New Formulation of Iterative Substructuring Methods Without Lagrange Multipliers: Neumann–Neumann and FETI

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This article is devoted to introduce a new approach to iterative substructuring methods that, without recourse to Lagrange multipliers, yields positive definite preconditioned formulations of the Neumann–Neumann and FETI types. To my knowledge, this is the first time that such formulations have been made without resource to Lagrange multipliers. A numerical advantage that is concomitant to such multipliers-free formulations is the reduction of the degrees of freedom associated with the Lagrange multipliers. Other attractive features are their generality, directness, and simplicity. The general framework of the new approach is rather simple and stems directly from the discretization procedures that are applied; in it, the differential operators act on discontinuous piecewise-defined functions. Then, the Lagrange multipliers are not required because in such an environment the functions-discontinuities are not an anomaly that need to be corrected. The resulting algorithms and equations-systems are also derived with considerable detail. © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 24: 845–878, 2008

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I. INTRODUCTION

Mathematical models of many systems of interest, including very important continuous systems of Engineering and Science, lead to a great variety of partial differential equations whose solution methods are based on the computational processing of large-scale algebraic systems. Furthermore, the incredible expansion experienced by the existing computational hardware and software has made amenable to effective treatment an ever increasing diversity and complexity of problems, posed by engineering and scientific applications.

Parallel computing is outstanding among the new computational tools, especially at present when further increases in hardware speed apparently have reached insurmountable barriers [1].

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Therefore, the emergence of parallel computing during the last 20 years or so, prompted on the part of the computational-modeling community a continued and systematic effort with the purpose of harnessing it for the endeavor of solving partial differential equations. Very early after such an effort began, it was recognized that domain decomposition methods (DDM) were the most effective technique for applying parallel computing to the solution of partial differential equations [2], since such an approach drastically simplifies the coordination of the many processors that carry out the different tasks and also reduces very much the requirements of information-transmission between them.

There are many approaches to DDM. One of the first to be studied, heralded by some well-known papers by P.L. Lions [3, 4], was the Schwarz Alternating Method that after further developments and generalizations led to the two-level overlapping methods and other related methods [5–8]. References presenting technical, as well as some historical details of such developments are abundant [9–17]. That not withstanding, in recent times much of the effort has gone into iterative substructuring methods in nonoverlapping partitions, such as Neumann–Neumann, Dirichlet–Dirichlet (preconditioned FETI), and FETI [17], and that is the subject of this present article.

Standard Neumann–Neumann or Dirichlet–Dirichlet—we refer to the preconditioned FETI method, following the nomenclature of Toselli and Widlund [17]—formulations do not lead to positive definite transformations in a direct manner. However, the Conjugate Gradient Method which is the basis of the most effective iterative procedures does require the positive definiteness of the transformations involved. A fundamental feature of iterative substructuring methods is that, after a domain partition has been introduced, they use what are essentially discontinuous piecewise-defined functions as base functions for representing the approximate solutions of the partial differential equations.

In this respect, it should be mentioned that at present the standard treatment of discontinuous functions is based on the use of Lagrange multipliers (see [18], for a review of this topic). However, recently Herrera presented a general theory of partial differential equations in discontinuous piecewise defined functions, in which discontinuous functions are the natural environment—not an anomaly—and which handles discontinuous functions directly without recourse to Lagrange multipliers [18], whereas mixed methods are incorporated as particular results of the theory (see Section VII of [19] for a derivation of mixed methods in this framework). Avoiding the introduction of Lagrange multipliers has clear numerical advantages such as reducing of the number of degrees of freedom to be handled. Thus, this article is devoted to present some significant improvements in the basic formulations of the iterative substructuring methods, achieved in this manner.

Accordingly, in this article we introduce a new approach to iterative substructuring methods that, without recourse to Lagrange multipliers, yields positive-definite preconditioned formulations such as Neumann–Neumann and FETI. The main distinguishing features of the new formulations, besides the avoidance of Lagrange multipliers, are their directness, simplicity, and generality. Such formulations are developed in full detail at the discrete level, but in order to place them in the realm of and compare with other well-known methods, some algorithms at the continuous level are also presented. Apparently, the range of applicability of the new formulations is wide and research is underway to apply it to other problems such as the biharmonic equation and to equations systems as those of elasticity and mixed methods.

Fundamental pieces of the general framework in which the new approach is based are two positive definite transformations, in terms of which many problems can be formulated. Actually, the structure of such a general framework is rather simple and most of the article is devoted to explain the incorporation of the discretization procedures into the general framework. In standard formulations, the treatment of vertices is an important challenge; however, in the new approach here

introduced, when the discretization steps are carried out as indicated in Section IV, the treatment of vertices is straightforward. This is explained in detail. On the other hand, the general framework is applicable in a direct manner, only when the differential operators are (strictly) positive definite, so a procedure for incorporating operators that only are non-negative (Laplacian-like) is also supplied. In the case of FETI, dual-primal formulations are a kind of procedures that are used to deal with vertices. In some situations they possess several attractive features and, therefore, in this article it is shown how to accommodate them in the general framework of the new algorithms here presented.

As for the theory of differential equations in discontinuous piecewise-defined functions, it is appropriate to mention that a basic feature of many numerical methods is the use, after a partition of the problem-domain has been introduced, of trial and test functions that are piecewise-defined; i.e., they are defined separately in each one of the partition-subdomains. In this respect, it must be pointed-out that the most general class of piecewise-defined functions includes functions that are fully discontinuous (by this we mean that the function itself has a jump discontinuity) across the internal boundary (i.e., that which separates the partition subdomains from each other). Indeed, such functions are defined independently in each one of the partition subdomains and due to this definition-independence the limits, from one and from the other side of the common boundary of two partition-subdomains, need not coincide. Thus, a truly general and systematic theory of Finite Element Methods (FEM) should be formulated in function spaces in which trial and test functions can be fully discontinuous across the internal boundary and such is the approach used in the author's theory [18]. It includes as a particular case discontinuous Galerkin (dG) methods and permits moving smoothly, without interruption, from the standard finite element method based on continuous piecewise-defined functions, to the dG methods. Furthermore, in that theory trial and test functions are piecewise-defined functions, which are fully discontinuous and an important advantage is that: "The theory of partial differential equations formulated in function spaces in which both trial and test functions are fully discontinuous avoids the introduction of Lagrange multipliers" [18]. The elimination of Lagrange multipliers, in turn, yields significant reductions in the number of degrees of freedom involved in the problems, which is an important practical advantage (see [20] that refers to some of the inconveniences of introducing Lagrange multipliers). Thus, methods such as dG methods, Trefftz methods [21–25], domain decomposition methods (DDM) [13, 17, 26–28], collocation methods, and matrix condensation should benefit from these results.

The author's theory of partial differential equations in discontinuous piecewise-defined functions stems from a long line of research, which spans from 1985 to the present [29–42], which has been devoted to the study of partial differential equations in discontinuous functions. When partial differential equations are formulated in discontinuous piecewise-defined functions, the well-posed problems are boundary value problems with prescribed jumps (BVPJ), in which the boundary conditions are complemented by suitable jump conditions to be satisfied across the internal boundary associated with the domain partition. The existence of solutions for such problems was discussed in [18]. An important element of that theory is a kind of Green's formulas applicable to discontinuous problems, introduced in 1985 and sometimes referred to as Green-Herrera formulas [30–34] (see also [19]). Their relevance is twofold; first, they supply more explicit expressions for the distributional derivatives and, second, they extend the notion of distributional derivative in a way that permits applying fully discontinuous trial and test functions simultaneously, something that is not possible when the standard theory of distributions is used. Apparently, it had been this latter fact what had prevented, until recently, the development of more direct approaches to partial differential equations formulated in discontinuous piecewise-defined functions.

The present article is organized as follows: After the Introduction, the notations and some preliminary notions, the new algorithms at the continuous level are presented in Section III. Then, in Section IV, to establish in a precise and general manner conditions under which the new algorithms are applicable, the discretization procedures are formulated axiomatically. Immediately after, in Sections V and VI, the positive definite transformations of the general framework are introduced and the basic problem is formulated in terms of them. In addition, in Section VI, the new algorithms are introduced in full generality and in a form that is immediately applicable at the discrete level. Section VII is devoted to explain their application at the continuous level. Since the general algorithms depend on the transformations mentioned earlier, it is important to develop effective means for evaluating them and this is done in Section VIII. The procedures for dealing with operators that are only non-negative (Laplacian-like) are explained in Section IX, while Section X is devoted to dual-primal methods. Finally, as an illustration, in Section XI all these procedures are implemented using piecewise-linear functions. The article closes with a Section on conclusions.

II. PIECEWISE-DEFINED FUNCTIONS

In what follows, $\Omega \subset R^m$ will be a domain, in the sense of Ciarlet [43], and $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$ a domain partition of Ω; i.e., it is assumed that

i. Ω_α, for $\alpha = 1, \ldots, E$ is a subdomain of Ω,

ii. $$\Omega_\alpha \cap \Omega_\beta = \phi, \quad \text{whenever } \alpha \neq \beta \tag{2.1}$$

iii. $$\Omega \subset \bigcup_{\alpha=1}^{E} \overline{\Omega}_\alpha \tag{2.2}$$

The notations $\partial\Omega$ and $\partial\Omega_\alpha, \alpha = 1, \ldots, E$, are adopted for the boundaries of Ω and Ω_α, respectively. Clearly, $\partial\Omega \subset \bigcup_{\alpha=1}^{E} \partial\Omega_\alpha$. In addition, $\Gamma \subset \bigcup_{\alpha=1}^{E} \partial\Omega_\alpha$ is defined to be the closed complement of $\partial\Omega$, with respect to $\bigcup_{\alpha=1}^{E} \partial\Omega_\alpha$, and will be called the internal boundary, whereas $\partial\Omega$ is referred to as the outer boundary. Observe that the internal boundary is also characterized by

$$\Gamma = \bigcup_{\alpha \neq \beta} \partial\Omega_\alpha \cap \partial\Omega_\beta \tag{2.3}$$

It is assumed that almost everywhere (a.e.) on Γ there is defined a unique unit normal vector denoted by n, whose sense is chosen arbitrarily; and then the positive side of Γ is defined to be that towards which the unit normal vector points.

In what follows, two functions, u and w, whose domain of definition is contained in Ω, are identified when the following condition is satisfied:

- The union of the set of points in which $u \neq w$, with the symmetric difference of their domains of definition has Lebesgue measure zero

Given the partition $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$, by a piecewise-defined function we mean a sequence of functions $\{w, \ldots, w_E\}$ such that for each $\alpha = 1, \ldots, E$, the function w_α is defined in Ω_α [18]. Given a function w defined in Ω, there is unique piecewise-defined function, $\{w_1, \ldots, w_E\}$, such that

$$w_\alpha = w|_{\Omega_\alpha}; \quad \alpha = 1, \ldots E \tag{2.4}$$

Here, $w|_{\Omega_\alpha}$ stands for the restriction of w to Ω_α. This establishes a one-to-one correspondence between functions defined in Ω and piecewise-defined functions. In what follows, we identify both the functions w defined almost everywhere (a.e). in Ω and their corresponding sequences $\{w, \ldots, w_E\}$. Given a function w defined in Ω, the sequence $\{w_1, \ldots, w_E\}$ will be referred to as the piecewise representation of w and the functions $w_\alpha, \alpha = 1, \ldots, E$, are the local components of w.

Given a family $\{D(\Omega_1), \ldots, D(\Omega_E)\}$ of linear spaces of functions defined in $\Omega_1, \ldots, \Omega_E$, respectively, we define a linear space $\hat{D}(\Omega)$ given by

$$\hat{D}(\Omega) \equiv D(\Omega_1) \oplus \ldots \oplus D(\Omega_E) \tag{2.5}$$

Let $\{w, \ldots, w_E\}$ be the piecewise representation of any w, then $w \in \hat{D}(\Omega)$ if and only if $w_\alpha \in D(\Omega_\alpha)$ for every $\alpha = 1, \ldots, E$. Let $D(\Omega)$ be a linear space of functions defined in Ω, then we define a linear space of piecewise defined functions, $\hat{D}(\Omega)$, given by Eq. (2.5) where $D(\Omega_\alpha)$ for each $\alpha = 1, \ldots, E$ is the linear space whose elements are the restrictions to Ω_α of functions belonging to $D(\Omega)$. In such a case, the mapping of $D(\Omega)$ into $\hat{D}(\Omega) \equiv D(\Omega_1) \oplus \ldots \oplus D(\Omega_E)$ which associates to each $w \in D(\Omega)$ its piecewise representation $\{w_1, \ldots, w_E\} \in D(\Omega_1) \oplus \ldots \oplus D(\Omega_E)$ is a bijection that will be referred to as the natural immersion of $D(\Omega)$ into $D(\Omega_1) \oplus \ldots \oplus D(\Omega_E)$ [18]. In what follows, we identify these two linear spaces and write

$$D(\Omega) \subset D(\Omega_1) \oplus \ldots \oplus D(\Omega_E) \tag{2.6}$$

When considering a function w defined in Ω, its definition on Γ is immaterial because the Lebesgue measure of Γ is zero. That not withstanding, if the trace of w_α is defined a.e. on $\partial\Omega_\alpha$, for $\alpha = 1, \ldots, E$, then such a trace is also defined in Γ and it makes sense to refer to it. In particular, if the traces of w_α are defined on $\partial\Omega_\alpha$ for every $\alpha = 1, \ldots, E$, then they define two functions a.e. on Γ, denoted by (w_+, w_-), corresponding to the traces from the positive and negative sides of Γ, respectively. This permits defining on Γ, the jump and the average of such a piecewise-defined function by

$$[\![w]\!] \equiv w_+ - w_- \quad \text{and} \quad \dot{w} \equiv \frac{1}{2}(w_+ + w_-) \tag{2.7}$$

respectively. Then, the following identities are fulfilled:

$$w_+ = \dot{w} + \frac{1}{2}[\![w]\!] \quad \text{and} \quad w_- = \dot{w} - \frac{1}{2}[\![w]\!] \tag{2.8}$$

It should be mentioned that in many applications such as elasticity and mixed methods, the functions w_α are vector-valued, i.e, they take values in the euclidean space R^m.

The "Sobolev space of piecewise-defined functions of integer order $p \geq 0$," is defined by

$$\hat{H}^p(\Omega, \Pi) \equiv H^p(\Omega_1) \oplus \ldots \oplus H^p(\Omega_E) \tag{2.9}$$

Here, $H^p(\Omega_\alpha)$ is the Hilbertian-Sobolev space of order p of functions defined in Ω_α. A function $\hat{u} \equiv \{u_1, \ldots, u_E\} \in H^0(\Omega)$ belongs to $\hat{H}^p(\Omega, \Pi)$ if and only if the norm

$$\|\hat{u}\|_{p,\Omega,\Pi} \equiv \left(\sum_{\alpha=1}^{E} \|u_\alpha\|^2_{p,\Omega_\alpha} \right)^{1/2} \tag{2.10}$$

is well-defined. Here, the subscripts Ω and Π have been included to emphasize the fact that such norm depends not only on the domain Ω considered, but on the partition Π, as well. When $\hat{H}^p(\Omega)$ is equipped with the norm of Eq. (2.10), and the corresponding inner product, it becomes a Hilbert space.

The following properties that were established in [18], are noticed:

1. When $w \in H^p(\Omega)$, then the restriction of w to Ω_α, w_α, has the property that $w_\alpha \in H^p(\Omega_\alpha)$. Therefore:

$$H^p(\Omega) \subset \hat{H}^p(\Omega) \tag{2.11}$$

2. When $u \in \hat{H}^1(\Omega)$, then

$$[\![u]\!] = 0, \quad \text{on } \Gamma \Leftrightarrow u \in H^1(\Omega) \tag{2.12}$$

3. When $u \in \hat{H}^2(\Omega)$, then

$$[\![u]\!] = \left[\!\left[\frac{\partial u}{\partial n}\right]\!\right] = 0, \quad \text{on } \Gamma \Leftrightarrow u \in H^2(\Omega) \tag{2.13}$$

The identity

$$\sum_{\alpha=1}^{E} \int_{\partial\Omega_\alpha} u_\alpha w_\alpha n_i dx = \int_{\partial\Omega} uwn_i dx - \int_\Gamma [\![uw]\!] n_i dx = \int_{\partial\Omega} uwn_i dx - \int_\Gamma (\dot{u}[\![w]\!] + \dot{w}[\![u]\!]) n_i dx \tag{2.14}$$

can be easily verified. Here, n_i is anyone of the unit normal-vector components. Furthermore, for the integrals over $\partial\Omega_\alpha$ and over $\partial\Omega$, the outer normal vectors (outer to Ω_α and Ω, respectively) have been used, while for the integral over Γ the unit normal vector that points towards its positive side (according to the convention that is used throughout this article and that was explained before) has been applied.

III. THE NEW ALGORITHMS AT THE CONTINUOUS LEVEL

Our main interest and, to a large extent, the practical value of the new algorithms lay at the discrete level. However, in order to place them in the realm of, and compare with other well-known methodologies, this Section is devoted to present the new algorithms at the continuous level. In particular, they are formulated for a simple example that has been used by many authors to introduce DDM (see, for example, [12, 13, 17]). The presentation of this Section is brief and it does not cover all the technical details; however, in the following Sections a full discussion, including many technical details, is presented at the discrete level. It should be mentioned that at the continuous level the new formulations are closely related with the standard formulations, as it is explained in the Appendix.

Consider the Poisson equation in a domain Ω, with zero Dirichlet boundary-data; i.e.,

$$\begin{aligned} -\Delta\overline{u} &= f_\Omega, \quad \text{in } \Omega \\ \overline{u} &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

The space $\overline{D}$, where the solution $\overline{u}$ is sought, is defined as:

$$\overline{D} = \{v \in H^2(\Omega) | \text{trace } v = 0, \text{ on } \partial\Omega\} \tag{3.2}$$

Another space to be used in the sequel is:

$$\tilde{D} \equiv \{v \in \hat{H}^2(\Omega, \Pi) | \text{trace } v = 0, \text{ on } \partial\Omega\} \tag{3.3}$$

When the datum f_Ω is such that the solution $\overline{u}$ belongs to $\overline{D}$, then this problem is equivalent to the following boundary-value problem with prescribed jumps (BVPJ): Find $\tilde{u} \in \tilde{D}$ such that [18]:

$$\begin{aligned} -\Delta\tilde{u} &= f_\Omega; \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ [\![\tilde{u}]\!] = \left[\!\!\left[\frac{\partial\tilde{u}}{\partial n}\right]\!\!\right] &= 0, \text{ in } \Gamma \end{aligned} \tag{3.4}$$

More precisely, $\tilde{u} \in \tilde{D}$ fulfills Eq. (3.4) if and only if $\tilde{u} = u$. The boundary conditions do not appear in Eq. (3.4) because they have been incorporated in the definition of the space $\tilde{D}$.

A. A manner of approaching this problem is to introduce an auxiliary function $\tilde{u}_P \in \tilde{D}$ that fulfills

$$\begin{aligned} -\Delta\tilde{u}_P &= f_\Omega; \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ [\![\tilde{u}_P]\!] = 0 \quad &\text{and } \dot{\tilde{u}}_P = 0, \text{ on } \Gamma \end{aligned} \tag{3.5}$$

Then, if $u \equiv \tilde{u} - \tilde{u}_P$ one obtains for $u \in \tilde{D}$ the equations

$$\begin{aligned} -\Delta u &= 0; \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ [\![u]\!] = 0; \quad \left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] &= -\left[\!\!\left[\frac{\partial\tilde{u}_P}{\partial n}\right]\!\!\right], \text{ in } \Gamma \end{aligned} \tag{3.6}$$

B. Another option is to replace Eq. (3.5) by

$$\begin{aligned} -\Delta\tilde{u}_P &= f_\Omega; \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ \left[\!\!\left[\frac{\partial\tilde{u}_P}{\partial n}\right]\!\!\right] = 0 \quad &\text{and} \quad \widehat{\frac{\partial\tilde{u}_P}{\partial n}} = 0, \text{ on } \Gamma \end{aligned} \tag{3.7}$$

In which case

$$\begin{aligned} -\Delta u &= 0; \quad \text{in } \Omega_\alpha, \ \alpha = 1, \ldots, E \\ [\![u]\!] = -[\![u_P]\!]; \quad &\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] = 0, \text{ in } \Gamma \end{aligned} \tag{3.8}$$

Independently of which approach is followed, one seeks for a function of the linear space

$$D \equiv \{u \in \tilde{D} | -\Delta u = 0; \text{ in } \Omega_\alpha, \alpha = 1, \ldots, E\} \tag{3.9}$$

Corresponding to the approaches A and B, one obtains the Neumann–Neumann and the Dirichlet–Dirichlet (throughout this article we use these terms for the preconditioned FETI; see [17]) algorithms, respectively. In the following Sections, these algorithms are presented at the discrete level in a manner that can be applied in any number of dimensions and any number of partition-subdomains, including partitions with vertices and differential operators that are positive but not positive definite; in particular, in many situations of practical interest the Laplace operator falls in this latter category.

A. The Neuman–Neuman Algorithm

1. Construct $\tilde{u}_P \in \tilde{D}$ and define $u \in D$ as in option A.
2. Construct $u_{21} \in D$ such that

$$\left[\!\left[\frac{\partial u_{21}}{\partial n} \right]\!\right] = -\left[\!\left[\frac{\partial \tilde{u}_P}{\partial n} \right]\!\right] \quad \text{and} \quad \widehat{\frac{\partial u_{21}}{\partial n}} = 0, \text{ on } \Gamma \tag{3.10}$$

3. In turn, $r^0 \in D$ is such that

$$[\![r^0]\!] = 0 \quad \text{and} \quad \hat{r^0} = \widehat{u_{21}}, \text{ on } \Gamma \tag{3.11}$$

Let $p^0 \equiv r^0$ and $u^0 \equiv 0$.
Do for $n = 0, 1, 2, \ldots$
4. Construct $\psi^n \in D$ such that

$$\left[\!\left[\frac{\partial \psi^n}{\partial n} \right]\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial \psi^n}{\partial n}} = \widehat{\frac{\partial p^n}{\partial n}}, \text{ on } \Gamma \tag{3.12}$$

5. $$\alpha^n \equiv \frac{p^n \cdot p^n}{p^n \cdot p^n + \psi^n \cdot \psi^n} \tag{3.13}$$
6. $$u^{n+1} = u^n + \alpha^n p^n \tag{3.14}$$
7. Furthermore, construct $q^n \in D$ such that

$$[\![q^n]\!] = 0 \text{ and } \hat{q^n} = \widehat{\psi^n}, \text{ on } \Gamma \tag{3.15}$$

8. $$r^{n+1} = r^n - \alpha^n q^n \tag{3.16}$$
9. $$\beta^n \equiv \frac{r^{n+1} \cdot r^{n+1}}{r^n \cdot r^n} \tag{3.17}$$
10. $$p^{n+1} = r^{n+1} + \beta^n p^n \tag{3.18}$$
11. $$n \to n+1 \quad \text{and} \quad \text{Go to 4} \tag{3.19}$$

B. The Dirichlet–Dirichlet Algorithm

1. Construct $\tilde{u}_P \in \tilde{D}$ and define $u \in D$ as in option B.
2. Construct $u_{11} \in D$ such that

$$[\![u_{11}]\!] = -[\![\tilde{u}_P]\!] \quad \text{and} \quad \hat{u_{11}} = 0, \text{ on } \Gamma \tag{3.20}$$

3. In turn, $r^0 \in D$ is such that

$$\left[\!\left[\frac{\partial r^0}{\partial n} \right]\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial r^0}{\partial n}} = \widehat{\frac{\partial u_{11}}{\partial n}}, \text{ on } \Gamma \tag{3.21}$$

Let $p^0 \equiv r^0$ and $u^0 \equiv 0$.
Do for $n = 0, 1, 2, \ldots$
4. Construct $\psi^n \in D$ such that

$$[\![\psi^n]\!] = 0 \quad \text{and} \quad \hat{\psi^n} = \hat{p^n}, \text{ on } \Gamma \tag{3.22}$$

5. $$\alpha^n \equiv \frac{p^n \cdot p^n}{p^n \cdot p^n + \psi^n \cdot \psi^n} \tag{3.23}$$
6. $$u^{n+1} = u^n + \alpha^n p^n \tag{3.24}$$
7. Furthermore, construct $q^n \in D$ such that

$$\left[\!\left[\frac{\partial q^n}{\partial n} \right]\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial q^n}{\partial n}} = \widehat{\frac{\partial \psi^n}{\partial n}}, \text{ on } \Gamma \tag{3.25}$$

8. $$r^{n+1} = r^n - \alpha^n q^n \tag{3.26}$$
9. $$\beta^n \equiv \frac{r^{n+1} \cdot r^{n+1}}{r^n \cdot r^n} \tag{3.27}$$
10. $$p^{n+1} = r^{n+1} + \beta^n p^n \tag{3.28}$$
11. $$n \to n+1 \quad \text{and} \quad \text{Go to 4} \tag{3.29}$$

As mentioned in the Introduction, the great interest that exists at present—and in the last 20 years or so—in DDM stems mainly from the fact that DDM is a very effective tool for applying parallel computation resources in the mathematical modeling of continuous systems and in particular, in the numerical solution of partial differential equations. Furthermore, the effectiveness of DDM is very much enhanced when it is applied by means of the conjugate gradient method (CGM), which is only possible when the matrices involved are symmetric and positive definite. However, the standard Neumann–Neumann and Dirichlet–Dirichlet formulations do not lead in a direct manner to positive-definite transformations (see, for example, [12, 13, 17]). In the case of FETI, which is based on the Dirichlet–Dirichlet formulation (preconditioned FETI), Lagrange multipliers are introduced. Thus, a very interesting property of the new formulations is that they lead in a direct manner to positive-definite transformations, without recourse to Lagrange multipliers.

IV. AXIOMATIC DISCRETIZATION

A. Let $\overline{D}$ be a finite-dimensional Hilbert space of functions defined in Ω, of dimension $\overline{N}$, while $\Pi = \{\Omega, \ldots, \Omega_E\}$ is a partition. Define, for each $\alpha = 1, \ldots, E$,

$$D(\Omega_\alpha) \equiv \{v | v = u|_{\Omega_\alpha} \text{ and } u \in \overline{D}\} \tag{4.1}$$

Then, write

$$\tilde{D} \equiv D(\Omega_1) \oplus \ldots \oplus D(\Omega_E) \tag{4.2}$$

Therefore, $\tilde{D}$ is a space of piecewise-defined functions and, under the natural Immersion of $\overline{D}$ into $\tilde{D}$, we have $\overline{D} \subset \tilde{D}$. A function $\tilde{w} \in \tilde{D}(\Omega)$ is said to have local support when there exists an $\alpha \in \{1, \ldots, E\}$ such that the support of $\tilde{w}$ is contained in the closure of Ω_α. Given any function $\overline{w} \in \overline{D}$, we say that a function $\tilde{w} \in \tilde{D}$ is a heir of $\overline{w}$, when $\tilde{w}$ is the restriction of $\overline{w}$ to a partition subdomain. Clearly, all the heirs of a function $\overline{w} \in \overline{D}$ have local support. As for the inner products in these spaces, it is assumed that they satisfy:

$$u \cdot w = \sum_{\alpha=1}^{E} u_\alpha \cdot w_\alpha$$

Whenever, $u = \{u_1, \ldots, u_E\}$ and $w = \{w_1, \ldots, w_E\}$.

B. Let $\overline{\mathcal{B}} \subset \overline{D}$ be a basis of $\overline{D}$ and write

$$\overline{\mathcal{B}} = \{\overline{w}^1, \ldots, \overline{w}^{\overline{N}}\} \tag{4.3}$$

Then, for each $i = 1, \ldots, \overline{N}, \mathcal{B}^i \subset \overline{D}$ will be the collection of heirs of $\overline{w}^i$. Furthermore, we write

$$\mathcal{B} \equiv \bigcup_{i=1}^{\overline{N}} \mathcal{B}^i \subset \tilde{D} \tag{4.4}$$

Clearly, the elements of $\mathcal{B}$ have local support, and it will be assumed that $\mathcal{B}$, as defined here, is a linearly independent basis of $\tilde{D}$.

The collection of sets $\{\mathcal{B}^1, \ldots, \mathcal{B}^{\overline{N}}\}$ is classified into two subfamilies: $\{\mathcal{B}_I^1, \ldots, \mathcal{B}_I^{N_I}\} \subset \{\mathcal{B}^1, \ldots, \mathcal{B}^{\overline{N}}\}$ and $\{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\} \subset \{\mathcal{B}^1, \ldots, \mathcal{B}^{\overline{N}}\}$; they are defined by the following conditions: $\mathcal{B}^i \in \{\mathcal{B}_I^1, \ldots, \mathcal{B}_I^{N_I}\}$ if and only if the cardinality of $\mathcal{B}^i$ is one, and $\mathcal{B}^i \in \{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\}$ if and only if the cardinality of $\mathcal{B}^i$ is greater than one. Then

$$\{\mathcal{B}^1, \ldots, \mathcal{B}^{\overline{N}}\} = \{\mathcal{B}_I^1, \ldots, \mathcal{B}_I^{N_I}\} \cup \{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\} \tag{4.5}$$

We define

$$\mathcal{B}_I \equiv \bigcup_{i=1}^{N_I} \mathcal{B}_I^i \quad \text{and} \quad \mathcal{B}_\Gamma \equiv \bigcup_{i=1}^{\overline{N}_\Gamma} \mathcal{B}_\Gamma^i \tag{4.6}$$

So that

$$\mathcal{B} = \mathcal{B}_I \cup \mathcal{B}_\Gamma \tag{4.7}$$

Next, an additional family of sets $\mathcal{B}_\Gamma^i, i = 1, \ldots, \overline{N}_\Gamma$ is introduced; each set $\overline{\mathcal{B}}_\Gamma^i$ is defined by replacing the set $\mathcal{B}_\Gamma^i$, above, by an equivalent linearly independent set (equivalent, in the sense that each one of $\mathcal{B}_\Gamma^i$ and $\overline{\mathcal{B}}_\Gamma^i$ spans the same linear space). The notation

$$\overline{\mathcal{B}}_\Gamma^i \equiv \{w_M^i, w_{J1}^j, \ldots, w_{Jm(i)}^j\} \tag{4.8}$$

is adopted. Furthermore, w_M^i is defined to be the mother function $\overline{w}^i \in \overline{\mathcal{B}} \subset \overline{D}$; i.e.,

$$w_M^i \equiv \overline{w}^i \in \overline{\mathcal{B}} \subset \overline{D} \tag{4.9}$$

Thus, the set

$$\mathcal{B}_J^i \equiv \{w_{J1}^i, \ldots, w_{Jm(i)}^i\} \tag{4.10}$$

is an algebraic complementary set of $\{\overline{w}^i\}$, with the property that $\overline{\mathcal{B}}_\Gamma^i$, when it is defined by Eq. (4.8), spans the same linear space as $\mathcal{B}_\Gamma^i$. Further definitions are:

$$\mathcal{B}_{\Gamma M} \equiv \{w_M^1, \ldots, w_M^{\overline{N}_\Gamma}\}, \quad \mathcal{B}_{\Gamma J} \equiv \bigcup_{i=1}^{\overline{N}_\Gamma} \mathcal{B}_J^i, \quad \text{and} \quad \overline{\mathcal{B}}_\Gamma \equiv \mathcal{B}_{\Gamma M} \cup \mathcal{B}_{\Gamma J} \tag{4.11}$$

Observe that $\mathcal{B}_{\Gamma M} \subset \overline{\mathcal{B}} \subset \overline{D}$. Clearly, with such definitions both $\overline{\mathcal{B}}_\Gamma$ and $\mathcal{B}_\Gamma$, span the same linear subspaces; however, a significant difference between $\overline{\mathcal{B}}_\Gamma$ and $\mathcal{B}_\Gamma$ is that all the elements of $\mathcal{B}_\Gamma$ have local support, which is not true for $\overline{\mathcal{B}}_\Gamma$. An additional property is:

$$\overline{\mathcal{B}} = \mathcal{B}_{\Gamma M} + \mathcal{B}_I \tag{4.12}$$

The subspaces spanned by the sets of functions $\mathcal{B}_I$, $\mathcal{B}_\Gamma$, $\mathcal{B}_{\Gamma J}$, and $\mathcal{B}_{\Gamma M}$ will be denoted by $\tilde{D}_I$, $\tilde{D}_\Gamma$, $\tilde{D}_{\Gamma 1}$, and $\tilde{D}_{\Gamma 2}$, respectively. We let N_I, N_Γ, and $\tilde{N}$ be the dimensions of $\tilde{D}_I$, $\tilde{D}_\Gamma$, and $\tilde{D}$, respectively. Then

$$\tilde{N} = N_\Gamma + N_I \tag{4.13}$$

Furthermore,

$$\left.\begin{aligned} \tilde{D} \equiv \tilde{D}_I + \tilde{D}_\Gamma, \quad \tilde{D}_I \cap \tilde{D}_\Gamma = \{0\} \\ \tilde{D}_\Gamma \equiv \tilde{D}_{\Gamma 1} + \tilde{D}_{\Gamma 2}, \quad \tilde{D}_{\Gamma 1} \cap \tilde{D}_{\Gamma 2} = \{0\} \end{aligned}\right\} \tag{4.14}$$

And

$$\overline{D} = \tilde{D}_I + \tilde{D}_{\Gamma 2} \tag{4.15}$$

Even more, Eq. (4.14) implies that every function $\tilde{v} \in \tilde{D}$ and every function $\tilde{v}_\Gamma \in \tilde{D}_\Gamma$ can be written in a unique manner in the following alternative forms

$$\tilde{v} = \tilde{v}_\Gamma + \tilde{v}_I, \quad \text{with } \tilde{v}_\Gamma \in \tilde{D}_\Gamma \text{ and } \tilde{v}_I \in \tilde{D}_I \tag{4.16}$$

$$\tilde{v}_\Gamma = \tilde{v}_J + \tilde{v}_M, \quad \text{with } \tilde{v}_J \in \tilde{D}_{\Gamma 1} \text{ and } \tilde{v}_M \in \tilde{D}_{\Gamma 2} \tag{4.17}$$

$$\tilde{v} = \tilde{v}_J + \tilde{v}_M + \tilde{v}_I, \quad \text{with } \tilde{v}_J \in \tilde{D}_{\Gamma 1}, \tilde{v}_M \in \tilde{D}_{\Gamma 2} \text{ and } \tilde{v}_I \in \tilde{D}_I \tag{4.18}$$

Given any function $\tilde{v} \in \tilde{D}$, we associate with it unique functions $\tilde{v}_\Gamma \in \tilde{D}_\Gamma$, $\tilde{v}_J \in \tilde{D}_{\Gamma 1}$, $\tilde{v}_M \in \tilde{D}_{\Gamma 2}$ and $\tilde{v}_I \in \tilde{D}_I$, whose definitions are given by Eqs. (4.16) and (4.18).

V. THE INTERMEDIATE AXIOMS

In this Section, we establish a general scheme that can be applied when there are subspaces of $\tilde{D} \supset \overline{D}$, which satisfy Eqs. (4.14) and (4.15), even if they are not constructed in the manner explained in Section IV. In particular, such an application will made in Section X.

Then the space $D \subset \tilde{D}$ is defined to be the orthogonal complement, with respect to $\tilde{D}$, of $\tilde{D}_I \subset \tilde{D}$; i.e., $D \equiv (\tilde{D}_I)^\perp$. Or, more explicitly:

$$D \equiv \{v \in \tilde{D} | v \cdot w = 0, \forall w \in \tilde{D}_I\} \tag{5.1}$$

Then,

$$\tilde{D} = D + \tilde{D}_I \quad \text{and} \quad D \cap \tilde{D}_I = \{0\} \tag{5.2}$$

The notation $\text{proj}_D : \tilde{D} \to D$ is here introduced for the projection operation of vectors of $\tilde{D}$ into D. Recall, from Eq. (4.14):

$$\tilde{D} = \tilde{D}_\Gamma + \tilde{D}_I \quad \text{and} \quad \tilde{D}_\Gamma \cap \tilde{D}_I = \{0\} \tag{5.3}$$

Hence,

$$\text{proj}_D \tilde{D}_\Gamma = D \tag{5.4}$$

Furthermore, the mapping $\text{prop}_D : \tilde{D}_\Gamma \to D$ is a bijection (i.e., one-to-one).

In what follows, the orthogonal complements of subspaces of D will be taken with respect to D. Using such notation, additional definitions are:

$$D_{11} \equiv \text{proj}_D \tilde{D}_{\Gamma 1} \quad \text{and} \quad D_{12} \equiv \text{proj}_D \tilde{D}_{\Gamma 2} \tag{5.5}$$

together with

$$D_{21} \equiv (D_{11})^\perp \quad \text{and} \quad D_{22} \equiv (D_{12})^\perp. \tag{5.6}$$

Then

$$D = D_{11} + D_{12} \quad \text{and} \quad D_{11} \cap D_{12} = \{0\} \tag{5.7}$$

since

$$D = \text{proj}_D \tilde{D}_{\Gamma 1} + \text{proj}_D \tilde{D}_{\Gamma 2} \tag{5.8}$$

by virtue of Eq. (4.14).

VI. THE GENERAL FRAMEWORK

In this Section, we establish a general framework in terms of which many iterative substructuring methods can be formulated.

Axiom VI.1. *The only assumption (or axiom) of this framework is that there is a Hilbert space, D, and a pair of (closed) subspaces of D, $\{D_{11}, D_{12}\}$, with the property that*

$$D = D_{11} + D_{12} \quad \textit{and} \quad D_{11} \cap D_{12} = \{0\} \tag{6.1}$$

In view of Eq. (5.7) the system of Section V, fulfills this Axiom. As for notation, the inner product of D will be denoted by $u \cdot w$ whenever $u \in D$ and $w \in D$.

Definition VI.1. *Let*

$$D_{21} = (D_{11})^\perp \quad \textit{and} \quad D_{22} = (D_{12})^\perp \tag{6.2}$$

Theorem VI.1. *Assume the Axiom and Definition 6.1 hold. Then:*

$$\left.\begin{aligned} D &= D_{11} + D_{21} \quad \textit{and} \quad D_{11} \cap D_{21} = \{0\} \\ D &= D_{12} + D_{22} \quad \textit{and} \quad D_{12} \cap D_{22} = \{0\} \\ D &= D_{21} + D_{22} \quad \textit{and} \quad D_{21} \cap D_{22} = \{0\} \end{aligned}\right\} \tag{6.3}$$

Proof. In the proof of this result we do not need to assume that D is finite-dimensional, so its validity is more general. The first two equalities of Eq. (6.3) are straightforward, because of the orthogonality properties of the subspaces involved. As for the third one, assume $u \in (D_{21} \cap D_{22})$

then $u \in (D_{11})^{\perp}$ and simultaneously $u \in (D_{12})^{\perp}$; therefore, $u \cdot w = 0$ for every $w \in D$, in view of Eq. (6.1). Hence $u = 0$. On the other hand, the subspace $\hat{D} \equiv D_{21} + D_{22} \subset D$ is closed. Consider $(\hat{D})^{\perp}$. Then

$$D = \hat{D} + (\hat{D})^{\perp} \tag{6.4}$$

If $u \in (\hat{D})^{\perp}$, then $u \in (D_{21})^{\perp} = D_{11}$ and $u \in (D_{22})^{\perp} = D_{12}$. Hence, $u \subset D_{11} \cap D_{12} = \{0\}$ and $(\hat{D})^{\perp} = \{0\}$. From Eq. (6.4) it follows that

$$D = \hat{D} = D_{21} + D_{22} \tag{6.5}$$

Eqs. (6.1) and (6.3) together, imply that every function $u \in D$ can be written in a unique manner as

$$u = u_{11} + u_{12} = u_{21} + u_{22}, \quad \text{with } u_{\alpha\beta} \in D_{\alpha\beta}; \quad \alpha, \beta = 1, 2 \tag{6.6}$$

Many iterative substructuring methods can be cast in terms of either one of the two abstract problems that are formulated next. ∎

Problem 1. *In this problem $u_{21} \in D_{21}$ is the datum: "Given $u_{21} \in D_{21}$, find $u \in D_{12}$ such that $u = u_{21} + u_{22}$, for some $u_{22} \in D_{22}$."*

Problem 2. *In this problem $u_{11} \in D_{11}$ is the datum: "Given $u_{11}, \in D_{11}$, find $u \in D_{22}$ such that $u = u_{11} + u_{12}$, for some $u_{12} \in D_{12}$."*

Depending on the manner in which the subspaces of D are chosen, these problems lead to generalized versions of the Neumann–Neumann and the Dirichlet–Dirichlet (preconditioned FETI [17]) approaches.

From Eq. (6.6), it follows that

$$u_{2\alpha} = \sum_{\beta=1}^{2} (u_{1\beta})_{2\alpha} \quad \text{and} \quad u_{1\alpha} = \sum_{\beta=1}^{2} (u_{2\beta})_{1\alpha}; \ \alpha = 1, 2 \tag{6.7}$$

Define, for each $\alpha, \beta = 1, 2$ and each $u \in D$, the mappings $\tau_{\alpha\beta} : D_{1\alpha} \to D_{2\beta}$ and $\mu_{\alpha\beta} : D_{2\alpha} \to D_{1\beta}$ by

$$\tau_{\alpha\beta} u \equiv (u_{1\alpha})_{2\beta} \quad \text{and} \quad \mu_{\alpha\beta} u \equiv (u_{2\alpha})_{1\beta} \tag{6.8}$$

Lemma VI.1. *When $u \in D_{12}$ and $w \in D_{22}$, one has:*

$$w \cdot \tau_{22} u = -u \cdot \mu_{22} w \tag{6.9}$$

Proof. Let $u \in D_{12}$ and $w \in D_{22}$ be given. Then $u_{12} = u$ and $w_{22} = w$. Furthermore, we recall the relations $u = u_{21} + u_{22}$ and $w = w_{11} + w_{12}$ that will be used in the sequel. Then

$$w \cdot \tau_{22} u = (u_{12})_{22} \cdot w = u_{22} \cdot w = -u_{21} \cdot w = -u_{21} \cdot w_{12} = -u \cdot (w_{22})_{12} = -u \cdot \mu_{22} w \tag{6.10}$$

∎

Corollary VI.1. *Define the transformation* $T_D : D_{12} \to D_{12}$, *for every* $u \in D_{12}$ *by*

$$T_D u \equiv -\mu_{22}\tau_{22}u \tag{6.11}$$

and the transformation $T_N : D_{22} \to D_{22}$, *for every* $u \in D_{22}$ *by*

$$T_N u \equiv -\tau_{22}\mu_{22}u \tag{6.12}$$

Then, each one of these transformations is non-negative definite.

Proof. Recall that both transformations $\tau_{22} : D_{12} \to D_{22}$ and $\mu_{22} : D_{22} \to D_{12}$ are one-to-one (i.e., nonsingular). Applying Eq. (6.9), it follows that when $u \in D_{12}$ and $u \neq 0$ one has

$$0 \leq \tau_{22}u \cdot \tau_{22}u = -u \cdot \mu_{22}\tau_{22}u \tag{6.13}$$

Similarly, from Eq. (6.9) it follows that when $u \in D_{22}$ and $u \neq 0$ one has

$$0 \leq \mu_{22}u \cdot \mu_{22}u = -u \cdot \tau_{22}\mu_{22}u \tag{6.14}$$

■

Theorem VI.2. *Formulation of Problems 1 and 2*

Let I be the identical transformation; then:

A. A function $u \in D_{12}$ is solution of Problem 1, if and only if

$$(I + T_D)u = \mu_{12}u_{21} \tag{6.15}$$

B. A function $u \in D_{22}$ is solution of Problem 2, if and only if

$$(I + T_N)u = \tau_{12}u_{11} \tag{6.16}$$

Here, the transformations $(I + T_D) : D_{12} \to D_{12}$ and $(I + T_N) : D_{22} \to D_{22}$ are positive definite.

Proof. In view of Corollary 6.1, each one of the transformations $I + T_D$ and $I + T_N$, is positive definite. When $u \in D_{12}$,

$$u = u_{21} + u_{22} = u_{21} + \tau_{22}u \tag{6.17}$$

Therefore,

$$u = u_{12} = \mu_{12}u_{21} + \mu_{22}\tau_{22}u \tag{6.18}$$

That is

$$u - \mu_{22}\tau_{22}u = \mu_{12}u_{21} \tag{6.19}$$

Then Eq. (6.15) is clear. The proof of the second part of this Theorem is similar, since in our developments the roles of the pairs (D_{11}, D_{12}) and (D_{21}, D_{22}) can be interchanged. ■

Since both $I + T_D$ and $I + T_N$ are positive definite transformations, the conjugate gradient method (CGM) is applicable to these problems. To obtain the new algorithms the following sequence can be used:

Let u^0 be given (or $u^0 = 0$) and set $r^0 = b - Au^0$, $p^0 = r^0$. For $n = 0, 1, \ldots$ let:

$$
\begin{aligned}
&1. \quad \alpha^n = \frac{p^n \cdot p^n}{p^n \cdot Ap^n} \\
&2. \quad u^{n+1} = u^n + \alpha^n p^n \\
&3. \quad r^{n+1} = r^n - \alpha^n Ap^n \\
&4. \quad \beta^n = \frac{r^{n+1} \cdot r^{n+1}}{r^n \cdot r^n} \\
&5. \quad p^{n+1} = r^{n+1} + \beta^n p^n \\
&\quad n = n + 1 \text{ and Go to 1}
\end{aligned}
\tag{6.20}
$$

When we apply it to Eq. (6.15) and use

$$(I - \mu_{22}\tau_{22})u = \mu_{12}u_{21} \tag{6.21}$$

So that $A = I - \mu_{22}\tau_{22} = I + T_D$ and $b = \mu_{12}u_{21}$, then the general scheme takes the form:

Set $p^0 \equiv r^0 = b = \mu_{12}u_{2l}$ and $u^0 \equiv 0$.

Do for $n = 0, 1, 2, \ldots$

$$
\begin{aligned}
&1. \quad \psi^n = \tau_{22}p^n \\
&2. \quad \alpha^n = \frac{p^n \cdot p^n}{p^n \cdot p^n + \psi^n \cdot \psi^n} \\
&3. \quad u^{n+1} = u^n + \alpha^n p^n \\
&4. \quad q^n = \mu_{22}\psi^n \\
&5. \quad r^{n+1} = r^n - \alpha^n q^n \\
&6. \quad \beta^n = \frac{r^{n+1} \cdot r^{n+1}}{r^n \cdot r^n} \\
&7. \quad p^{n+1} = r^{n+1} + \beta^n p^n \\
&\quad n = n + 1 \text{ and Go to 1}
\end{aligned}
\tag{6.22}
$$

The algorithm for the equation

$$(I - \tau_{22}\mu_{22})u = \tau_{12}u_{11} \tag{6.23}$$

can be obtained in similar manner.

VII. DERIVATION OF THE ALGORITHMS AT THE CONTINUOUS LEVEL

The new algorithms presented in Section III, correspond to applications at the continuous level of the general algorithms of Section VI. Here, a brief explanation of such a procedure is presented leaving out technical details. As said before, our focus is on problems at the discrete level.

Consider the following problem: find $u \in H^2(\Omega)$ such that

$$-\Delta u = f_\Omega, \text{ in } \Omega \text{ in} \tag{7.1}$$

Together with

$$u = 0, \text{ on } \partial\Omega \tag{7.2}$$

Let $\Pi = \{\Omega_1, \Omega_2\}$ be a partition of Ω. Then, an alternative formulation of that problem is [18]: find $u \in \hat{H}^2(\Omega)$ such that

$$-\Delta u = f_\Omega, \text{ in } \Omega_\alpha, \quad \alpha = 1, 2 \tag{7.3}$$

and fulfills

$$u = 0, \text{ on } \partial\Omega \tag{7.4}$$

Together with

$$[\![u]\!] = 0 \quad \text{and} \quad \left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] = 0, \text{ on } \Gamma \tag{7.5}$$

Let

$$\begin{cases} \overline{D} \equiv \{w \in H^2(\Omega) | \text{trace}(w) = 0, \text{ on } \partial\Omega\} \\ \tilde{D} \equiv \{w \in \hat{H}^2(\Omega) | \text{trace}(w) = 0, \text{ on } \partial\Omega\} \end{cases} \tag{7.6}$$

Furthermore, we introduce the space $D \subset \tilde{D}$ as:

$$D \equiv \{w \in \tilde{D} | -\Delta u = 0, \text{ in } \Omega_\alpha, \alpha = 1, 2\} \tag{7.7}$$

Additional definitions are:

$$D_{11} \equiv \{w \in D | \dot{w} = 0, \text{ on } \Gamma\} \quad \text{and} \quad D_{12} \equiv \{w \in D | [\![w]\!] = 0, \text{ on } \Gamma\} \tag{7.8}$$

$$D_{21} \equiv \left\{ w \in D \left| \frac{\widehat{\partial w}}{\partial n} = 0, \text{ on } \Gamma \right. \right\} \quad \text{and} \quad D_{22} \equiv \left\{ w \in D \left| \left[\!\!\left[\frac{\partial w}{\partial n}\right]\!\!\right] = 0, \text{ on } \Gamma \right. \right\} \tag{7.9}$$

Then, every function $w \in D$ can be written in a unique manner as

$$w = w_{11} + w_{12}, \text{ with } w_{11} \in D_{11} \,\&\, w_{12} \in D_{12} \tag{7.10}$$

Furthermore, D_{21} and D_{11} as well as D_{22} and D_{12} are orthogonal with respect to the inner product

$$u \cdot w \equiv \sum_{\alpha=1}^{2} \int_{\Omega_\alpha} \nabla w \cdot \nabla u dx \tag{7.11}$$

This can be seen using the relation

$$\sum_{\alpha=1}^{2} \int_{\Omega_\alpha} \nabla w \cdot \nabla u dx = -\int_\Gamma \left\{ \dot{w} \left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] + [\![w]\!] \frac{\widehat{\partial u}}{\partial n} \right\} dx \tag{7.12}$$

which can be shown applying Eq. (2.14). The transformation $\tau_{22} : D_{12} \to D_{22}$ is characterized by: Given a function $w \in D$ such that

$$[\![w]\!] = 0, \text{ on } \Gamma \tag{7.13}$$

Then $(\tau_{22} w) \in D$ is such that

$$\left[\!\!\left[\frac{\partial(\tau_{22} w)}{\partial n}\right]\!\!\right] = 0 \quad \text{and} \quad \frac{\partial(\tau_{22} w)}{\partial n} = \frac{\widehat{\partial w}}{\partial n}, \text{ on } \Gamma \tag{7.14}$$

Similarly, the transformation $\mu_{22} : D_{22} \to D_{12}$ is characterized by: Given a function $w \in D$ such that

$$\left[\!\left[\frac{\partial w}{\partial n}\right]\!\right] = 0, \text{ on } \Gamma \tag{7.15}$$

Then $(\mu_{22} w) \in D$ is such that

$$[\![\mu_{22} w]\!] = 0 \quad \text{and} \quad \mu_{22} w = \dot{w}, \text{ on } \Gamma \tag{7.16}$$

Clearly, evaluating $\tau_{22} w$ requires solving a problem with Neumann conditions on Γ in each one of the partition-subdomains, Ω_1 and Ω_2, while evaluating $\mu_{22} w$ requires solving a problem with Dirichlet conditions on Γ in each one of the partition-subdomains, Ω_1 and Ω_2. Hence, $T_D \equiv -\mu_{22}\tau_{22}$ involves a Neumann problem followed by a Dirichlet one, while $T_N \equiv -\tau_{22}\mu_{22}$ involves a Dirichlet problem followed by a Neumann one. For a comparison of this approach with standard formulations at the continuous level see the Appendix.

VIII. PROCEDURES FOR EVALUATING THE TRANSFORMATION COMPONENTS

To apply the general scheme explained in Section VI, effective procedures for evaluating the transformations

$$\tau_{\alpha\beta} : D_{1\alpha} \to D_{2\beta} \quad \text{and} \quad \mu_{\alpha\beta} : D_{2\alpha} \to D_{1\beta} \tag{8.1}$$

are required. This in turn is accomplished if $u_{\alpha\beta} \in D_{\alpha\beta}$, $\alpha, \beta = 1, 2$, such that

$$u = u_{11} + u_{12} = u_{21} + u_{22} \tag{8.2}$$

can be evaluated effectively, when $u \in D$ is given.

In this Section, we continue working with the assumptions of Section VI, but in addition we assume the following, which in particular is fulfilled by the system of Section IV: There exist linearly independent subsets:

$$\left.\begin{array}{c} \mathcal{B} \subset \tilde{D}, \quad \mathcal{B}_I \subset \tilde{D}_I, \quad \mathcal{B}_\Gamma \subset \tilde{D}_\Gamma, \\ \overline{\mathcal{B}}_\Gamma \subset \tilde{D}_\Gamma, \quad \mathcal{B}_{\Gamma M} \subset \tilde{D}_{\Gamma 2}, \quad \mathcal{B}_{\Gamma J} \subset \tilde{D}_{\Gamma 1} \end{array}\right\} \tag{8.3}$$

Which satisfy:

$$\mathcal{B} = \mathcal{B}_I \cup \mathcal{B}_\Gamma \quad \text{and} \quad \overline{\mathcal{B}}_\Gamma = \mathcal{B}_{\Gamma M} \cup \mathcal{B}_{\Gamma J} \tag{8.4}$$

The span of each one of the subsets $\mathcal{B}_\Gamma$ and $\overline{\mathcal{B}}_\Gamma$, is $\tilde{D}_\Gamma$; however, a distinguishing property of $\mathcal{B}_\Gamma$ is that its members have local support. We explain first a procedure for computing $u_{11} \in D_{11}$ and $u_{12} \in D_{12}$.

According to Eq. (4.18), we can write

$$u = \tilde{u}_J + \tilde{u}_M + \tilde{u}_I, \quad \text{where } \tilde{u}_J \in \tilde{D}_{\Gamma 1}, \tilde{u}_M \in \tilde{D}_{\Gamma 2} \text{ and } \tilde{u}_I \in \tilde{D}_I \tag{8.5}$$

Furthermore:

$$\begin{aligned} u_{11} &= (\tilde{u}_{11})_J + (\tilde{u}_{11})_I, \quad \text{where } (\tilde{u}_{11})_J \in \tilde{D}_{\Gamma 1} \text{ and } (\tilde{u}_{11})_I \in \tilde{D}_I \\ u_{12} &= (\tilde{u}_{12})_M + (\tilde{u}_{12})_I, \quad \text{where } (\tilde{u}_{12})_M \in \tilde{D}_{\Gamma 2} \text{ and } (\tilde{u}_{12})_I \in \tilde{D}_I \end{aligned} \tag{8.6}$$

Equations (8.2) and (8.6) together imply that

$$(\tilde{u}_{11})_J = \tilde{u}_J \quad \text{and} \quad (\tilde{u}_{12})_M = \tilde{u}_M \tag{8.7}$$

Therefore,

$$\begin{aligned} u_{11} &= \tilde{u}_J + (\tilde{u}_{11})_J, \quad \text{where } (\tilde{u}_{11})_I \in \tilde{D}_I \\ u_{12} &= \tilde{u}_M + (\tilde{u}_{12})_I, \quad \text{where } (\tilde{u}_{12})_I \in \tilde{D}_I \end{aligned} \tag{8.8}$$

Then, $(\tilde{u}_{11})_I \in \tilde{D}_I$ can be determined by the system of equations:

$$(\tilde{u}_{11})_I \cdot \tilde{w} = -\tilde{u}_J \cdot \tilde{w}, \quad \forall \tilde{w} \in \mathcal{B}_I \tag{8.9}$$

while $(\tilde{u}_{12})_I \in \tilde{D}_I$ is determined by

$$(\tilde{u}_{12})_I \cdot \tilde{w} = -\tilde{u}_M \cdot \tilde{w}, \quad \forall \tilde{w} \in \mathcal{B}_I \tag{8.10}$$

These equations, because $u_{11} \in D$ and $u_{12} \in D$, which is orthogonal to $\tilde{D}_I$. Each one of Eqs. (8.9) and (8.10) constitutes a sequence of "E" independent local problems.

On the other hand, when $u \in D$ is given, the function $u_{21} \in D_{21}$ is characterized by

$$\begin{aligned} u_{21} \cdot w &= 0, \quad \forall w \in \tilde{D}_I \\ (u_{21} - u) \cdot w &= 0, \quad \forall w \in D_{12} \\ u_{21} \cdot w &= 0, \quad \forall w \in D_{11} \end{aligned} \tag{8.11}$$

Using the facts that every $w \in D_{12}$ is

$$w = \tilde{w}_M + \tilde{w}_I, \quad \text{where } \tilde{w}_M \in \tilde{D}_{\Gamma 2} \text{ and } \tilde{w}_I \in \tilde{D}_I \tag{8.12}$$

and that every $w \in D_{11}$ is

$$w = \tilde{w}_J + \tilde{w}_I, \quad \text{where } \tilde{w}_J \in \tilde{D}_{\Gamma 1} \text{ and } \tilde{w}_I \in \tilde{D}_I \tag{8.13}$$

it is seen that the system of equations of Eq. (8.11) is equivalent to

$$\begin{aligned} u_{21} \cdot \tilde{w}_I &= 0, \quad \forall \tilde{w}_I \in \tilde{D}_I \\ u_{21} \cdot \tilde{w}_M &= u \cdot \tilde{w}_M, \quad \forall \tilde{w}_M \in \tilde{D}_{\Gamma 2} \\ u_{21} \cdot \tilde{w}_J &= 0, \quad \forall \tilde{w}_J \in \tilde{D}_{\Gamma 1} \end{aligned} \tag{8.14}$$

Furthermore, $\tilde{D}_\Gamma = \tilde{D}_{\Gamma 1} + \tilde{D}_{\Gamma 2}$ and therefore Eq. (8.14) is satisfied, if and only if:

$$\begin{aligned} u_{21} \cdot \tilde{w}_I &= 0, \quad \forall \tilde{w}_I \in \tilde{D}_I \\ u_{21} \cdot \tilde{w}_\Gamma &= u \cdot (\tilde{w}_\Gamma)_M, \quad \forall \tilde{w} \in \tilde{D}_\Gamma \end{aligned} \tag{8.15}$$

Here, it is understood that every $\tilde{w}_\Gamma \in \tilde{D}_\Gamma$ has been written as

$$\tilde{w}_\Gamma = (\tilde{w}_\Gamma)_J + (\tilde{w}_\Gamma)_M; \quad \text{with } (\tilde{w}_\Gamma)_J \in \tilde{D}_{\Gamma 1} \text{ and } (\tilde{w}_\Gamma)_M \in \tilde{D}_{\Gamma 2} \tag{8.16}$$

Finally, introducing the bases $\mathcal{B}_I$, and $\mathcal{B}_\Gamma$ of $\tilde{D}_I$ and $\tilde{D}_\Gamma$, respectively, Eq. (8.15) is replaced by

$$\begin{aligned} &u_{21} \cdot \tilde{w}_I = 0, \quad \forall \tilde{w}_I \in \mathcal{B}_I \\ &u_{21} \cdot \tilde{w}_\Gamma = u \cdot (\tilde{w}_\Gamma)_M, \quad \forall \tilde{w}_\Gamma \in \mathcal{B}_\Gamma \end{aligned} \tag{8.17}$$

Using the fact that all the functions $\tilde{w}_\Gamma \in \mathcal{B}_\Gamma$ have local support, it is seen that Eq. (8.17) constitutes a sequence of "E" independent local systems of equations. In a similar manner, it is shown that $u_{22} \in D_{22}$ satisfies

$$\begin{aligned} &u_{22} \cdot \tilde{w}_I = 0, \quad \forall \tilde{w}_I \in \mathcal{B}_I \\ &u_{22} \cdot \tilde{w}_\Gamma = u \cdot (\tilde{w}_\Gamma), \quad \forall \tilde{w}_\Gamma \in \mathcal{B}_\Gamma \end{aligned} \tag{8.18}$$

Here, again Eq. (8.18) constitutes a sequence of "E" independent local systems of equations. Finally, observe that generally only one of the two systems (8.17) and (8.18) need to be solved since $u = u_{21} + u_{22}$. A similar remark applies in the case of the pair of Eqs. (8.9) and (8.10).

Application of the CGM algorithms of Section VI, also requires computing u_{11} or u_{21}. The standard discrete version of the original problem, using continuous functions exclusively, is: find $\overline{u} \in \overline{D}$ such that

$$\overline{u} \cdot \overline{w} = \int_\Omega \overline{w} f_\Omega dx, \forall \overline{w} \in \overline{D} \tag{8.19}$$

This is equivalent to: find $\overline{u} \in \tilde{D}$ such that

$$\begin{aligned} \overline{u} \cdot \tilde{w}_I &= \int_\Omega \tilde{w}_I f_\Omega dx, \forall \tilde{w}_I \in \tilde{D}_I \\ \overline{u} \cdot \tilde{w}_M &= \int_\Omega \tilde{w}_M f_\Omega dx, \forall \tilde{w}_M \in \tilde{D}_{\Gamma 2} \\ \widetilde{(\overline{u})}_J &= 0 \end{aligned} \tag{8.20}$$

Let $\tilde{u}_P \in \tilde{D}$ be any function that satisfies

$$\begin{cases} \tilde{u}_P \cdot \tilde{w}_I = \displaystyle\int_\Omega \tilde{w}_I f_\Omega dx, \forall \tilde{w}_I \in \tilde{D}_I \\ (\tilde{u}_P)_J = 0 \end{cases} \tag{8.21}$$

And define $u \equiv \overline{u} - \tilde{u}_P$. Then it is seen that

$$\begin{cases} u \in D \\ u \cdot \tilde{w}_M = \displaystyle\int_\Omega \tilde{w}_M f_\Omega dx - \tilde{u}_P \cdot \tilde{w}_M, \forall \tilde{w}_M \in \tilde{D}_{\Gamma 2} \\ \tilde{u}_J = -(\tilde{u}_P)_J = 0 \end{cases} \tag{8.22}$$

Then, $u \in D_{12}$ and one can apply Eq. (8.17) to obtain u_{21}. A second option is to define $\tilde{u}_P \in \tilde{D}$ to be a function that satisfies

$$\begin{cases} \tilde{u}_P \cdot \tilde{w}_I = \displaystyle\int_\Omega \tilde{w}_I f_\Omega dx, \forall \tilde{w}_I \in \tilde{D}_I \\ \tilde{u}_P \cdot \tilde{w}_\Gamma = \displaystyle\int_\Omega \tilde{w}_\Gamma f_\Omega dx, \forall \tilde{w}_\Gamma \in \tilde{D}_\Gamma \end{cases} \tag{8.23}$$

In this case

$$u \cdot \tilde{w}_M = 0, \forall \tilde{w}_M \in \tilde{D}_{\Gamma 2} \quad \text{and} \quad \tilde{u}_J = -(\tilde{u}_P)_J \tag{8.24}$$

so that $u \in D_{22}$, by virtue of Eq. (8.15), while Eqs. (8.8) and (8.9) can be used to obtain u_{11}.

IX. LAPLACIAN-LIKE OPERATORS

In the application of the developments of previous Sections to iterative substructuring methods, there is usually a bilinear form $a(u, w)$ that satisfies

$$a(u, w) \equiv \sum_{\alpha=1}^{E} a_\alpha(u_\alpha, w_\alpha), \quad \forall u, w \in \tilde{D} \tag{9.1}$$

and is positive definite in $\overline{D}$. Here, for every $u \in \tilde{D}$ and $w \in \tilde{D}$, $u_\alpha \equiv u|_{\Omega_\alpha}$ and $w_\alpha \equiv w|_{\Omega_\alpha}$. When, $a(u, w)$ is also positive definite in $\tilde{D}$, then the procedures of previous sections can be applied using $a(u, w)$ as inner product in $\tilde{D}$. However, in some cases, such as when treating Laplace operator, it may happen that although $a(u, w)$ is non-negative in $\tilde{D}$, it does not satisfy the positive-definiteness condition there. This section is devoted to explain the modifications that are required in order to deal with problems of that kind.

We adopt the following assumptions:

1. $\tilde{D}$ is a finite-dimensional Hilbert space and

$$a(u, u) \geq 0, \quad \forall u \in \tilde{D} \tag{9.2}$$

2. When $u \in \overline{D}$ and $u \neq 0$,

$$a(u, u) > 0 \tag{9.3}$$

3. Let $D_C \subset \tilde{D}$ be the null subspace of $a(u, w)$; i.e.,

$$D_C \equiv \{v \in \tilde{D} | a(v, v) = 0\} \tag{9.4}$$

and $(\cdot, \cdot)$ is an inner product defined in D_C, with the property that

$$(u, w) = \sum_{\alpha=1}^{E} (u_\alpha, w_\alpha), \quad \forall u, w \in D_C \tag{9.5}$$

When Eq. (9.1) is satisfied, there exists a linearly independent basis of D_C that will be denoted by $\mathcal{B}_C = \{w_C^1, \ldots, w_C^{N_C}\} \subset D_C$, such that each one of its members has local support. Furthermore, the above assumptions imply

$$\overline{D} \cap D_C = \{0\} \tag{9.6}$$

This permits choosing an algebraic complement of D_C, $\tilde{D}_R \subset \tilde{D}$, which enjoys the following properties:

- $$\tilde{D} = D_C + \tilde{D}_R \quad \text{and} \quad D_C \cap \tilde{D}_R = \{0\} \tag{9.7}$$
- $a(u, w)$ is symmetric and positive definite in $\tilde{D}_R \times \tilde{D}_R$; and
- $$\overline{D} \subset \tilde{D}_R \tag{9.8}$$

Then every function $u \in \tilde{D}$ can be written in a unique manner as

$$u = u_C + u_R, \quad \text{where } u_C \in D_C \text{ and } u_R \in \tilde{D}_R \tag{9.9}$$

Observe also that

$$\mathcal{B}_I \subset \tilde{D}_I \subset \overline{D} \subset \tilde{D}_R \tag{9.10}$$

Using the representation of Eq. (9.9), we define the inner product:

$$u \cdot w = (u_C, w_C) + a(u, w), \quad \forall u, w \in \tilde{D} \tag{9.11}$$

When the definition of Eq. (9.11) is adopted the following identities that will be used in the sequel, are fulfilled:

$$u \cdot w = (u_C, w_C) + a(u_R, w_R), \quad \forall u, w \in \tilde{D} \tag{9.12}$$

and

$$a(u, w) = a(u, w_R) = u \cdot w_R = u_R \cdot w_R, \quad \forall u, w \in \tilde{D} \tag{9.13}$$

Furthermore, it can be verified that the operation defined by Eq. (9.11) is indeed an inner product, in which the subspaces D_C and $\tilde{D}_R$ constitute an orthogonal pair of subspaces, which are complementary with respect to $\tilde{D}$. Then the developments of Sections III–VI will be applied to the space $\tilde{D}$, provided with such an inner product.

The only points that require further explanation are the procedures of Section VIII, for evaluating the transformation components. To this end we recall the unique representation of every $V \in \tilde{D}$:

$$V = V_\Gamma + V_I, \quad V_\Gamma \in \tilde{D}_\Gamma, \quad \text{and} \quad V_I \in \tilde{D}_I \tag{9.14}$$

Then, we define

$$\left.\begin{aligned} \tilde{D}_{\Gamma C} &\equiv \{V_\Gamma \in \tilde{D}_\Gamma | \exists V \in D_C \ni V = V_\Gamma + V_I\} \\ \tilde{D}_{\Gamma R} &\equiv \{V_\Gamma \in \tilde{D}_\Gamma | \exists V \in D_R \ni V = V_\Gamma + V_I\} \end{aligned}\right\} \tag{9.15}$$

These definitions together with Eq. (9.7) imply that

$$\tilde{D}_\Gamma = \tilde{D}_{\Gamma C} + \tilde{D}_{\Gamma R} \quad \text{and} \quad \tilde{D}_{\Gamma C} \cap \tilde{D}_{\Gamma R} = 0 \tag{9.16}$$

In what follows the notations $\mathcal{B}_{\Gamma C}$ and $\mathcal{B}_{\Gamma R}$ will be used for a linear independent basis of $\tilde{D}_{\Gamma C}$ and $\tilde{D}_{\Gamma R}$, respectively.

Next, we revise the application of Eqs. (8.9), (8.10), (8.17), and (8.18). Consider first Eqs. (8.9) and (8.10), taking into account that $\mathcal{B}_I \subset \tilde{D}_R$, they can be written, respectively, as:

$$a((\tilde{u}_{11})_I, \tilde{w}) = -a(\tilde{u}_J, \tilde{w}), \quad \forall \tilde{w} \in \mathcal{B}_I \tag{9.17}$$

and

$$a((\tilde{u}_{12})_I, \tilde{w}) = -a(\tilde{u}_M, \tilde{w}), \quad \forall \tilde{w} \in \mathcal{B}_I \tag{9.18}$$

In this form they are directly suitable for their application, if they are used jointly with Eq. (8.8).

Consider now Eq. (8.17), the first part of it can be written as

$$a((\tilde{u}_{21})_I, \tilde{w}) = 0, \quad \forall \tilde{w} \in \mathcal{B}_I \tag{9.19}$$

whereas the second part of is

$$\left.\begin{aligned} u_{21} \cdot \tilde{w}_{\Gamma C} &= u \cdot (\tilde{w}_{\Gamma C})_M, \quad \forall \tilde{w}_{\Gamma C} \in \mathcal{B}_{\Gamma C} \\ u_{21} \cdot \tilde{w}_{\Gamma R} &= u \cdot (\tilde{w}_{\Gamma R})_M, \quad \forall \tilde{w}_{\Gamma R} \in \mathcal{B}_{\Gamma R} \end{aligned}\right\} \tag{9.20}$$

These latter equations can be transformed into

$$((u_{21})_C, w_C) = a(u, (\tilde{w}_{\Gamma C})_M), \quad \forall w_C \in \mathcal{B}_C \tag{9.21}$$

$$a(u_{21}, \tilde{w}_{\Gamma R}) = a(u, (\tilde{w}_{\Gamma R})_M), \quad \forall \tilde{w}_{\Gamma R} \in \mathcal{B}_{\Gamma R} \tag{9.22}$$

Here, $\tilde{w}_{\Gamma C} \in \tilde{D}_{\Gamma C}$ is defined by the condition

$$w_C = \tilde{w}_{\Gamma C} + \tilde{w}_I \tag{9.23}$$

Eqs. (9.19) and (9.21) need to be complemented with

$$u_{21} = (\tilde{u}_{21})_\Gamma + (\tilde{u}_{21})_I, \quad \text{where } (\tilde{u}_{21})_\Gamma \in \tilde{D}_\Gamma \text{ and } (\tilde{u}_{21})_I \in \tilde{D}_I \tag{9.24}$$

An important property that should be noticed is that the systems of equations defined by Eqs. (9.21) and (9.22), respectively, are independent of each other.

X. DUAL-PRIMAL METHODS

The dual-primal methods are procedures that permit dealing with partition-vertices. The basic idea of such methods consists in keeping undivided the functions associated with such vertices and treating them as internal nodes. An effect of such a procedure is, however, to couple the systems of equations corresponding to partition-subdomains that share a vertex, which may be inconvenient in some instances. For completeness, in this section we incorporate dual-primal methods in our framework.

The collection of sets $\{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\} \subset \{\mathcal{B}^1, \ldots, \mathcal{B}^{\overline{N}}\}$, of Section IV, is divided into two subfamilies, $\{\mathcal{B}_\Delta^1, \ldots, \mathcal{B}_\Delta^{N_\Delta}\} \subset \{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\}$ and $\{\mathcal{B}_\pi^1, \ldots, \mathcal{B}_\pi^{N_\pi}\} \subset \{\mathcal{B}_\Gamma^1, \ldots, \mathcal{B}_\Gamma^{\overline{N}_\Gamma}\}$; they are defined by the following conditions: $\mathcal{B}_\Gamma^i \in \{\mathcal{B}_\Delta^1, \ldots, \mathcal{B}_\Delta^{N_\Delta}\}$ if and only if the cardinality of $\mathcal{B}_\Gamma^i$ is two, and

$\mathcal{B}_\Gamma^i \in \{\mathcal{B}_\pi^1, \ldots, \mathcal{B}_\pi^{N_\pi}\}$ if and only if the cardinality of $\mathcal{B}_\Gamma^i$ is greater than two. Each one of the sets of the collection $\{\mathcal{B}_\Delta^1, \ldots, \mathcal{B}_\Delta^{N_\Delta}\}$ will be said to be a "dual set," whereas each one of the collection $\{\mathcal{B}_\pi^1, \ldots, \mathcal{B}_\pi^{N_\pi}\}$ will be said to be a "primal set." Clearly, functions of a primal set correspond to functions associated with vertices.

Let $\mathcal{B}_\pi^i$ be a primal set, then we define $\overline{\mathcal{B}}_\pi^i = \{w_M^i\}$ where $w_M^i \in \overline{D}$ is the mother function of the primal set, $\mathcal{B}_\pi^i$. Furthermore,

$$\mathcal{B}_\pi \equiv \bigcup_{i=1}^{\overline{N}_\pi} \overline{\mathcal{B}}_\pi^i = \left\{w_M^1, \ldots, w_M^{N_\pi}\right\} \tag{10.1}$$

and

$$\mathcal{B}_\eta \equiv \mathcal{B}_\pi \cup \mathcal{B}_I \tag{10.2}$$

On other hand, when the cardinality of $\mathcal{B}_\Gamma^i$ is two, so that it is a dual set, we proceed in a similar fashion to that of Section IV, and define:

$$\mathcal{B}_\Delta \equiv \bigcup_{i=1}^{N_\Delta} \mathcal{B}_\Delta^i, \quad \overline{\mathcal{B}}_\Delta^i \equiv \left\{w_M^i, w_J^i\right\}, \quad \overline{\mathcal{B}}_\Delta \equiv \bigcup_{i=1}^{N_\Delta} \overline{\mathcal{B}}_\Delta^i, \quad \text{and} \quad \mathcal{B} \equiv \mathcal{B}_\Delta \cup \mathcal{B}_\eta \tag{10.3}$$

as well as

$$\mathcal{B}_{\Delta M} \equiv \left\{w_M^1, \ldots, w_M^N\right\} \quad \text{and} \quad \mathcal{B}_{\Delta J} \equiv \left\{w_J^1, \ldots, w_J^N\right\} \tag{10.4}$$

Then: $\mathcal{B}_{\Delta M} \subset \overline{\mathcal{B}} \subset \overline{D}$; each set $\overline{\mathcal{B}}_\Delta$ and $\mathcal{B}_\Delta$, spans the same linear subspace and a conspicuous property is that all the elements of $\mathcal{B}_\Delta$ have local support, while the same is not true of $\overline{B}_\Delta$. Furthermore:

$$\overline{\mathcal{B}} = \mathcal{B}_{\Delta M} \cup \mathcal{B}_\eta \tag{10.5}$$

The subspaces spanned by the sets of functions $\mathcal{B}, \mathcal{B}_\eta, \mathcal{B}_\Delta, \mathcal{B}_{\Delta J}$, and $\mathcal{B}_{\Delta M}$ be denoted by $\tilde{D}, \tilde{D}_\eta, \tilde{D}_\Sigma, \tilde{D}_{\Sigma 1}$, and $\tilde{D}_{\Sigma 2}$, respectively. We are now in essentially the same position as in Section IV, if the following replacements are made:

$$\left.\begin{array}{c} \tilde{D}_\eta \to \tilde{D}_1, \quad \tilde{D}_\Sigma \to \tilde{D}_\Gamma, \quad \tilde{D}_\eta \to \tilde{D}_\eta \\ \tilde{D}_{\Sigma 1} \to \tilde{D}_{\Gamma 1}, \quad \tilde{D}_{\Sigma 2} \to \tilde{D}_{\Gamma 2} \end{array}\right\} \tag{10.6}$$

Since corresponding to the assumptions of Section V we have:

$$\left.\begin{array}{c} \tilde{D} \equiv \tilde{D}_\eta + \tilde{D}_\Sigma \quad \text{and} \quad \tilde{D}_\eta \cap \tilde{D}_\Sigma = \{0\} \\ \tilde{D}_\Sigma \equiv \tilde{D}_{\Sigma 1} + \tilde{D}_{\Sigma 2} \quad \text{and} \quad \tilde{D}_{\Sigma 1} \cap \tilde{D}_{\Sigma 2} = \{0\} \\ \overline{D} = \tilde{D}_{\Sigma 2} + \tilde{D}_J \end{array}\right\} \tag{10.7}$$

Therefore, we define

$$D \equiv (\tilde{D}_\eta)^\perp \tag{10.8}$$

Just as in Section V, once $D \subset \tilde{D}$ has been defined, the orthogonal complements will be taken with respect to D. And the following definitions are adopted:

$$D_{11} \equiv \text{proj}_D \tilde{D}_{\Sigma 1} \quad \text{and} \quad D_{12} \equiv \text{proj}_D \tilde{D}_{\Sigma 2} \tag{10.9}$$

together with

$$D_{21} \equiv (D_{11})^{\perp} \quad \text{and} \quad D_{22} \equiv (D_{12})^{\perp} \tag{10.10}$$

Then

$$D = D_{11} + D_{12} \quad \text{and} \quad D_{11} \cap D_{12} = \{0\} \tag{10.11}$$

The most important difference with respect to the developments of Section III, is that there all the functions of the space $\tilde{D}_I$, have local support, while that is not the case for the subspace $\tilde{D}_\gamma$, as here defined. Because of this fact, there will be some coupling between the equations corresponding to different subdomains that share a common vertex.

XI. APPLICATION WITH LINEAR FUNCTION

This Section is devoted to illustrate the application of our axiomatic framework. To make the presentation friendlier, we start with a very simple illustration and increase the complexity of the subjects treated progressively, while following the order of Sections IV–VI and VIII for our presentation.

A. The discretization process

In domain decomposition methods of the kind discussed in this Section, one has actually two partitions; on one hand, the partition $\Pi \equiv \{\Omega_1, \ldots, \Omega_E\}$ considered in previous Sections, which characterizes the DDM applied in Ω, and a triangulation of Ω in the sense of Ciarlet [43], whose elements need not be triangles, but can also be rectangles or parallelepipeds. For greater clarity we reserve the terms domain decomposition, or simply partition for the first one, whereas the second one will be "the triangulation." The notation Γ refers to the internal boundary associated with the domain decomposition of Ω, whereas Γ_t will be the boundary separating the triangulation-elements from each other. It is assumed throughout that $\Gamma \subset \Gamma_t$.

In this Section, $\overline{D}$ is the finite-dimensional Hilbert space whose elements are piecewise-linear functions defined in a triangulation of Ω, continuous in Ω and which vanish on $\partial\Omega$, so that $\overline{D} \subset H_0^1(\Omega)$. The subset $\overline{\mathcal{B}} \subset \overline{D}$ is the collection of functions of $\overline{D}$, which satisfy the condition of vanishing at every node of the triangulation except at one, where it takes the value one. Then $\overline{\mathcal{B}}$ is a linearly independent basis of $\overline{D}$, as required in Section IV. Furthermore, we define:

$$D(\Omega_\alpha) \equiv \{v | v = u|_{\Omega_\alpha} \text{ and } u \in \overline{D}\} \tag{11.1}$$

together with

$$\tilde{D} \equiv D(\Omega_1) \oplus \cdots \oplus D(\Omega_E) \tag{11.2}$$

We first consider the case when $E = 2$, for which the domain decomposition has only two partition-subdomains and adopt the notation $\Pi = \{\Omega_+, \Omega_-\}$, as illustrated in Fig. 1. For simplicity, in the illustrative figures the triangulation is made of squares and Γ is a straight line. According to Section IV, the elements of $\mathcal{B}$ are the same as those of $\overline{\mathcal{B}}$, except for the functions associated with the nodes lying on Γ, which have been divided into two by the domain decomposition Π; clearly, each one of the heirs of such functions has local support, since its support is contained either in the closure of Ω_+ or of Ω_-. We write $\mathcal{B}_\Gamma^i = \{w_+^i, w_-^i\}$ for the pair of heirs associated

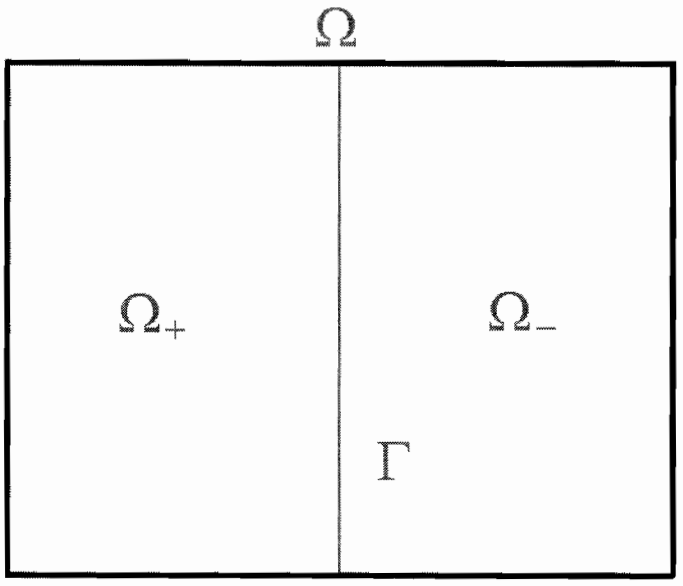

FIG. 1. Partition with two subdomains.

with each node lying on Γ, with an obvious notation for the restrictions to Ω_+ and Ω_- of each $\overline{w}^i \in \overline{\mathcal{B}}$. Therefore,

$$\overline{w}^i = w_+^i + w_-^i \tag{11.3}$$

The union of all the pairs $\mathcal{B}_\Gamma^i$, is the set of functions $\mathcal{B}_\Gamma$; i.e.,

$$\mathcal{B}_\Gamma \equiv \bigcup_{i=1}^{N_\Gamma} \mathcal{B}_\Gamma^i \tag{11.4}$$

According to Section IV, the functions associated with the internal nodes of each one the partition-subdomains, which have not been divided, constitute the set $\mathcal{B}_I \subset \overline{\mathcal{B}} \subset \mathcal{B}$. It can be seen that

$$\mathcal{B} = \mathcal{B}_I \cup \mathcal{B}_\Gamma \tag{11.5}$$

Observe that once $\overline{\mathcal{B}}$ and the partition Π are given, the set $\mathcal{B}_\Gamma$ as defined above is uniquely determined and, furthermore, that all the members of $\mathcal{B}_\Gamma$ have local support. However, none of these properties is enjoyed by the set $\mathcal{B}_\Gamma$. Indeed, according to Section IV, $\overline{\mathcal{B}}_\Gamma^i \equiv \{w_M^i, w_J^i\}$, where

$$w_M^i \equiv \overline{w}^i = w_+^i + w_-^i \tag{11.6}$$

Clearly, w_M^i does not have local support. As for w_J^i, all that is required is that it be such that the pair $\{w_M^i, w_J^i\}$ span the same linear space as the pair $\{w_+^i, w_-^i\}$, and this condition does not define it uniquely. A possible choice that fulfills this condition is

$$w_J^i = w_+^i - w_-^i \tag{11.7}$$

The correspondence between the pairs $\{w_+^i, w_-^i\}$ and w_M^i, w_J^i that satisfies Eqs. (11.6) and (11.7) is one-to-one and the inverse transformation is:

$$w_+^i = \frac{1}{2}\left(w_M^i + w_J^i\right) \quad \text{and} \quad w_-^i = \frac{1}{2}\left(w_M^i - w_J^i\right) \tag{11.8}$$

However, there are many other options that produce definitions of w_J^i that are compatible with the axioms of Section IV, among them:

$$w_J^i = w_+^i \tag{11.9}$$

or

$$w_J^i = w_-^i \tag{11.10}$$

When Eq. (11.9) is adopted:

$$w_+^i = w_J^i \quad \text{and} \quad w_-^i = w_M^i - w_J^i \tag{11.11}$$

In this Section we mostly stick to the definition of Eq. (11.7), which leads to developments that are closer to standard approaches. Further comments on alternative definitions will be made elsewhere.

From Section IV, we have

$$\mathcal{B}_{\Gamma M} = \left\{w_M^1, \ldots, w_M^{\overline{N}_\Gamma}\right\}, \quad \mathcal{B}_{\Gamma J} = \left\{w_J^1, \ldots, w_J^{\overline{N}_\Gamma}\right\}, \quad \text{and} \quad \overline{\mathcal{B}}_\Gamma = \mathcal{B}_{\Gamma M} \cup \mathcal{B}_{\Gamma J} \tag{11.12}$$

The subspaces $\tilde{D}_I$, $\tilde{D}_\Gamma$, $\tilde{D}_{\Gamma 1}$, and $\tilde{D}_{\Gamma 2}$ are spanned by the sets of functions $\mathcal{B}_I$, $\mathcal{B}_\Gamma$, $\mathcal{B}_{\Gamma J}$, and $\mathcal{B}_{\Gamma M}$, respectively. Then, from Section V, Eqs. (5.4)–(5.8), we have

$$\text{proj}_D \tilde{D}_\Gamma = D \tag{11.13}$$

together with

$$\begin{aligned} D_{11} &\equiv \text{proj}_D \tilde{D}_{\Gamma 1} \quad \text{and} \quad D_{12} \equiv \text{proj}_D \tilde{D}_{\Gamma 2} \\ D_{21} &\equiv (D_{11})^\perp \quad \text{and} \quad D_{22} \equiv (D_{12})^\perp \end{aligned} \tag{11.14}$$

Here, the orthogonal complements are taken with respect to D. The space D is made of functions which satisfy a weak version of the homogeneous differential equation in each one of the subdomains separately. A more detailed description and a more complete interpretation, of such weak solutions can be obtained applying the Green-Herrera formulas developed in previous papers by Herrera and his collaborators [30–34]. The subspace $D_{12} \subset D$ is uniquely defined, independently of the definition of w_J^i that is adopted, and it is constituted by the functions of D that are continuous across Γ. Then the subspace $D_{22} \subset D$ is also uniquely defined, since according to Section VI, it is the orthogonal complement of D_{12}. On the other hand, the definition of the subspace $D_{11} \subset D$ does depend on which of Eqs. (11.7), (11.9), or (11.10) is adopted. However, independently of which of these equations is chosen, the members of D_{11} are generally discontinuous. In particular, if Eq. (11.7) is chosen, as we shall do in what follows, D_{11} is constituted by the members of D whose average across Γ is zero; if, on the other hand, Eq. (11.9) is chosen, then D_{11} is constituted

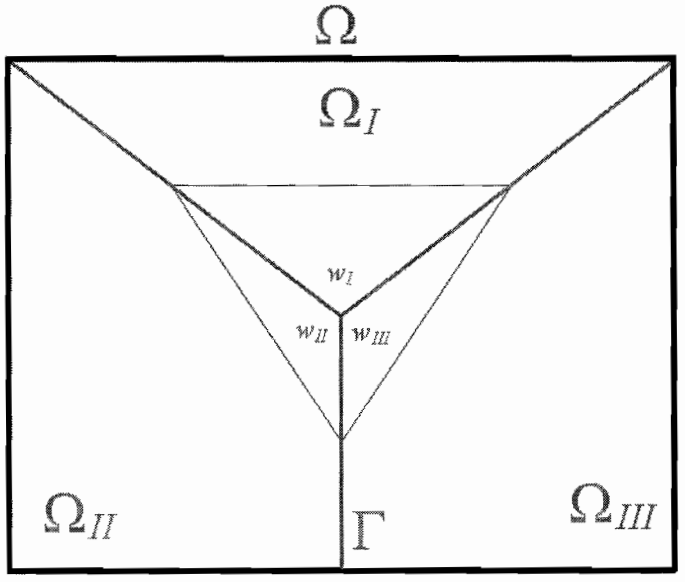

FIG. 2. Partition with three subdomains.

by the members of D that vanish identically on Ω_-. In turn, D_{21} is defined in terms D_{11}, as its orthogonal complement (recall Section VI).

B. The treatment of vertices

The general method described in Sections IV–VIII permits dealing with cases in which the partition has vertices. By a vertex, we mean a node on Γ where more than two partition-subdomains meet. Consider for example the case when $E \geq 3$, and assume that three partition subdomains meet at a node on Γ (Fig. 2). Let $\overline{w}^i \in \overline{\mathcal{B}}_\Gamma$ be the continuous piecewise linear function that takes the value one at such a node; then, $\overline{w}^i$ has three heirs so that $\mathcal{B}^i_\Gamma = \{w^i_I, w^i_{II}, w^i_{III}\}$, where w^i_I, w^i_{II}, and w^i_{III} are the restrictions of the function $\overline{w}^i$ to each one of the three partition-subdomains. In such a case, one necessarily has $\overline{\mathcal{B}}^i_\Gamma = \{w^i_M, w^i_{J1}, w^i_{J2}\}$ with $w^i_M \equiv \overline{w}^i$, while the pair $\{w^i_{J1}, w^i_{J2}\}$ is an algebraic complementary set of $w^i_M = \overline{w}^i$. Then, a definition that satisfies the axioms of Section IV is

$$w^i_{J1} \equiv w^i_I, \quad w^i_{J2} \equiv w^i_{II} \quad \text{and} \quad w^i_M \equiv \overline{w}^i = w^i_I + w^i_{II} + w^i_{III} \tag{11.15}$$

Although other choices are possible. This equation permits carrying out the transformation $\mathcal{B}^i_\Gamma \to \overline{\mathcal{B}}^i_\Gamma$ that is used in the application of the methods here presented; it is given by

$$w^i_I = w^i_{J1}, \quad w^i_{II} = w^i_{J2} \quad \text{and} \quad w^i_{III} = w^i_M - w^i_{J1} - w^i_{J2} \tag{11.16}$$

This permits expressing each one of the functions of $\mathcal{B}^i_\Gamma$, which have local support, in the form given in Eq. (4.15); more precisely, using the notation of Eq. (4.15), we have

$$\left.\begin{aligned} \left(w^i_I\right)_J &= w^i_I \quad \text{and} \quad \left(w^i_I\right)_M = 0 \\ \left(w^i_{II}\right)_J &= w^i_{II} \quad \text{and} \quad \left(w^i_{II}\right)_M = 0 \end{aligned}\right\} \tag{11.17}$$

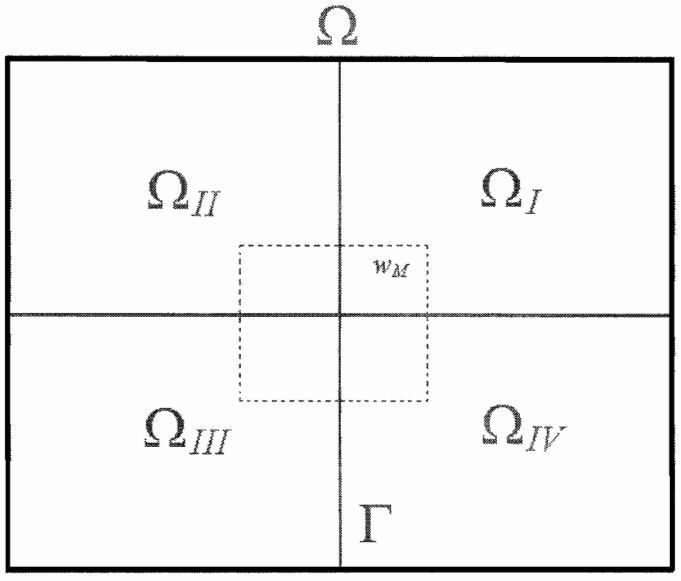

FIG. 3. Partition with four subdomains.

Together with

$$\left(w^i_{III}\right)_J = -w^i_I - w^i_{II} \quad \text{and} \quad \left(w^i_{III}\right)_M = w^i_M = w^i_I + w^i_{II} + w^i_{III} \tag{11.18}$$

The relations of Eqs. (11.17) and (11.18) are required for the application of Eqs. (8.17) and (8.18).

If four partition subdomains meet at a point of Γ (Fig. 3), then $\mathcal{B}^i_\Gamma \equiv \{w^i_I, w^i_{II}, w^i_{III}, w^i_{IV}\}$ and one can take $\overline{\mathcal{B}}^i_\Gamma \equiv \{w^i_M, w^i_{J1}, w^i_{J2}, w^i_{J3}\}$, with

$$w^i_{J1} \equiv w^i_I, \quad w^i_{J2} \equiv w^i_{II}, \quad w^i_{J3} \equiv w^i_{III} \quad \text{and} \quad w^i_M \equiv \overline{w}^i = w^i_I + w^i_{II} + w^i_{III} + w^i_{IV} \tag{11.19}$$

The inverse transformation for this case is:

$$\left.\begin{aligned} \left(w^i_I\right)_J &= w^i_I \quad \text{and} \quad \left(w^i_I\right)_M = 0 \\ \left(w^i_{II}\right)_J &= w^i_{II} \quad \text{and} \quad \left(w^i_{II}\right)_M = 0 \\ \left(w^i_{III}\right)_J &= w^i_{III} \quad \text{and} \quad \left(w^i_{III}\right)_M = 0 \end{aligned}\right\} \tag{11.20}$$

Together with

$$\left(w^i_{IV}\right)_J = -w^i_I - w^i_{II} - w^i_{III} \quad \text{and} \quad \left(w^i_{IV}\right)_M = w^i_M = w^i_I + w^i_{II} + w^i_{III} + w^i_{IV} \tag{11.21}$$

C. The local systems of equations

In this Subsection, we explain the procedures and discuss the work required for applying the systems of equations that were developed in Section VIII; in particular, how to apply Eqs. (8.9), (8.10), (8.17), and (8.18) when a function $u \in D$ is given. To make more concrete some of the general arguments that follow, the reader may take as an illustration the case of a two partition-subdomains.

Firstly, it is necessary to express the function $u \in D$ in the form of Eq. (4.18):

$$u = \tilde{u}_J + \tilde{u}_M + \tilde{u}_I; \quad \text{with } \tilde{u}_J \in \tilde{D}_{\Gamma 1}, \tilde{u}_M \in \tilde{D}_{\Gamma 2} \text{ and } \tilde{u}_I \in \tilde{D}_I \tag{11.22}$$

Equation (8.9) splits into independent systems of equations—for the values of u_{11} at internal nodes—one for each partition-subdomain. For each one of these systems of equations, the system-matrix is a square matrix of size equal to the number of internal nodes of each partition-subdomain. The same argument applies to Eq. (8.10). A very similar analysis applies to the systems of Eqs. (8.17) and (8.18), since they also split into independent systems of equations, one system for each partition subdomain, in spite of the fact that now functions associated with nodes that lay on Γ also occur in those systems. Observe the systems corresponding to different partition-subdomains remain indeed uncoupled, because the functions $\tilde{w}_\Gamma \in \mathcal{B}_\Gamma$ have local support. Furthermore, in Eqs. (8.17) and (8.18), it maybe noticed that the exchange of information between different subdomains is accomplished through their right-hand sides, where the functions $(\tilde{w}_\Gamma^i)_M$ and $(\tilde{w}_\Gamma^i)_J$ occur. Indeed, although every function $\tilde{w}_\Gamma \in \mathcal{B}_\Gamma$ has local support, the mother function $(\tilde{w}_\Gamma)_M = \overline{w}_\Gamma \in \overline{\mathcal{B}}$ has a support whose intersection with subdomains different to that on which the support of $\tilde{w}_\Sigma$ lies, is nonvoid. When the mother function $\overline{w}_\Gamma$ has only two heirs, then $(w_\Gamma)_M \equiv (\overline{w}_\Gamma)_+ + (\overline{w}_\Gamma)_-$ has a support that intersects more than one subdomain. More generally, in vertices, where the number of heirs is greater than two the mother function intersects all of them. Something similar happens with $(w_\Gamma)_J$, as Eqs. (11.18) and (11.21) illustrate.

In conclusion, Eqs. (8.9), (8.10), (8.17) and (8.18) are local systems of equations that can be applied separately in each one of the partition subdomains. The exchange of information between different subdomains is accomplished through the right-hand members of those equations, while the equations systems themselves remain uncoupled. In achieving this advantageous feature, the double expression of functions—in terms of the two bases $\mathcal{B}_\Gamma$ and $\overline{\mathcal{B}}_\Gamma$—plays an essential role.

D. The treatment of singular bilinear forms

Let us consider the case of the Laplacian operator. Then,

$$a(u, w) \equiv \sum_{\alpha=1}^{E} \int_{\Omega_\alpha} \nabla u \cdot \nabla w dx \tag{11.23}$$

When all the partition-subdomains touch the outer domain-boundary, $\partial\Omega$, then $a(u, w)$ is positive definite. However, if for some $\alpha = 1, \ldots, E$, the closure of the subdomain Ω_α does not intersect the outer boundary in $\partial\Omega$, then $a(u, w)$ is only non-negative. In this latter case, let the nonvoid subfamily of such partition-subdomains be

$$\{\Omega_1^0, \ldots, \Omega_{E_0}^0\} \subset \{\Omega_1, \ldots, \Omega_E\} \tag{11.24}$$

Then, the subspace $D_C \subset D$ defined by Eq. (9.4) has the basis $\mathcal{B}_C \equiv \{w_C^1, \ldots, w_C^{N_c}\}$, where $N_C = E_0$ and for each $\alpha = 1, \ldots, E_0$ the function support of w_C^α is contained in the closure of Ω_α^0. Furthermore, the restriction of w_C^α to Ω_α^0, can be taken to be identically equal to 1.

Thereby, we point out a related property. For this purpose, given any $w_C^\alpha \in \mathcal{B}_C$, define $\tilde{w}_{\Gamma C}^\alpha \in \tilde{D}_\Gamma$ as the unique function that satisfies the representation:

$$w_C^\alpha = \tilde{w}_{\Gamma C}^\alpha + \tilde{w}_I, \quad \text{with } \tilde{w}_{\Gamma C}^\alpha \in \tilde{D}_\Gamma \text{ and } \tilde{w}_I \in \tilde{D}_I \tag{11.25}$$

When Ω_α^0 is a polygonal subdomain, it can be seen that such a function is the unique linear combination of functions of $\mathcal{B}_\Gamma$, with support in Ω_α^0, that is 1 on $\partial\Omega_\alpha^0$ identically. Using this fact, after defining for each $\alpha = 1, \ldots, E_0$:

$$\mathcal{B}_{\Gamma\alpha}^0 \equiv \left\{\tilde{w}_\Gamma \in \mathcal{B}_\Gamma | \text{ support } (\tilde{w})_\Gamma \subset \overline{\Omega}_\alpha^0\right\} \tag{11.26}$$

it can be seen that

$$\tilde{w}_{\Gamma C}^\alpha = \sum_{\tilde{w}_\Gamma \in \mathcal{B}_{\Gamma\alpha}^0} \tilde{w}_\Gamma \tag{11.27}$$

To define the family $\mathcal{B}_{\Gamma R}$ of Section IX, for each $\alpha = 1, \ldots, E_0$, let $\mathcal{E}_{\Gamma\alpha}^0 \subset \mathcal{B}_{\Gamma\alpha}^0$ be a subfamily of $\mathcal{B}_{\Gamma\alpha}^0$ that is linearly independent $\tilde{w}_{\Gamma C}^\alpha$, then we can define

$$\mathcal{B}_{\Gamma R} \equiv \bigcup_{\alpha=1}^{E_0} \mathcal{E}_{\Gamma\alpha}^0 \tag{11.28}$$

In turn, and in view of Eq. (11.27), $\mathcal{E}_{\Gamma\alpha}^0$ can be constructed by removing just one function from each set $\mathcal{B}_{\Gamma\alpha}^0, \alpha = 1, \ldots, E_0$.

All that remains is to review the application of Eqs. (9.17), (9.18), (9.21), and (9.22). The Eqs. (9.17) and (9.18) do not differ from Eqs. (8.9) and (8.10), and can be applied in the same manner as these latter equations. As for Eqs. (9.21) and (9.22), a point that deserves mention is that special care must be exercised when evaluating $(\tilde{w}_{\Gamma C}^\alpha)_M$ and $(\tilde{w}_{\Gamma C}^\alpha)_J$; they can be obtained expressing $\tilde{w}_{\Gamma C}^\alpha$ as a linear combination of functions $\tilde{w}_\Gamma \in \mathcal{B}_\Gamma$ and then deriving $(\tilde{w}_\Gamma)_M$ and $(\tilde{w}_\Gamma)_J$ for each one of such functions.

E. Application of dual-primal methods

Other procedures for dealing with vertices are dual-primal methods, To derive this kind of method in the general framework presented in this article all that is required is to treat functions associated with vertices as if they were associated with internal nodes, as it was explained in Section X. As mentioned there, this leads to coupling between partition-subdomains that share a common vertex.

XII. CONCLUSIONS

At present, the standard treatment of partial differential equations defined on discontinuous functions is based on the use of Lagrange multipliers. In particular, this is the approach followed in the iterative substructuring methods that are available. Such a procedure, however, is not direct and has the inconvenient feature of increasing the number of degrees of freedom to be handled. In this article, it has been shown that more direct approaches are feasible. In particular, using the theory of partial differential equations in discontinuous piecewise-defined functions [18], without recourse to Lagrange multipliers, positive-definite preconditioned formulations of the Neumann–Neumann, and Dirichlet–Dirichlet (preconditioned FETI) types were developed. Apparently, the new algorithms can be applied not only to second order partial differential equations, such as Laplace equation, but also other equations; however, further research in such potential applications is required. On the other hand, the conditions under which the new algorithms are applicable to an equation or a system are established with precision in the article.

APPENDIX

The standard Neumann–Neumann algorithm that is usually formulated for Laplace equation is closely related to its new version, presented in Section III. However, from the standard Neumann–Neumann algorithm it is not possible obtain a positive definite transformation. And something similar happens with the corresponding Dirichlet–Dirichlet formulations. So, the purpose of this Appendix is to exhibit in a more explicit manner the differences between the standard and the new formulations that explain such a standard-formulation limitation.

In our notation the standard Neumann–Neumann algorithm, as given for example in [17], for a two-subdomains partition is

$$\left.\begin{aligned} &\mathcal{L}u^{n+1/2} = f_\Omega, \quad \text{in } \Omega_\alpha, \alpha = 1,2 \\ &[\![u^{n+1/2}]\!] = 0 \quad \text{and} \quad \widehat{u^{n+1/2}} = u_\Gamma^n, \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.1}$$

Together with

$$\left.\begin{aligned} &\mathcal{L}\psi^{n+1} = 0, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &\left[\!\!\left[\frac{\partial \psi^{n+1}}{\partial n} \right]\!\!\right] = 2\left[\!\!\left[\frac{\partial u^{n+1/2}}{\partial n} \right]\!\!\right] \quad \text{and} \quad \widehat{\frac{\partial \psi^{n+1}}{\partial n}} = 0, \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.2}$$

And the function u_Γ^{n+1} on Γ, is defined to be

$$u_\Gamma^{n+1} = u_\Gamma^n - 2\theta\widehat{\psi^n}, \quad \text{with } \theta \in (0, \theta_{\max}) \tag{A1.3}$$

With a suitable $\theta \in (0, \theta_{\max})$.

The positive definite transformation used for the application of the CGM in Section III, in the case of the Neumann–Neumann algorithm, can be derived replacing Eqs. (A1.1) and (A1.2) by

$$\left.\begin{aligned} &\mathcal{L}u^{n+1/2} = f_\Omega, \quad \text{in } \Omega_\alpha, \alpha = 1,2 \\ &[\![u^{n+1/2}]\!] = 0 \quad \text{and} \quad \widehat{u^{n+1/2}} = u_\Gamma^n, \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.4}$$

and

$$\left.\begin{aligned} &\mathcal{L}\psi^{n+1} = 0, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &\left[\!\!\left[\frac{\partial \psi^{n+1}}{\partial n} \right]\!\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial \psi^{n+1}}{\partial n}} = \widehat{\frac{\partial u^{n+1/2}}{\partial n}} \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.5}$$

Similarly, the standard Dirichlet–Dirichlet formulation (or preconditioned FETI [17]) in our notation is:

$$\left.\begin{aligned} &\mathcal{L}u^{n+1/2} = f_\Omega, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &\left[\!\!\left[\frac{\partial u^{n+1/2}}{\partial n} \right]\!\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial u^{n+1/2}}{\partial n}} = \lambda^n, \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.6}$$

Together with

$$\left.\begin{aligned} &\mathcal{L}\psi^{n+1} = 0, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &[\![\psi^{n+1}]\!] = 0 \quad \text{and} \quad \widehat{\psi^{n+1}} = [\![u^{n+1/2}]\!], \ \text{on } \Gamma \end{aligned}\right\} \tag{A1.7}$$

The function λ^{n+1} on Γ, is defined to be

$$\lambda^{n+1} = \lambda^n + \theta \left[\!\left[\frac{\partial \psi^n}{\partial n} \right]\!\right], \quad \text{with } \theta \in (0, \theta_{\max}) \tag{A1.8}$$

The positive definite transformation used for the application of the CGM in Section III, in the case of the Dirichlet–Dirichlet algorithm, can be derived replacing Eqs. (A1.6) and (A1.7) by

$$\left.\begin{aligned} &\mathcal{L}u^{n+1/2} = f_\Omega, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &\left[\!\left[\frac{\partial u^{n+1/2}}{\partial n} \right]\!\right] = 0 \quad \text{and} \quad \widehat{\frac{\partial u^{n+1/2}}{\partial n}} = \lambda^n, \text{ on } \Gamma \end{aligned}\right\} \tag{A1.9}$$

Together with

$$\left.\begin{aligned} &\mathcal{L}\psi^{n+1} = 0, \quad \text{in } \Omega_\alpha, \alpha = 1, \ldots, E \\ &[\![\psi^{n+1}]\!] = 0 \quad \text{and} \quad \widehat{\psi^{n+1}} = \widehat{u^{n+1/2}}, \text{ on } \Gamma \end{aligned}\right\} \tag{A1.10}$$

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