}''} I_{2i} & \text{and} & N_{(R'')*} = \bigcap_{i\in\mathbf{N}''} N_{(R_i)*} \supset \sum_{i\in\mathbf{N}'} I_{2i} \end{array}\right\}. \tag{A1.19}$$

II. Each one of the subspaces $\Sigma_{i\in\mathbf{N}'}, I_{1i}$, $\Sigma_{i\in\mathbf{N}''} I_{1i}$, $\Sigma_{i\in\mathbf{N}'} I_{2i}$, and $\Sigma_{i\in\mathbf{N}''} I_{2i}$ is TH-complete for $N_{(R')*}$, $N_{(R'')*}$, $N_{R'}$, and $N_{R''}$, with respect to R, respectively.
III. Each one of the pairs of subspaces, $(N_{R'}, N_{(R'')*})$ and $(N_{R''}, N_{(R')*})$, is a comletely regular conjugate pair. Furthermore, the pair of operators (R', R'') decomposes R.

Furthermore, for every $i \notin \mathbf{N}'$, one has

IV. R' is a boundary operator for every R_i.
V. $(R')^*$ is a boundary operator for every $(R_i)^*$.
VI. R_i is a boundary operator for every R', and
VII. $(R_i)^*$ is a boundary operator for every $(R')^*$.

Proof. Before proving this result, it is mentioned that the assertions of Theorem A1.1 remain valid if the primes and the bi-primes are interchanged. To start the proof, it is seen that the definitions of Eq. (A3.8) imply

$$\bigcap_{i\in\mathbf{N}'} N_{R_i} \supset \sum_{i\in\mathbf{N}''} I_{1i}. \tag{A1.20}$$

Furthermore, the relation

$$N_{R'} \supset \bigcap_{i\in\mathbf{N}'} N_{R_i} \tag{A1.21}$$

follows from Eq. (A1.18). On the other hand,

$$\langle R'u, w\rangle = \sum_{i\in\mathbf{N}'} \langle R_i u, w\rangle = 0, \qquad \forall w \in \hat{D}_2 \subset D_2 \Rightarrow u \in N_{R_i}, \quad \forall i \in \mathbf{N}'. \tag{A1.22}$$

Hence,

$$N_{R'} \subset \bigcap_{i\in\mathbf{N}'} N_{R_i}. \tag{A1.23}$$

And the first of the equations (A1.19) is established. The remaining of the proof of Property I) is similar. As for Property II), we only prove that $\Sigma_{i\in\mathbf{N}'} I_{2i}$ is TH-complete for $N_{R'}$. Assume

$$\langle Ru, w\rangle = 0, \qquad \forall w \in \sum_{j\in\mathbf{N}'} I_{2j}. \tag{A1.24}$$

Then, $u \in \cap_{j\in\mathbf{N}'} N_{R_j} = N_{R'}$. Next, we prove Property III). Clearly,

$$R = R' + R''. \tag{A1.25}$$

Now, assume $(u, w) \in N_{R'} \times N_{(R'')^*}$; then,

$$\langle Ru, w\rangle = \langle R'u, w\rangle + \langle R''u, w\rangle = 0. \tag{A1.26}$$

Thus, the pair $(N_{R'}, N_{(R'')^*})$ is conjugate. Furthermore, $N_{(R'')^*}$ is TH-complete for $N_{R'}$, with respect to R, since $N_{(R'')^*} \supset \sum_{i\in \mathbf{N}'} I_{2i}$. Similarly, $N_{R'}$ is TH-complete for $N_{(R'')^*}$ and the pair $(N_{R'}, N_{(R'')^*})$ is completely regular. Additional arguments of the same kind yield Property III). Of Properties IV) to VII), we only prove IV) and VI). To prove IV), we the fact that

$$\langle Ru, w\rangle = \langle R_iu, w\rangle = 0, \qquad \forall w \in I_{2i} \subset \sum_{j\in \mathbf{N}''} I_{2j} \subset N_{(R')^*} \Rightarrow u \in N_{R_i} = 0. \tag{A1.27}$$

Then the result is clear. To prove VI), we use the fact that

$$\langle Ru, w\rangle = \langle R'u, w\rangle = 0, \qquad \forall w \in \sum_{j\in \mathbf{N}'} I_{2j} \subset N_{(R_i)^*} \Rightarrow u \in N_{R'} = 0. \tag{A1.28}$$

Then the result is clear.

Corollary A1.1. *Let* $\mathbf{F} \equiv \{R_i | i = 1, \ldots, n\}$ *be a decomposition of* R *and let* $\mathbf{F}' \equiv \{R_j | j \in \mathbf{N}' \subset \mathbf{N}\}$ *be any subfamily of* $\mathbf{F}$. *Assume* $R_i \notin \mathbf{F}'$, *then*

$$\sum_{j\in \mathbf{N}'} \langle R_ju, w\rangle = 0, \qquad \forall w \in N_{R_i} \tag{A1.29}$$

implies $R_ju = 0$ *for every* $j \in \mathbf{N}'$.

Proof. Let $\mathbf{N}'' \subset \mathbf{N}$ be the complement of $\mathbf{N}'$ relative to $\mathbf{N}$. Then $i \in \mathbf{N}''$. Therefore $R_i \in \mathbf{F}''$ is a boundary operator for $\sum_{j\in \mathbf{N}'} R_j$ and the Eq. (A1.29) implies that

$$\left(\sum_{j\in \mathbf{N}'} R_j\right)u = 0. \tag{A1.30}$$

This equation in turn implies that $R_ju = 0$ for every $j \in \mathbf{N}'$, by virtue of Eq. (A1.19).

A2. Derivation of the Results Used in the Article

First, we prove Remark 6.1 of Section 6. When Eq. (6.2) is a Green's formula, then the pair of operators $\{B, -C^*\}$ decomposes $R \equiv P - Q^*$ and Eq. (6.3) is obtained from Eq. (A1.12), while Eq. (6.4) is obtained from Eq. (A1.13). Next, we prove Lemma 7.1 of Section 7, with the addition of Corollary 7.1.

Proof of Lemma 7.1 of Section 7. Part I) (of Lemma 7.1) is implied by Theorem A1.1 (Parts VI) and VII)). As for Parts II) and III), they are implied by Part 1) of Lemma A1.1. The equation

$$\langle (P - B)u, w\rangle = 0, \qquad \forall w \in D_2 \tag{A2.1}$$

implies

$$\langle Pu, w \rangle = 0, \qquad \forall w \in N_{R^*} \subset N_{B^*}. \tag{A2.2}$$

Hence, $Pu = 0$ and $\langle Bu, w \rangle = 0$, $\forall w \in D_2$, since N_{R^*} is TH-complete for P. Therefore, $N_{P-B} \subset N_P \cap N_B$ and the relation $N_{P-B} = N_P \cap N_B$ is clear. The second part of Eq. (7.5) is similar. This establishes Part IV). As for Part V), the relations in Eq. (7.6) correspond to two particular cases of Part I) of Theorem A1.1. To prove Part VI) it is enough to show that pair of operators $\{(B + J), -(C + K)^*\}$ decomposes R. In view of Definition A1.2, we need to prove that each one of the pairs $\{N_{B+J}, N_{C+K}\}$ and $\{N_{(C+K)^*}, N_{(B+J)^*}\}$ is a completely regular conjugate pair, with respect to R. And this is implied by Part III) of Theorem A1.1.

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