New Matrix General Formulas of Dual-Primal Domain Decomposition Methods without Recourse to Lagrange Multipliers

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ABSTRACT

Nowadays parallel computing is the most effective means for increasing computational speed. In turn, the domain decomposition methods (DDM) are most efficient for applying parallel-computing to the solution of partial differential equations. The non-overlapping class of such methods, which are especially effective, is constituted mainly by the Schur complement and the non-preconditioned FETI (or Neumann) methods, here grouped generically as the one-way methods, together with the Neumann-Neumann and the preconditioned FETI methods, here grouped generically as the round-trip methods. More recently, such methods have been improved by the introduction of the dual-primal methods, in which a relatively small number of continuity constraints across the interfaces are enforced. However, the treatment of round-trip algorithms up to now has been done with recourse to Lagrange multipliers exclusively. Recently, however, Herrera and his collaborators have introduced a more direct treatment, the "multipliers-free method", in which the differential operators are applied to discontinuous functions, and the matrices are applied to 'discontinuous vectors'. The multipliers-free method possesses significant advantages, many of them derived from the directness of its approach; among them, it allows the development of more explicit and general expressions of the algorithm matrices, which are here reported. Such matrix-expressions in turn allow the development of more robust and simple computational codes.

1.- INTRODUCTION

Nowadays parallel computing is the most effective means for increasing computational speed. In turn, DDM is most efficient for applying parallel-computing to the solution of partial differential equations [1]. Since the time when the international community began to intensively study DDM, attention has shifted from overlapping to non-overlapping methods, mainly because they are very effective for many problems. At a first stage, the application of such an approach produced the *Schur-complement method*, which corresponds to formulating Dirichlet problems in each one of the subdomains, and the *non-preconditioned FETI method*, which corresponds to formulating Neumann problems at each one of the partition subdomains. The performance of these

[1] Alternativas en Computación, S.A. de C.V.

methods, however, is not satisfactory in many problems and can be drastically improved by applying appropriate *preconditioners* [2].

Some of the most efficient preconditioned non-overlapping methods are obtained by using the *Neumann method* as a preconditioner of the *Schur complement method*; or, conversely using the *Schur complement method* as a preconditioner of the *Neumann method*. In particular, when the Schur complement method is preconditioned with the Neumann method, a procedure that is known in the international literature as the *Neumann-Neumann method* is obtained, while the *preconditioned FETI* is obtained when the *Neumann method* is preconditioned with the *Schur complement method*. A feature that is common to all these methods is that one goes from the space of continuous functions to that with continuous normal derivatives and back; or the same, but in reverse order. Thus, in this article they are generically called the *'round-trip methods'* and, in order to make this terminology more inclusive, we group the *Schur complement* and the *non-preconditioned FETI methods* into the category of *'one-way methods'*.

More recently, the *dual-primal Neumann-Neumann and FETI methods* were introduced [3, 4]; they are modifications that further improve the efficiency of the algorithms described above. In them, a relatively small number of continuity constraints across the interfaces are enforced. They have been very successful because they enhance in a rather significant manner the condition number of the matrices involved in the problems and accelerate the convergence rates [2].

The treatment of r*ound-trip* algorithms, until recently, had been exclusively done with recourse to *Lagrange multipliers*. However, Herrera and his coworkers, in a sequence of papers [5-8] have introduced a *multipliers-free* approach to all these methods, including *dual-primal methods*. Actually, the whole series constitutes a *'general theory of partial differential operators acting on discontinuous functions, and of matrices acting on discontinuous vectors'*. The theory *of partial differential operators acting on discontinuous functions* is based on extensive previous work that was revised and integrated in the first article of that series [5] (see this latter publication for extensive references to that work).

In the second paper [6], the general ideas and theoretical background that allows the formulation of a very general class of substructuring methods without recourse to Lagrange multipliers were introduced. The results of that article were developed working in finite-dimensional function spaces. In the third paper [7], on the contrary, the developments there were done directly in the matrices, regardless of the function-spaces from which they derive, introducing for that purpose a new approach to deal with matrices acting on *'discontinuous vectors'*. The most conspicuous feature of this methodology permitted the development of the matrices associated with the different algorithms in a more explicit manner and also extending the procedures to include *dual-primal methods*. The series culminates with a fourth paper in which general explicit matrix-expressions of the different algorithms are given [8]. The following features of such results should be highlighted: the different algorithms are expressed in terms of *Schur-complement* matrices, $\underline{\underline{S}}$, which involve *internal-boundary nodes* exclusively; such explicit matrix-expressions are given for all the algorithms mentioned above: one-way and round-trip algorithms; the case when the *null-subspace* of $\underline{\underline{S}}$ is non-trivial is included in the fourth paper of the series [8]; they equally apply to a single equation and to systems of equations; by simple substitutions parallel-processing codes can be developed for a wide variety of systems governed by differential equations or systems of such equations. Thus, code-development is simplified and the effort required is significantly reduced. As previously mentioned, the Laplace equation and the equations of static elasticity are included, and many more.

2.- DUAL AND PRIMAL NODES Equation Section 2

Let the set of *'original nodes'* be $\Omega \equiv \{1,...,d\}$, while the family $\{\Omega_1,...,\Omega_E\} \subset \Omega$ is a *cover of* Ω; i.e.,

$$\Omega = \bigcup_{\alpha=1}^{E} \Omega_\alpha \tag{2.1}$$

For the time being, we consider pairs $\underline{p} \equiv (p,\alpha)$, such that $p \in \Omega$ and $\alpha \in \{1,...,E\}$. Then, we define

$$\bar{\Omega}^T \equiv \left\{ \underline{p} = (p,\alpha) \middle| p \in \Omega_\alpha \right\} \text{ and } Z^T(p) \equiv \left\{ \alpha \in \{1,...,E\} \middle| (p,\alpha) \in \bar{\Omega}^T \right\} \tag{2.2}$$

and the *total multiplicity of* p, $m^T(p)$, is the cardinality of $Z^T(p)$. We shall distinguish two classes of *original nodes*: when $m^T(p)=1$, $p\in\Omega$ is said to be an *'interior node'* and when $m^T(p)>1$, it is said to be a *'(internal-)boundary node'*; the sets of *interior nodes* and *boundary nodes*, which are disjoint, will be denoted by Ω^I and Ω^Γ, respectively.

We choose a set $\Omega_0\subset\Omega^\Gamma$ that will be kept fixed in what follows; so, the all the developments that follow are relative to it. Then, we define:

$$\begin{cases} \mathrm{I}\equiv\left\{\underline{p}\equiv(p,\alpha)\middle|\, p\in\Omega^I\right\}\subset\bar{\Omega}^T \\ \pi\equiv\left\{\underline{p}\equiv(p,0)\middle|\, p\in\Omega_0\right\} \\ \Delta\equiv\left\{\underline{p}\equiv(p,\alpha)\middle|\, p\in\Omega^\Gamma-\Omega_0\right\}\subset\bar{\Omega}^T \end{cases} \tag{2.3}$$

Together with

$$\bar{\Omega}\equiv\mathrm{I}\cup\pi\bigcup\Delta \; and \; \Pi\equiv\mathrm{I}\cup\pi \tag{2.4}$$

Notice that $\bar{\Omega}=\bar{\Omega}^T$, when $\Omega_0=\varnothing$. The members of $\bar{\Omega}$ will be said to be the *derived nodes*, which may be *internal*, *primal* or *dual*, depending on whether they belong to I, π or Δ, respectively. For every $p\in\Omega$, we write

$$\mathrm{Z}(p)\equiv\left\{\alpha\in\{0,1,...,E\}\middle|(p,\alpha)\in\bar{\Omega}\right\} \tag{2.5}$$

the *multiplicity* $m(p)$ of p is the cardinality of $\mathrm{Z}(p)$; in particular, $m(p)=1$ if and only if $p\in\Pi$.

3.- VECTORS AND CONTINUOUS VECTORS Equation Section 3

Notice that every real-valued function defined either in Ω or in $\bar{\Omega}$, is a vector. The linear spaces $\tilde{D}(\Omega)$ and $\tilde{D}(\bar{\Omega})$ will be constituted by the functions (i.e., vectors) defined in Ω and in $\bar{\Omega}$, respectively. Similarly, $\tilde{D}(\mathrm{I})$, $\tilde{D}(\pi)$, $\tilde{D}(\Pi)$ and $\tilde{D}(\Delta)$ will be the linear subspaces of $\tilde{D}(\bar{\Omega})$ whose functions vanish outside I, π, Π and Δ, respectively. Then, it can be seen that

$$\tilde{D}(\bar{\Omega})=\tilde{D}(\Pi)\oplus\tilde{D}(\Delta)=\tilde{D}(\mathrm{I})\oplus\tilde{D}(\pi)\oplus\tilde{D}(\Delta) \tag{3.1}$$

Here, as in what follows, the symbol $\oplus$ stands for the *direct sum* of two linear spaces. When $\underline{p}=(p,\alpha)\in\bar{\Omega}$, we write $u_{(p,\alpha)}\equiv u(p,\alpha)$; here, $u(p,\alpha)$ stands for the value of the function $\underline{u}$ at the *derived node* (p,α).

The *'Euclidean inner product'*, which is the only one to be considered in this Section, is defined to be

$$\begin{cases} \hat{\underline{u}}\bullet\hat{\underline{w}}\equiv\sum\limits_{p\in\Omega}\hat{\underline{u}}(p)\hat{\underline{w}}(p),\forall\hat{\underline{u}},\hat{\underline{w}}\in\tilde{D}(\Omega) \\ \underline{u}\bullet\underline{w}\equiv\sum\limits_{\underline{p}\in\bar{\Omega}}\underline{u}(\underline{p})\underline{w}(\underline{p})=\sum\limits_{p\in\Omega}\sum\limits_{\alpha\in Z(p)}u(p,\alpha)w(p,\alpha),\forall\underline{u},\underline{w}\in\tilde{D}(\bar{\Omega}) \end{cases} \tag{3.2}$$

A proper treatment of the matrices that occur when dealing with systems of partial differential equations, such as those of elasticity, requires introducing vector-valued functions. In such cases, the values at nodes, $\hat{\underline{u}}(p)$ and $\underline{u}(p,\alpha)$, are themselves vectors and Eq.(3.2) must be replaced by

$$\begin{cases} \hat{\underline{u}}\bullet\hat{\underline{w}}\equiv\sum\limits_{p\in\Omega}\hat{\underline{u}}(p)\odot\hat{\underline{w}}(p),\forall\hat{\underline{u}},\hat{\underline{w}}\in\tilde{D}(\Omega) \\ \underline{u}\bullet\underline{w}\equiv\sum\limits_{\underline{p}\in\bar{\Omega}}\underline{u}(\underline{p})\odot\underline{w}(\underline{p})=\sum\limits_{p\in\Omega}\sum\limits_{\alpha\in Z(p)}\underline{u}(p,\alpha)\odot\underline{w}(p,\alpha),\forall\underline{u},\underline{w}\in\tilde{D}(\bar{\Omega}) \end{cases} \tag{3.3}$$

Here, the symbol $\odot$ stands for the *inner product* of the vector space where the vectors $\hat{\underline{u}}(p)$ and $\underline{u}(p,\alpha)$ lie. The orthogonality relations to be used in this Section are understood to be with respect to the Euclidean inner product.

The *natural immersion* of $\tilde{D}(\Omega)$ into $\tilde{D}(\bar{\Omega})$ is denoted by $\tau:\tilde{D}(\Omega)\to\tilde{D}(\bar{\Omega})$, where $\tau\hat{\underline{u}}\in\tilde{D}(\bar{\Omega})$ is defined, for every $\hat{\underline{u}}\in\tilde{D}(\Omega)$, by

$$(\tau\hat{\underline{u}})(p,\alpha)=\hat{\underline{u}}(p),\forall p\in\Omega\ and\ \forall\alpha\in Z(p) \tag{3.4}$$

The subspace of *continuous vectors*, $\tilde{D}_{12}(\bar{\Omega})\subset\tilde{D}(\bar{\Omega})$, is defined to be the image of $\tilde{D}(\Omega)$ under the *natural immersion*. *Continuous vectors* $\underline{u}\in\tilde{D}_{12}(\bar{\Omega})$ are characterized by the fact that

$$u(p,\alpha)=u(p,\beta),\forall\alpha,\beta\in Z(p) \tag{3.5}$$

Two matrices $\underline{\underline{a}}:\tilde{D}(\bar{\Omega})\to\tilde{D}(\bar{\Omega})$ and $\underline{\underline{j}}:\tilde{D}(\bar{\Omega})\to\tilde{D}(\bar{\Omega})$ are now introduced, which are defined by

$$\underline{\underline{a}}\underline{u}=Proj_{\tilde{D}_{12}}\underline{u}\ and\ \underline{\underline{j}}=\underline{\underline{I}}-\underline{\underline{a}} \tag{3.6}$$

Here, $\underline{\underline{I}}$ is the identity matrix and the projection on $\tilde{D}_{12}(\bar{\Omega})$ is taken with respect to the *Euclidean* inner product. Define $\tilde{D}_{11}(\bar{\Omega})$ to be the orthogonal complement of $\tilde{D}_{12}(\bar{\Omega})$, then

$$\left.\begin{aligned}\tilde{D}_{11}(\bar{\Omega})&=\underline{\underline{j}}\tilde{D}(\bar{\Omega})\\ \tilde{D}_{12}(\bar{\Omega})&=\underline{\underline{a}}\tilde{D}(\bar{\Omega})\end{aligned}\right\}\ and\ \tilde{D}(\bar{\Omega})=\tilde{D}_{11}(\bar{\Omega})\oplus\tilde{D}_{12}(\bar{\Omega}) \tag{3.7}$$

Both $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are projection operators, the latter being the projection onto $\tilde{D}_{11}(\bar{\Omega})$; they enjoy a good number of nice properties that we refrain from describing in detail (such details can be seen in [ ]). The construction of the matrix $\underline{\underline{a}}$ is relatively simple; writing

$$\underline{\underline{a}}\equiv\left(a_{(i,\alpha)(j,\beta)}\right) \tag{3.8}$$

then,

$$a_{(i,\alpha)(j,\beta)}=\frac{1}{m(i)}\delta_{ij} \tag{3.9}$$

As for $\underline{\underline{j}}$, it is easily obtained applying Eq.(3.6).

Equation Section 4

4.- THE DISCONTINUOUS MULTIPLIERS-FREE MATRIX-FORMULATION

In what follows we consider two symmetric matrices:

$$\underline{\underline{\hat{A}}}:\tilde{D}(\Omega)\to\tilde{D}(\Omega)\ and\ \underline{\underline{A}}:\tilde{D}(\bar{\Omega})\to\tilde{D}(\bar{\Omega}) \tag{4.1}$$

The matrix $\underline{\underline{\hat{A}}}$ will be referred to as the *'original matrix'*. We write:

$$\underline{\underline{\hat{A}}}\equiv\left(\hat{A}_{pq}\right),\ where\ p,q\in\Omega \tag{4.2}$$

It will be assumed throughout the paper that:

1. $\underline{\underline{\hat{A}}}:\tilde{D}(\Omega)\to\tilde{D}(\Omega)$ is positive definite; and
2. $$\hat{A}_{pq}=0,\ whenever\ p\in\Omega^{I}\cap\Omega_{\alpha},\ q\in\Omega^{I}\cap\Omega_{\beta}\ and\ \alpha\neq\beta \tag{4.3}$$

3. The matrices $\underline{\underline{\widehat{A}}}$ and $\underline{\underline{A}}$ are related by:

$$\underline{\hat{w}} \cdot \underline{\underline{\widehat{A}}}\,\underline{\hat{u}} = \tau(\underline{\hat{w}}) \cdot \underline{\underline{A}}\,\tau(\underline{\hat{u}}), \forall \underline{\hat{u}}, \underline{\hat{w}} \in \tilde{D}(\Omega) \tag{4.4}$$

This latter relation does not determine $\underline{\underline{A}}$ uniquely when $\underline{\underline{\widehat{A}}}$ is given, but a convenient procedure for constructing one such matrix $\underline{\underline{A}}$ is presented in [8]. Then, the matrix $\underline{\underline{A}}$ is positive definite on the subspace of continuous vectors.

We write

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \tag{4.5}$$

Where the notation is such that

$$\begin{cases} \underline{\underline{A}}_{\Pi\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Pi), \quad \underline{\underline{A}}_{\Pi\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Pi) \\ \underline{\underline{A}}_{\Delta\Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Delta), \quad \underline{\underline{A}}_{\Delta\Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Delta) \end{cases} \tag{4.6}$$

The *'dual-primal Schur complement matrix'*, $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$, is defined by

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi} \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{A}}_{\Pi\Delta} \tag{4.7}$$

When $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ is positive definite, then $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$ is also positive definite. Then, the bilinear form associated with $\underline{\underline{S}}$ defines an inner product on $\tilde{D}(\Delta)$, referred to as the *energy inner product*. The subspaces $\tilde{D}_{21}(\Delta) \subset \tilde{D}(\Delta)$ and $\tilde{D}_{22}(\Delta) \subset \tilde{D}(\Delta)$ are defined to be

$$\tilde{D}_{21}(\Delta) \equiv \tilde{D}_{11}^{\perp}(\Delta) \; and \; \tilde{D}_{22}(\Delta) \equiv \tilde{D}_{12}^{\perp}(\Delta) \tag{4.8}$$

Here, the orthogonal complements (as subspaces of $\tilde{D}(\Delta)$) are taken with respect to the *energy inner product*. We notice, for later use that

$$\tilde{D}_{22}(\Delta) = \left\{ \underline{w} \in \tilde{D}(\Delta) \middle| \underline{\underline{aS}}\,\underline{w} = 0 \right\} \tag{4.9}$$

Furthermore, the image of the transformation $\underline{\underline{S}}^{-1}\,\underline{\underline{j}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$ is $\tilde{D}_{22}(\Delta)$, since

$$\underline{\underline{aS}}\,\underline{\underline{S}}^{-1}\,\underline{\underline{j}} = \underline{\underline{0}} \tag{4.10}$$

This will permit us to introduce, in Section 5, the transformation

$$\underline{\underline{S}}^{-1}\underline{\underline{j}}:\tilde{D}_{22}(\Delta)\to\tilde{D}_{22}(\Delta) \tag{4.11}$$

Definition 4.1.- Let $\underline{\hat{f}}\in\tilde{D}(\Omega)$ be given, and define $\underline{f}_{\Delta 2}\in\tilde{D}_{12}(\Delta)$ by

$$\underline{f}_{\Delta 2}\equiv\underline{\underline{m}}^{-1}\tau\left(\underline{\hat{f}}\right)_{\Delta}-\underline{\underline{a}}\,\underline{\underline{A}}_{\Delta\Pi}\underline{\underline{A}}_{\Pi\Pi}^{-1}\tau\left(\underline{\hat{f}}\right)_{\Pi} \tag{4.12}$$

Then, the *'original problem'* consists in searching for a function $\underline{\hat{u}}\in\tilde{D}(\Omega)$ that satisfies

$$\underline{\underline{\hat{A}}}\,\underline{\hat{u}}=\underline{\hat{f}} \tag{4.13}$$

While the *'transformed problem'* consists in searching for a function $\underline{u}_{\Delta}\in\tilde{D}(\Delta)$ such that

$$\underline{\underline{a}}\,\underline{\underline{A}}\underline{u}_{\Delta}=\underline{f}_{\Delta 2}\ and\ \underline{\underline{j}}\,\underline{u}_{\Delta}=0 \tag{4.14}$$

Applying these definitions, the following result was shown in [ ]: The vector

$$\underline{\hat{u}}\equiv\tau^{-1}\left\{\left(\underline{\underline{I}}-\underline{\underline{A}}_{\Pi\Pi}^{-1}\underline{\underline{A}}_{\Pi\Delta}\right)\underline{u}_{\Delta}+\underline{\underline{a}}\,\underline{\underline{A}}_{\Delta\Pi}\underline{\underline{A}}_{\Pi\Pi}^{-1}\tau\left(\underline{\hat{f}}\right)_{\Pi}\right\} \tag{4.15}$$

it solution of *the original problem* if and only if the vector $\underline{u}\in\tilde{D}(\bar{\Omega})$ is solution of the *transformed problem.*

Equation Section 5

5.- GREEN-HERRERA FORMULA FOR MATRICES

For differential operators acting in discontinuous functions, this kind of formulas were introduced by Herrera, in 1983 (see, [5] for a recent review of this subject). However, their extension to matrices acting in *discontinuous vectors* discussed in this Section, is very recent [7].

To this end, the following definitions are introduced

$$\underline{\underline{L}}\equiv\begin{pmatrix}\underline{\underline{A}}_{\Pi\Pi}&\underline{\underline{A}}_{\Pi\Delta}\\0&0\end{pmatrix}\ and\ \underline{\underline{R}}\equiv\begin{pmatrix}0&0\\\underline{\underline{A}}_{\Delta\Pi}&\underline{\underline{A}}_{\Delta\Delta}\end{pmatrix} \tag{5.1}$$

And write the identity:

$$\underline{\underline{R}}=\underline{\underline{a}}\,\underline{\underline{R}}+\underline{\underline{j}}\,\underline{\underline{R}} \tag{5.2}$$

This implies, since $\underline{\underline{A}}=\underline{\underline{A}}^{T}$, that

$$\underline{\underline{L}}+\underline{\underline{a}}\,\underline{\underline{R}}+\underline{\underline{j}}\,\underline{\underline{R}}=\underline{\underline{L}}^{T}+\underline{\underline{R}}^{T}\underline{\underline{a}}+\underline{\underline{R}}^{T}\underline{\underline{j}} \tag{5.3}$$

We notice that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{R}}$ are contained in $\tilde{D}(\Pi)$ and $\tilde{D}(\Delta)$, respectively, while those of $\underline{\underline{aR}}$ and $\underline{\underline{jR}}$ are contained in $\tilde{D}_{12}(\Delta)$ and $\tilde{D}_{11}(\Delta)$, also respectively; so, they are linearly independent. The identity:

$$\underline{\underline{L}}+\underline{\underline{aR}}-\underline{\underline{R}}^T\underline{\underline{j}}=\underline{\underline{L}}^T+\underline{\underline{R}}^T\underline{\underline{a}}-\underline{\underline{jR}} \tag{5.4}$$

which follows from Eq.(5.3), will be referred to as *'Green-Herrera formula for matrices'*.

Taking $\underline{f}_{\Delta 2}\in D_{12}(\Delta)$ to be the vector that was introduced when defining the *transformed problem*, the following problem formulation was introduced in [7]:

"Find a $\underline{u}\in\tilde{D}(\bar{\Omega})$ *that satisfies*

$$\left(\underline{\underline{L}}+\underline{\underline{aR}}-\underline{\underline{R}}^T\underline{\underline{j}}\right)\underline{u}=\underline{f}_{\Delta 2} \tag{5.5}"$$

Applying $\underline{\underline{j}}$ to this equation it obtained

$$\underline{\underline{jR}}^T\underline{\underline{ju}}=\underline{\underline{jR}}\,\underline{\underline{ju}}=\underline{\underline{jA}}_{\Delta\Delta}\underline{\underline{ju}}=0 \tag{5.6}$$

Recalling that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{aR}}$ are in $\tilde{D}(\Pi)$ and $\tilde{D}_{12}(\Delta)$, respectively, it seen that Eq.(5.5) implies that

$$\underline{\underline{Lu}}=0 \tag{5.7}$$

The *'harmonic functions space'*, is defined to be

$$D\equiv\left\{\underline{u}\in\tilde{D}(\bar{\Omega})\,\middle|\,L\underline{u}=0\right\} \tag{5.8}$$

Some important properties of harmonic functions are listed next.

A. *Harmonic functions are characterized by their dual values*. Indeed, if $\underline{u}\in D$, then

$$\underline{u}_\Pi=-\underline{\underline{A}}^{-1}_{\Pi\Pi}\underline{\underline{A}}_{\Pi\Delta}\underline{u}_\Delta \tag{5.9}$$

B. When $\underline{u}\in D$,

$$\underline{\underline{Au}}=\underline{\underline{Ru}}=\underline{\underline{Su}} \tag{5.10}$$

C. The matrix $\underline{\underline{S}}$ defines a transformation, $\underline{\underline{S}}:\tilde{D}(\Delta)\rightarrow\tilde{D}(\Delta)$, of $\tilde{D}(\Delta)$ into itself, which is symmetric (and positive definite, when $\underline{\underline{A}}:\tilde{D}(\bar{\Omega})\rightarrow\tilde{D}(\bar{\Omega})$ is positive definite).

Thus, our solution-search can be restricted to the subspace D. When $\underline{u}\in D$, the problem formulation of Eq. (5.5) reduces to the condition that

$$\left(\underline{\underline{a}}\,\underline{\underline{R}}-\underline{\underline{R}}^{T}\,\underline{\underline{j}}\right)\underline{u}=\underline{f}_{\Delta 2} \tag{5.11}$$

For *harmonic functions* Eq.(5.11) can be replaced by

$$\left(\underline{\underline{a}}\,\underline{\underline{S}}-\underline{\underline{S}}\,\underline{\underline{j}}\right)\underline{u}_{\Delta}=\underline{f}_{\Delta 2} \tag{5.12}$$

Now, the matrix $\underline{\underline{S}}:\tilde{D}(\Delta)\rightarrow\tilde{D}(\Delta)$ transforms the subspace of *internal-boundary vectors* into itself. Furthermore, $\underline{f}_{\Delta 2}\in\tilde{D}(\Delta)$. Hence, Eq.(5.12) can be used directly for obtaining $\underline{u}_{\Delta}\in\tilde{D}(\Delta)$, as we do in the next Section. Once, $\underline{u}_{\Delta}$ is constructed other part ($\underline{u}_{\Pi}\in\tilde{D}(\Pi)$) of the *harmonic vector* $\underline{u}=\left(\underline{u}_{\Delta}+\underline{u}_{\Pi}\right)\in D$ is given by Eq.(5.9).

Equation Section 6

6.- DUAL-PRIMAL METHODS

The following *dual-primal algorithms* are formulated using the results of Section 5. They cover a wide variety of non-overlapping DDMs.

1. One-way methods.

   1.1. Schur-complement method.- *"This problem consists in searching for a function* $\underline{u}_{\Delta}\in\tilde{D}_{12}(\Delta)$ *such that it satisfies*

   $$\underline{\underline{a}}\,\underline{\underline{S}}\,\underline{u}_{\Delta}=\underline{f}_{\Delta 2} \tag{6.1}"$$

   1.2. Non-preconditioned FETI method.- *"This problem consists in searching for a function* $\underline{u}_{\Delta}\in\tilde{D}_{22}(\Delta)$ *such that*

   $$\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{u}_{\Delta}=-\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_{\Delta 2} \tag{6.2}"$$

2. Roundtrip methods.

   2.1. Neumann-Neumann method.- *"Find a* $\underline{u}\in\tilde{D}_{12}(\Delta)$ *such that*

$$\underline{\underline{aS}}^{-1}\,\underline{\underline{aS}}\,\underline{u}_{\Delta} = \underline{\underline{aS}}^{-1}\,\underline{f}_{\Delta 2} \qquad (6.3)"$$

2.2. The *preconditioned FETI method*.- "Find a $\underline{u}_{\Delta} \in \tilde{D}_{22}(\Delta)$ such that

$$\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{u}_{\Delta} = -\underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\,\underline{f}_{\Delta 2} \qquad (6.4)"$$

Equation Section 7

7.- DISCUSSION

The algorithms of Section 6, except for the Algorithm 1.1 of the one-way methods (the *Schur-complement method*), require that the matrix $\underline{\underline{A}}$ be positive definite since they use the inverse of $\underline{\underline{S}}$. In [8], it is shown how to modify the matrix $\underline{\underline{S}}$ to make it positive definite, which also permits the application of all these algorithms when the *null-subspace of* $\underline{\underline{S}}$ is non-trivial.

The solution searches in Algorithms 1.1 and 2.1 are carried out in $\tilde{D}_{12}(\Delta)$ and the transformations

$$\underline{\underline{aS}} : \tilde{D}_{12}(\Delta) \to \tilde{D}_{12}(\Delta) \; and \; \underline{\underline{aS}}^{-1}\,\underline{\underline{aS}} : \tilde{D}_{12}(\Delta) \to \tilde{D}_{12}(\Delta) \qquad (7.1)$$

are transformations of $\tilde{D}_{12}(\Delta)$ in itself, and in this respect they are suitable for applying an iterative procedure. Furthermore, when restricted to $\tilde{D}_{12}(\Delta)$, $\underline{\underline{aS}}$ equals $\underline{\underline{aSa}}$, so the associate bilinear form is symmetric and positive definite. As for $\underline{\underline{aS}}^{-1}\,\underline{\underline{aS}}$, it is self-adjoint and positive definite with respect to the *energy inner product*; indeed, when such an inner product is used the associated bilinear corresponds to that of the matrix: $\underline{\underline{SaS}}^{-1}\,\underline{\underline{aS}}$, which is symmetric and positive definite.

The solution searches in Algorithms 1.2 and 2.2 are carried out in $\tilde{D}_{22}(\Delta)$ and the transformations

$$\underline{\underline{S}}^{-1}\,\underline{\underline{j}} : \tilde{D}_{22}(\Delta) \to \tilde{D}_{22}(\Delta) \; and \; \underline{\underline{S}}^{-1}\,\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}} : \tilde{D}_{22}(\Delta) \to \tilde{D}_{22}(\Delta) \qquad (7.2)$$

are transformations of $\tilde{D}_{22}(\Delta)$ in itself, as was shown in Section 4, and suitable for applying an iterative procedure in $\tilde{D}_{22}(\Delta)$. Also, the matrices $\underline{\underline{S}}^{-1}\,\underline{\underline{j}}$ and

$\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}$ are both positive definite and self-adjoint with respect to the *energy inner product*, since the associated bilinear forms correspond to such matrices

$$\underline{\underline{S}}\left(\underline{\underline{S}}^{-1}\underline{j}\right)=\underline{j}\ and\ \underline{\underline{S}}\left(\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{j}\right)=\underline{\underline{j}}\underline{\underline{S}}\underline{j} \tag{7.3}$$

All the matrix-formulations here presented are adequate for applying the *conjugate gradient method (CGM)* directly and suitable for its implementation fully in parallel [7, 8].

8.- CONCLUSIONS

A general formulation of non-overlapping domain decomposition methods (DDM) has been presented. Contrary to standard formulations in which discontinuous functions are treated as an *anomaly* that requires remediation by means of *Lagrange multipliers*, the formulations here are developed without recourse to such multipliers. To this end *"a general theory of partial differential operators acting on discontinuous functions, and of matrices acting on discontinuous vectors*, has been developed [5-8], which yields a more direct approach to DDM.

This theory possesses great generality and it is here applied to develop explicit formulas for the algorithm matrices that can be applied to any symmetric and positive differential equation, or systems of such equations. The matrix-formulas presented in this paper include the main non-overlapping DDMs, which we here group in two large categories: *one-way* (*Schur-complement* and *non-preconditioned FETI* methods) and *round-trip approaches* (*Neumann-Neumann* and *preconditioned FETI*).

The greater generality and simplicity of the *multipliers-free method* here presented must be high-lighted. Some of the advantages of its use are listed next:

- The significant simplification of the computational codes required for the algorithms implementation;

- The algorithms are derived directly from the problem-matrices, independently of the partial differential equations that originated them and the number of dimensions of the problem; thus, codes developed for its application to 2D-problems can be easily modified for its application to 3D-problems. In standard treatments the dimensions of the space are defined from the start;
- The use of the *average* and *jump* matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$, respectively, which are a conspicuous feature of the *multipliers-free method*, exhibits superior computational properties. In particular, the $\underline{\underline{j}}$ operator is the optimal choice for the ***B*** operator of the FETI methods [7]. In numerical experiments that have been carried out, very significant reductions in the number of iterations were achieved when the matrix $\underline{\underline{j}}$ was used instead of standard ***B*** operators;
- These matrices are generalizations of the *'average'* and *'jump'* of a function, which can be effectively applied at the discrete level (i.e., to vectors) not only at *internal-boundary nodes* but at edges and corners, as well. They are symmetric and non-negative projection matrices, on complementary subspaces. Furthermore, their construction is very simple; indeed, $\underline{\underline{a}}$ is the average over each node and, once $\underline{\underline{a}}$ is available, $\underline{\underline{j}}$ derives from it. Other useful property is $\underline{\underline{a}}\,\underline{\underline{j}} = \underline{\underline{j}}\,\underline{\underline{a}} = \underline{\underline{0}}$;
- The parallelism of standard formulations, using Lagrange multipliers, is impaired by the introduction of *dual-primal* preconditioners. In the *multipliers-free method*, on the contrary, the *dual-primal* preconditioners are introduced in a manner that is implemented fully in parallel;
- The unified theory implies a new expression for the *Steklov-Poincaré* operator for matrices that is completely akin with the meaning of the *Steklov-Poincaré* operator for differential operators [7,8], wining thereby, in theoretical consistency.
- Furthermore, in the numerical experiments thus far performed they exhibit better convergence properties than other formulations [7,8].

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