

Multipliers-Free Dual-Primal Domain Decomposition Methods for Nonsymmetric Matrices and Their Numerical Testing

Ismael Herrera, Robert A. Yates*

Instituto de Geofísica, Universidad Nacional Autónoma de México (UNAM), México, DF 14000, México

Received 29 September 2009; accepted 1 February 2010

Published online 26 April 2010 in Wiley Online Library (wileyonlinelibrary.com).

DOI 10.1002/num.20581

The most commonly used nonoverlapping domain decomposition algorithms, such as the FETI-DP and BDDC, require the introduction of discontinuous vector spaces. Most of the works on such methods are based on approaches that originated in Lagrange multipliers formulations. Using a theory of partial differential equations formulated in discontinuous piecewise-defined functions, introduced and developed by Herrera and his collaborators through a long time span, recently the authors have developed an approach to domain decomposition methods in which general problems with prescribed jumps are treated at the discrete level. This yields an elegant and general direct framework that permits analyzing the problems in greater detail. The algorithms derived using it have properties similar to those of well-established methods such as FETI-DP, but, in our experience, they are easier to implement. Also, they yield explicit matrix formulas that unify the different methods. Furthermore, this multipliers-free framework has permitted us to extend such formulas to make them applicable to nonsymmetric matrices. The extension of the unifying matrix formulas to nonsymmetric matrices is the subject matter of the present article. A conspicuous result is that in numerical experiments in 2D and 3D, the MF-DP algorithms for nonsymmetric matrices exhibit an efficiency of the same order as state-of-the-art algorithms for symmetric matrices, such as BDDC, FETI-DP, and MF-DP.

© 2010 Wiley Periodicals, Inc. *Numer Methods Partial Differential Eq* 27: 1262–1289, 2011

Keywords: domain decomposition methods; dual-primal; Lagrange multipliers; preconditioners; discontinuous Galerkin; FETI; Neumann–Neumann

I. INTRODUCTION

Two of the most commonly used nonoverlapping domain decomposition algorithms are the FETI-DP and the BDDC. The original finite element tearing and interconnecting method (FETI) of Farhat [1, 2] was later modified by the incorporation of a dual-primal approach to obtain FETI-DP [3, 4]. On the other hand, the balancing domain decomposition (BDD) preconditioner of

Correspondence to: Ismael Herrera, Instituto de Geofísica, Universidad Nacional Autónoma de México (UNAM), Apdo. Postal 22–220, México, DF 14000, México (e-mail: iherrera@unam.mx)

*Present address: Alternativas en Computación, S.A. de C.V

© 2010 Wiley Periodicals, Inc.

Mandel [5, 6] is an improved version of the Neumann–Neumann preconditioner that is due to De Roeck and Le Tallec [7], which in turn is based on the work of Glowinski, Wheeler, and others [8, 9]. The original BDD method was recently modified by Dohrman using a constrained minimization approach to obtain BDDC [10, 11]. The main advantage of FETI-DP and BDDC over the original FETI and BDD methods is that the more recent versions eliminate the need for solving singular systems.

In the case of FETI, one first formulates Neumann–Neumann local problems and then the Schur complement is applied to them as a preconditioner; this is very clearly explained by Toselli and Widlund in their book [12] (Chapter 6 is devoted to FETI and Neumann–Neumann methods), where the FETI method is referred to as the preconditioned FETI (see pp. 13–15 of [12]). In the case of BDD [5, 6], Mandel’s preconditioner is applied to the Schur-complement formulation (also known as Dirichlet–Dirichlet formulation). Thus, essentially the same problems are solved in both methods, except that they are solved in reverse order; therefore, these two methods are closely related. Indeed, it has recently been shown that the eigenvalues of the preconditioned BDDC and FETI-DP systems are almost identical [13, 14].

It is clear from the above that when applying either FETI or BDD algorithms, sooner or later, one has to solve discretized versions of Neumann problems formulated in each one of the subdomains of a domain decomposition. This introduces two kinds of complications: first, such problems do not possess a unique solution and, second, their solutions are discontinuous on the internal boundary when the normal derivative is continuous there (see for example [15]).

The BDD preconditioner of Mandel was very significant precisely because it introduced an effective manner of dealing with the first of these problems. However, as mentioned before, the more recent introduction of FETI-DP and BDDC eliminates the need to deal with singular problems. As for the second problem mentioned above, the need to deal with discontinuous solutions, two approaches are feasible:

APPROACH A: Treat the problem as one of constrained optimization, using a Lagrange multipliers formulation, where the condition that the solution be continuous is imposed as a constraint; or

APPROACH B: Formulate an equivalent problem in an enlarged function-space in which its members are generally discontinuous, albeit it contains the continuous function-space as a linear subspace.

In both approaches, one has to work in the discontinuous function-space (the space W , in Widlund’s notation, see [12], which is the cartesian product of the function-spaces of the substructures); the main difference, however, is that in the Approach A such a space remains in the background when the basic formulations are established, whereas in Approach B it remains in the foreground. In particular, for example, the basic matrix formulas for solving Neumann problems that are incorporated both in BDD and FETI were derived using the Approach A. In this respect, we would like to be more precise. FETI, for example, was originally formulated in terms of a collection of substructure spaces [1], but very soon after it was realized that it can be formulated in terms of the cartesian product of such spaces, which is the space W mentioned above. As the saddle point formulation and the Lagrange multipliers are only used to obtain the matrix formulation of the problem, at the end the numerical algorithms are derived from matrix formulas defined on the discontinuous function-space W . On the other hand, when Approach B is used, one defines a general class of problems in the enlarged space (i.e., the discontinuous function-space), which is constituted by problems with “prescribed jumps.” In particular, a continuous solution is obtained when the prescribed jump is zero. Independently of the relative merits of the two approaches, A and B are clearly different methodologies.

In our opinion Approach B, besides being more elegant, is more general, more direct and more enlightening. So, we thought it was a subject that deserved more study and research, especially since the DDM community has clearly given more attention to Approach A, up to now. Thus, we embarked on a line of research oriented to develop Approach B more fully. The results of this line of research have been reported in a sequence of articles [16–19]. In it, the authors have developed systematically a theory of domain decomposition methods, the “multipliers-free dual-primal DDM” (briefly: MF-DP), applying Approach B.

Domain decomposition methods have achieved a very impressive development during the last 20 or 25 years, most of it using Approach A (in this respect, two very good sources of information are the Proceedings of the International Congresses organized by the DDM organization [20] and the broad set of references contained in [12]). Through such developments a broad knowledge of domain decomposition methods has been attained; our work was made possible by that knowledge and, also, has found inspiration on it. In particular, for example, in Section III of the second article of the series [17], we introduced two preconditioned algorithms that were inspired by the continuous versions of Neumann–Neumann and the Dirichlet–Dirichlet (or preconditioned FETI) algorithms, as described by Toselli and Widlund (pp.10–15 of [12]), albeit we introduced a modification.

Indeed, in the standard versions [12, 15], the introduction of an acceleration parameter (the symbol θ is used for it in pages 10 and 13 of [12]) is required. This is due to the fact that in standard formulations u^{n+1} is not derived from u^n by means of a symmetric, positive definite transformation and, so, the conjugate gradient method (CGM) cannot be directly applied. The developments of the theory of MF-PD methods, on the other hand, have permitted us to formulate the same problem in a manner that u^{n+1} is derived from u^n by means of a symmetric and positive definite transformation. So, in our formulation, the direct application of CGM is feasible. Thereby, we observe that although CGM does not introduce explicitly an acceleration parameter, its use implies an optimal choice of the acceleration parameter required in standard formulations at the continuous level. This modification of such standard formulations was originally made in [17] (see, Section III and the Appendix, where the new formulations at the continuous level were presented and compared with the standard formulations, respectively) by application of the general abstract scheme of Section III of the present article, due to Herrera but then unpublished, where the “abstract form of the MF-DP algorithms” is stated.

For the case when the matrix of the original continuous problem is symmetric and positive definite, the MF-DP method has already been fully developed using a dual-primal approach, which, as is well known, has the advantage of avoiding the need of dealing with singular local problems. A thorough description of the MF-DP method in its present state is given in [19], where numerical experiments to test its efficiency were carried out. Up to now, the results of such numerical tests have been very encouraging and more extensive computational experiments are underway.

The present article is devoted to extend the MF-DP method to nonsymmetric matrices and to report the results of numerical tests of its efficiency when it is applied to such kinds of matrices. We achieve this by introducing a more general scheme where the matrices are generally nonsymmetric, but in which they can also be symmetric and positive definite. When this latter case occurs, the framework reduces to that discussed in previous articles of the series [16–19]; thus the new scheme is truly a generalization of the previous one. A brief description of its main features follows.

First, we introduce in Section III a scheme, mentioned before, that yields the abstract form of the MF-DP algorithms. A characteristic of this abstract form that we think is attractive is that in its framework many Dual-Primal nonoverlapping DDMs can be formulated in a unified yet explicit manner. Here, such explicit formulas are only derived for the algorithms that we had

called, in [19], the Neumann–Neumann and the preconditioned FETI algorithms, but in a form that is also applicable to general nonsymmetric matrices. In particular, when the original matrix is symmetric we recover the algorithms presented in [19]. However, many more algorithms are included in the general class of preconditioned algorithms of Section III; to obtain explicit formulas for them all that is required is to make a choice of the four operators $\sigma_{\alpha\beta}$, $\alpha, \beta = 1, 2$, satisfying the assumptions of that Section, and supply explicit expressions for each one of such four operators.

To profit from such a general scheme we have to develop a suitable framework, which is very similar to that developed in previous articles, but some adjustments had to be made. This is done in Sections IV to VIII. Then, in Section IX, given the original problem formulated in the space of continuous vectors—the \hat{W} space, in Widlund’s notation [12]—an equivalent problem is formulated in the space of discontinuous vectors. The original continuous problem is formulated in terms of the matrix $\hat{\underline{\underline{A}}}$, whereas the problem in discontinuous vectors is formulated in terms of the matrix $\underline{\underline{A}}^t$, and procedures for deriving $\underline{\underline{A}}^t$ from $\hat{\underline{\underline{A}}}$ are supplied. Using $\underline{\underline{A}}^t$, a general problem with prescribed jumps is formulated; this problem has the property that when the prescribed jump is zero the continuous solution of the original problem is obtained.

A special kind of Schur-complement matrix, the dual-primal Schur-complement matrix, is defined in Section X, where the dual-primal Schur-complement formulations are given. Then, the four algorithms previously mentioned: Schur MF-DP, FETI MF-DP, Neumann–Neumann MF-DP, and Preconditioned FETI MF-DP, are derived in Section XI, whereas Section XII is devoted to explain the numerical procedures and results. The Conclusions of the article are summarized in Section XIII. As the multipliers-free methodology is recent and not yet well known, some background about its origin and foundations is given in Section III.

The unified explicit matrix formulas obtained in the present article for nonsymmetric matrices, in form, are the same as those that were introduced in [19], except that now they can also be applied to nonsymmetric matrices:

$$\begin{cases} \underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}} = \underline{\underline{f}}_{\Delta 2} \text{ and } \underline{\underline{j}}\underline{\underline{u}} = 0; & \text{Schur MF-DP} \\ \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{u}} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\underline{f}}_{\Delta 2} \text{ and } \underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}} = 0; & \text{FETI-MF-DP} \end{cases} \quad (1.1)$$

for the non-preconditioned algorithms. For the preconditioned algorithms they are:

$$\begin{cases} \underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}}_{\Delta} = \underline{\underline{a}}\underline{\underline{S}}^{-1}\underline{\underline{f}}_{\Delta 2}; & \text{Neumann–Neumann MF-DP} \\ \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{u}} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{\underline{f}}_{\Delta 2}; & \text{Preconditioned FETI-MF-DP} \end{cases} \quad (1.2)$$

In Eqs. (1.1) and (1.2), $\underline{\underline{S}}$ is the dual-primal Schur complement matrix defined in Section X of the present article for the general case of possibly nonsymmetric matrices; for the special case when the matrix is symmetric and positive definite, this definition reduces to that we introduced in Section XI of [19]. To avoid confusion, we use the suffix MF-DP (for Multipliers-Free and Dual-Primal). The search in the Schur MF-DP algorithm and its preconditioned version, the Neumann–Neuman MF-DP, is carried out in the subspace of dual-vectors (i.e., vectors that vanish everywhere except at dual nodes) that are continuous (as already said, continuity is tantamount to $\underline{\underline{j}}\underline{\underline{u}} = 0$). The search in the FETI-MF-DP algorithm and its preconditioned version is carried out in the subspace of dual-vectors for which $\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}} = 0$. In Section IX of [19], a new definition of the Steklov–Poincaré operator at the discrete level was proposed and has been used in our

developments. As it is explained there, this new definition has clear advantages over the standard definitions, such as those presented in [12], p.3, and [15]. According to the new definition, $\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}}$ is the discretized version of the jump of the normal derivative across the internal boundary. Thus, $\underline{\underline{a}}\underline{\underline{S}}\underline{\underline{u}} = 0$ is tantamount to the condition that the normal derivative be continuous.

Among the features we have observed throughout our study and application of the “multipliers-free dual-primal method,” the following should be highlighted.

- The MF-DP method supplies a framework applicable to both symmetric and nonsymmetric matrices.
- In the case of symmetric matrices, the numerical efficiency of the preconditioned algorithms is as good as other state-of-the-art DDMs (see [19], where the comparisons are made with those obtained using FETI-DP, as reported in [12]). However, in our experience thus far, the computational properties of the multipliers-free DDMs have shown to be superior.
- In the case of nonsymmetric matrices, in the numerical experiments performed in the present article (see Section XII) they have worked nearly as efficiently as they do for symmetric ones (here, the efficiency is measured by the number of iterations required for convergence and such a number is of the same order for both symmetric and nonsymmetric matrices). Here, we remark that the treatment of non-symmetric matrices is considerably more difficult than that of non-symmetric ones (see, for example, Chapter 12 of [12]).
- Explicit matrix formulas, given in Eqs. (1.1) and (1.2), are supplied that unify the different methods. Some properties of these formulas worth noticing are: once the original matrix is given, they are uniquely determined; the very same formulas are applicable in both the symmetric and nonsymmetric cases; and they are equally applicable to a single linear differential equation or to a system of such equations.
- Code development is simplified.
- Very robust codes are obtained; for example, a code has been developed that has been applied in 2D and 3D problems (such a code was used to obtain the numerical results reported in Section XII of this article), something that is not possible when standard approaches are used.
- The MF-DP algorithms are 100% parallelizable, as it is shown in Section XII.

It is also worth mentioning that the jump matrix, $\underline{\underline{j}}$, introduced in previous articles, is probably the optimal choice of the matrix $\underline{\underline{B}}$, used to specify the continuity constraint in standard formulations [6, 7].

II. SOME BACKGROUND ON THE MULTIPLIERS-FREE THEORY

Some background material on the foundations of the multipliers free domain decomposition approach applied in this article was given in the first article of the series on which the MF-DP is based [16].

The origin of our approach can be traced back to a series of articles, published in 1985 [21–23], in which the development of a “general theory of partial differential equations in discontinuous piecewise-defined functions,” which supplies a framework suitable for discontinuous Galerkin (dG-) methods, was initiated. These articles, in turn, were based on a previous Algebraic Theory of Boundary Value Problems published in book form in the Pitman Advanced Publishing Program [24]. The general theory of partial differential equations in discontinuous piecewise-defined functions was further developed through a long time span: It was the basis of a discretization

method known as the localized adjoint method (LAM) [25]; the Eulerian–Lagrangian LAM (ELLAM) [26] was also based on it and has been recently presented in an integrated form in the first article of the series previously mentioned [16], where further references can be found.

Basic elements of that theory are a general boundary value problem with prescribed jumps, which is formulated for any linear differential operator \mathcal{L} , and a general Green’s formula introduced by Herrera in [21, 23] that can be used when such an operator is applied to discontinuous piecewise-defined functions [16]. In an explicit form, they may be found in several articles such as [26].

Such Green’s formulas, sometimes called Green–Herrera formulas, as we shall do in what follows, can be derived as it is explained next. Given \mathcal{L} , a domain Ω and a partition $\Pi \equiv \{\Omega_1, \dots, \Omega_E\}$ of Ω , with internal boundary Γ (Σ is used in [26]), are introduced. When \mathcal{L}^* is the formal adjoint of \mathcal{L} , the following equation is satisfied

$$w\mathcal{L}u - u\mathcal{L}^*w = \nabla \cdot \underline{\mathcal{D}}(u, w) \quad (2.1)$$

Here, $\underline{\mathcal{D}}(u, w)$ is a suitable vector-valued function, bilinear in the pair (u, w) . Therefore,

$$\int_{\Omega} \{w\mathcal{L}u - u\mathcal{L}^*w\} dx = \int_{\partial\Omega} \underline{\mathcal{D}}(u, w) \cdot \underline{n} dx - \int_{\Gamma} \|\underline{\mathcal{D}}(u, w)\| \cdot \underline{n} dx \quad (2.2)$$

Here, u and w are fully discontinuous piecewise-defined functions [16]. Then, Eq. (2.2) is equivalent to the following Green’s formula, sometimes called Green–Herrera formula:

$$\begin{aligned} \int_{\Omega} w\mathcal{L}u dx - \int_{\partial\Omega} \mathcal{B}(u, w) dx - \int_{\Gamma} \mathcal{J}(u, w) dx \\ = \int_{\Omega} u\mathcal{L}^*w dx - \int_{\partial\Omega} \mathcal{C}(w, u) dx - \int_{\Gamma} \mathcal{K}(w, u) dx \end{aligned} \quad (2.3)$$

Here,

$$\underline{\mathcal{D}}(u, w) \cdot \underline{n} = \mathcal{B}(u, w) - \mathcal{C}(w, u) \quad \text{and} \quad \|\underline{\mathcal{D}}(u, w)\| \cdot \underline{n} = \mathcal{J}(u, w) - \mathcal{K}(w, u) \quad (2.4)$$

The introduction of $\mathcal{B}(u, w)$ and $\mathcal{C}(w, u)$ is standard in the theory of partial differential equations (see, for example, Lions and Magenes [27]), whereas the introduction of $\mathcal{J}(u, w)$ and $\mathcal{K}(w, u)$ is only required when the problems are formulated in discontinuous piecewise-defined functions [16]. The Green–Herrera formula of Eq. (2.3) is applicable to any vector-valued linear differential operator that may have discontinuous coefficients. One deals with vector-valued differential operators when treating systems of differential equations; in particular, if such a system consists of only one equation, the functions are real-valued functions. A particular case of the application of the formula of Eq. (2.3), is when the coefficients of the differential operator are continuous; then, suitable definitions of $\mathcal{J}(u, w)$ and $\mathcal{K}(w, u)$ are

$$\mathcal{J}(u, w) \equiv -\underline{\mathcal{D}}(\llbracket u \rrbracket, \dot{w}) \cdot \underline{n} \quad \text{and} \quad \mathcal{K}(w, u) \equiv \underline{\mathcal{D}}(\dot{u}, \llbracket w \rrbracket) \cdot \underline{n} \quad (2.5)$$

Many ideas of domain decomposition methods (DDMs), when they are approached after discretization, have been inspired by concepts stemming from partial differential equations formulations of such methods before discretization. In the standard approach, the model to mimic is that of boundary value problems formulated on the Sobolev space of the domain of definition of

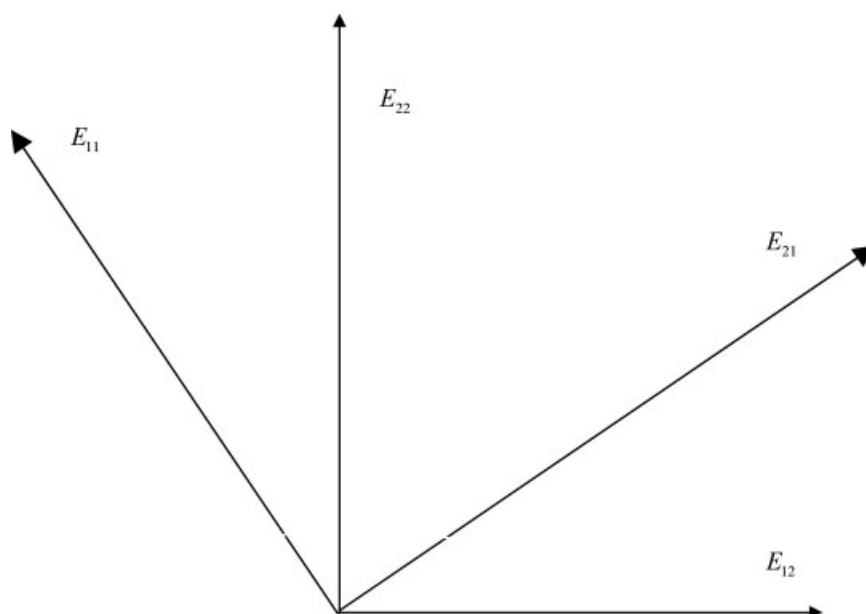


FIG. 1. A conjugate coordinate system.

the problem. On the other hand, we preferred to mimic boundary value problems with prescribed jumps formulated on the Sobolev space of discontinuous piecewise-defined functions. This was possible because we had available the theory of partial differential equations in discontinuous piecewise-defined-functions [16].

So, as a first step for developing the new approach, we extended the original Green-Herrera formulas for differential operators to Green-Herrera formulas for matrices, which act on discontinuous vectors; once this was done the similarity between the continuous and the discrete approaches was so apparent that the route to follow became evident. In particular, the new definition of the Steklov–Poincaré operator at the discrete level, which we introduced [19], was very useful in our developments.

The geometric situation is very simple and it is summarized in Fig. 1, which appears and is explained in detail in Section III of the present article, “A general class of preconditioned algorithms”; essentially the same figure appeared previously in references [18] and [19]. This figure contains four function-subspaces, $E_{\alpha\beta}$, with $\alpha, \beta = 1, 2$. For $\beta = 1, 2$, the subspaces $E_{1\beta}$ and $E_{2\beta}$ are orthogonal with respect to the Euclidean inner product. At this stage, it was clear that our theory was not limited to symmetric matrices and proceeded to construct the MF-DP method for nonsymmetric matrices that is presented in this article, which comes as an addition to the other four articles previously published [16–19]. The main difference between the cases when the matrix is symmetric (and positive definite) and when it is nonsymmetric is that in the former, for $\alpha = 1, 2$, the subspaces $E_{\alpha 1}$ and $E_{\alpha 2}$ are also orthogonal, but with respect to inner product induced by the Schur complement, whereas in the latter case such an inner product is not defined.

As the reader can see, the MF-DP for nonsymmetric matrices uses a functional analytic framework that we have introduced and used in previous articles (see [17–19]) to produce a very direct approach in which all the developments are done at the matrix level. Recently, a functional analytic framework was introduced and used, by Brenner and Sung [28], to discuss the connection between BDDC and FETI-DP. Independently of the merits of such an approach, at this stage we

only observe that the functional analytic framework used in the MF-DP method is considerably different from it.

Finally, a word on the notation we use. Such a notation has been developed steadily since the inception of our theory in 1985, and has many attractive features, such as introducing a systematic manner of denoting subspaces of any linear vector space and simplifying the algebraic manipulations. However, we would like to make our notation friendlier for the potential readers of our articles by transforming it into one closer to that used by the mainstream authors in DDM, but without losing its attractive properties mentioned above as they contribute to the advancement of DDM. A common practice, for example, is to use W for the whole space of discontinuous functions and the same symbol with various ad hoc decorations are used for its subspaces [12]. On the other hand, we have a systematic manner of denoting such subspaces. Other examples of the simplifications implied by our notation is that when it is used the interpolating (or restriction) operators, R_α , are not required.

Unfortunately, this is not an easy task and we are still working on it. This having been said, it should also be mentioned that the notation used in the unifying matrix formulas of Eqs. (1.1) and (1.2) is sufficiently close to that of the DDM mainstream that we think they can be easily understood by any reader sufficiently acquainted with the basics of domain decomposition methods.

III. THE ABSTRACT FORM OF THE PRECONDITIONED MF-DP ALGORITHMS

In this Section, we present a framework whose basic ideas were originally introduced by Herrera [17]. It supplies a general formulation that, without recourse to Lagrange Multipliers, permits deriving a unified approach to Dual-Primal Domain Decomposition Methods, which is not only applicable to symmetric matrices but also to nonsymmetric matrices as well. In this article, for the first time, the formulation for nonsymmetric matrices will be introduced.

The notation \oplus will be used for the direct sum of two linear spaces; i.e., when F , G , and H are linear spaces,

$$H = F \oplus G \quad (3.1)$$

if and only if

$$\begin{cases} H = F + G \\ \{0\} = F \cap G \end{cases} \quad (3.2)$$

Definition 3.1. When Eq. (3.1) is fulfilled, the pair of linear spaces (F, G) is said to be a “coordinate system” of H .

In what follows, E will be a *finite-dimensional Hilbert space*.

Definition 3.2. Let (E_{11}, E_{12}) and (E_{21}, E_{22}) be two coordinate systems of E . Then, the pair of coordinate systems $\{(E_{11}, E_{12}), (E_{21}, E_{22})\}$ is said to be “conjugate” when, for any $\alpha \neq \beta$,

$$E_{\alpha i} \cap E_{\beta i} = \{0\}, \text{ for } i = 1, 2 \quad (3.3)$$

A schematic representation of this Definition is given in Fig. 1.

Given a conjugate pair of coordinate systems $\{(E_{11}, E_{12}), (E_{21}, E_{22})\}$, we define four mappings:

$$\sigma_{\alpha i} : E \rightarrow E_{\alpha i}; i = 1, 2 \quad \text{and} \quad \alpha = 1, 2 \quad (3.4)$$

They are uniquely defined by the condition:

$$\sigma_{\alpha 1} + \sigma_{\alpha 2} = I; \text{ which holds for } \alpha = 1, 2 \quad (3.5)$$

Here, I is the identity mapping. It can be verified that such mappings are indeed uniquely defined when Eq. (3.5) is fulfilled. For each one of such mappings, write $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ for its restriction to $E_{\beta j} \subset E$. When we take $\beta \neq \alpha$ and $j \neq i$, this yields four mappings. To each one of them the following Lemma applies.

Lemma 3.1. *Given a conjugate pair of coordinate systems $\{(E_{11}, E_{12}), (E_{21}, E_{22})\}$ of E , consider the mappings $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ such that $\beta \neq \alpha$ and $j \neq i$. Then, for each one of them the null subspace of $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ is the set $\{0\}$.*

Proof. Assume $w \in E_{\beta j}$ and

$$\sigma_{\alpha i} w = 0 \quad (3.6)$$

Then, Eq. (3.6) implies $w \in E_{\alpha j}$. Hence, $w \in E_{\beta j} \cap E_{\alpha j} = \{0\}$ since $\beta \neq \alpha$. ■

Corollary 3.1. *Under the assumptions of Lemma 2.1, the dimensions of the linear spaces $E_{\alpha i}$ and $E_{\beta j}$ are equal, whenever $\beta \neq \alpha$ and $j \neq i$, and each one of the mappings $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ is bijective.*

Proof. Each one of the mappings $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ and $\sigma_{\beta j} : E_{\alpha i} \rightarrow E_{\beta j}$ is injective; therefore, if $d_{\alpha i}$ and $d_{\beta j}$ are the dimensions of $E_{\alpha i}$ and $E_{\beta j}$, respectively, then

$$d_{\alpha i} \geq d_{\beta j} \geq d_{\alpha i} \quad (3.7)$$

This implies $d_{\alpha i} = d_{\beta j}$ and, therefore, the mapping $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ is bijective. ■

Many domain decomposition methods can be cast in terms of the following abstract problem.

Problem A. In this problem, a conjugate pair of coordinate systems $\{(E_{11}, E_{12}), (E_{21}, E_{22})\}$ of E , is given. Furthermore, it is assumed that $\alpha \neq \beta$ and $i \neq j$. The problem consists in: “Given $g \in E_{\alpha i}$, find $u \in E_{\beta j}$ such that $\sigma_{\alpha i} u = g$.” Thus, in this problem $\sigma_{\alpha i} u \in E_{\alpha i}$ is prescribed

In view of Lemma 3.1, the existence of a solution of this problem is immediate since in that case $\sigma_{\alpha i} : E_{\beta j} \rightarrow E_{\alpha i}$ is a bijection. Taking $\alpha \neq i$ and $\beta \neq j$, we define:

The non-preconditioned algorithm: Find $u \in E_{\beta j}$, such that

$$\sigma_{\alpha i} u = g \quad (3.8)$$

and

The preconditioned algorithm: Find $u \in E_{\beta j}$, such that

$$\sigma_{\beta j} \sigma_{\alpha i} u = \sigma_{\beta j} g \quad (3.9)$$

Theorem 3.1. *Let a conjugate pair of coordinate systems $\{(E_{11}, E_{12}), (E_{21}, E_{22})\}$ of E , be given. Then, the non-preconditioned algorithms and the preconditioned algorithms are equivalent.*

Proof. It follows from the Lemma and the Corollary 3.1, since $\sigma_{\beta j} : E_{\alpha i} \rightarrow E_{\beta j}$ is a bijection. ■

IV. NODES AND THEIR CLASSIFICATION

Sections IV to VIII repeat briefly material that was presented in [19]. Thus, the reader is referred to [19] for further details. For definiteness, the set of “original-nodes” is assumed to be a set of natural numbers, $\Omega \equiv \{1, \dots, d\}$, whereas the family $\{\Omega_1, \dots, \Omega_E\}$ is a cover of Ω ; i.e.,

$$\Omega = \bigcup_{\alpha=1}^E \Omega_\alpha \quad (4.1)$$

We also consider pairs $\underline{p} \equiv (p, \alpha)$, such that $p \in \Omega$ and $\alpha \in \{1, \dots, E\}$. Then, we define

$$\bar{\Omega} \equiv \{ \underline{p} \equiv (p, \alpha) | p \in \Omega_\alpha \} \quad (4.2)$$

The pairs $\underline{p} \equiv (p, \alpha)$ that belong to $\bar{\Omega}$ are said to be “derived nodes,” which may be interior or boundary nodes (see [19]); they constitute the sets I and Γ , respectively. Given a $p \in \Omega$, $Z(p)$ is the set of derived nodes that originated from p . The subset $\pi \subset \Gamma$ is made of the primal nodes and the subset of dual nodes is defined to be $\Delta \equiv \Gamma - \pi$. Then, we define $\Pi \equiv I \cup \pi$. The following relations hold [7]:

$$\bar{\Omega} = I \cup \pi \cup \Delta = \Pi \cup \Delta \quad \text{and} \quad \emptyset = \Pi \cap \Delta = \pi \cap \Delta = \pi \cap I = \Delta \cap I \quad (4.3)$$

and

$$\pi = \emptyset \Rightarrow \Delta = \Gamma \quad (4.4)$$

V. VECTORS AND CONTINUOUS VECTORS

The vector spaces $\tilde{D}(\Omega)$ and $\tilde{D}(\bar{\Omega})$ are constituted by the functions defined in Ω and in $\bar{\Omega}$, respectively. Let $\underline{u} \in \tilde{D}(\bar{\Omega})$ and write $\underline{u}(p, \alpha)$ for its value at any derived node $(p, \alpha) \in \bar{\Omega}$; then, such a vector is said to be continuous when $\underline{u}(p, \alpha)$ is independent of α , for every derived node $(p, \alpha) \in \bar{\Omega}$. The set of continuous vectors constitute a linear subspace that is denoted by $\tilde{D}(\bar{\Omega})$. The notation: $\tilde{D}(\Pi) \subset \tilde{D}(\bar{\Omega})$ and $\tilde{D}(\Delta) \subset \tilde{D}(\bar{\Omega})$ is adopted for the linear subspaces of $\tilde{D}(\bar{\Omega})$ whose elements vanish outside Π and Δ , respectively. The subspaces $\tilde{D}(I)$, $\tilde{D}(\pi)$, and $\tilde{D}(\Gamma)$, of $\tilde{D}(\bar{\Omega})$, are defined similarly. Then,

$$\tilde{D}(\bar{\Omega}) = \tilde{D}(\Pi) \oplus \tilde{D}(\Delta) \quad (5.1)$$

Vectors of $\tilde{D}(\bar{\Omega})$ can be uniquely represented as

$$\underline{u} = (\underline{u}_\Pi, \underline{u}_\Delta) = \underline{u}_\Pi + \underline{u}_\Delta, \text{ with } \underline{u}_\Pi \in \tilde{D}(\Pi) \text{ and } \underline{u}_\Delta \in \tilde{D}(\Delta) \quad (5.2)$$

The natural immersion of $\tilde{D}(\Omega)$ into $\tilde{D}(\bar{\Omega})$, is defined to be the mapping $\tau : \tilde{D}(\Omega) \rightarrow \tilde{D}(\bar{\Omega}) \subset \tilde{D}(\Omega)$, which for every $\underline{\hat{u}} \in \tilde{D}(\Omega)$ and every $(p, \alpha) \in \bar{\Omega}$ satisfies

$$(\tau \underline{\hat{u}})(p, \alpha) = \underline{\hat{u}}(p) \quad (5.3)$$

Observe, that the inverse mapping of $\tau : \tilde{D}(\Omega) \rightarrow \tilde{D}(\bar{\Omega})$ is well defined; it will be denoted by $\tau^{-1} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\Omega)$.

VI. THE EUCLIDEAN INNER PRODUCTS

The “Euclidean inner product,” which is the only one to be considered in this article, is defined to be

$$\begin{cases} \underline{\hat{u}} \cdot \underline{\hat{w}} \equiv \sum_{p \in \Omega} \underline{\hat{u}}(p) \underline{\hat{w}}(p), \quad \forall \underline{\hat{u}}, \underline{\hat{w}} \in \tilde{D}(\Omega) \\ \underline{u} \cdot \underline{w} \equiv \sum_{p \in \Omega} \underline{u}(p) \underline{w}(p) = \sum_{q \in \Omega} \sum_{p \in \zeta(q)} \underline{u}(p) \underline{w}(p), \quad \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \end{cases} \quad (6.1)$$

The methods described in this article are not restricted, in their applicability, to a single differential equation, but they are equally applicable to systems of differential equations, such as those occurring in elasticity. A proper treatment in our scheme of those systems requires introducing vector-valued functions. In such cases, $\underline{\hat{u}}(p)$ and $\underline{u}(p)$ are themselves vectors and, when defining the Euclidean inner product, Eq. (6.1) must be replaced by

$$\begin{cases} \underline{\hat{u}} \cdot \underline{\hat{w}} \equiv \sum_{p \in \Omega} \underline{\hat{u}}(p) \odot \underline{\hat{w}}(p), \quad \forall \underline{\hat{u}}, \underline{\hat{w}} \in \tilde{D}(\Omega) \\ \underline{u} \cdot \underline{w} \equiv \sum_{p \in \Omega} \underline{u}(p) \odot \underline{w}(p) = \sum_{q \in \Omega} \sum_{p \in \zeta(q)} \underline{u}(p) \odot \underline{w}(p), \quad \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \end{cases} \quad (6.2)$$

Here, the symbol \odot stands for the inner product of the vector space where the vectors $\underline{\hat{u}}(p)$ and $\underline{u}(p)$ lie.

The multiplicity of an original node, p , equals the number of derived nodes of the form (p, α) ; it is denoted by $m(p)$. The auxiliary matrices $\underline{\hat{m}} : \tilde{D}(\Omega) \rightarrow \tilde{D}(\Omega)$ and $\underline{m} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$, are defined, for each $\underline{\hat{u}} \in \tilde{D}(\Omega)$ and each $\underline{u} \in \tilde{D}(\bar{\Omega})$, by

$$\begin{aligned} \underline{\hat{m}} \underline{\hat{u}}(p) &= m(p) \underline{\hat{u}}(p), \quad \forall p \in \Omega \\ \underline{m} \underline{u}(p) &= m(p) \underline{u}(p), \quad \forall p = (p, \alpha) \in \bar{\Omega} \end{aligned} \quad (6.3)$$

Both of them are diagonal matrices; more precisely, one is diagonal and the other one is block-diagonal. The values at the main diagonals of $\underline{\hat{m}}$ and \underline{m} are the multiplicities $m(p)$. Simple results whose proofs are straightforward are:

$$\tau \underline{\hat{m}} \underline{\hat{u}} = \underline{m} \tau \underline{\hat{u}} \quad \text{and} \quad \tau \underline{\hat{m}}^{-1} \underline{\hat{u}} = \underline{m}^{-1} \tau \underline{\hat{u}}, \quad \forall \underline{\hat{u}} \in \tilde{D}(\Omega) \quad (6.4)$$

together with

$$\underline{m} \tilde{D}(\bar{\Omega}) = \tilde{D}(\bar{\Omega}) = \underline{m}^{-1} \tilde{D}(\bar{\Omega}) \quad (6.5)$$

In [19], it was shown that each one of the following relations holds:

$$\left. \begin{aligned} \underline{\hat{u}} \cdot \underline{\hat{m}} \underline{\hat{w}} &= \tau(\underline{\hat{u}}) \cdot \tau(\underline{\hat{w}}) \\ \underline{\hat{u}} \cdot \underline{\hat{w}} &= \tau(\underline{\hat{u}}) \cdot \underline{\hat{m}}^{-1} \tau(\underline{\hat{w}}) \end{aligned} \right\}, \quad \forall \underline{\hat{u}}, \underline{\hat{w}} \in \tilde{D}(\Omega) \quad (6.6)$$

Furthermore, let $\underline{u} \in \bar{D}(\bar{\Omega})$ be such that for some $\underline{\hat{u}} \in \tilde{D}(\Omega)$ it fulfills

$$\underline{\hat{u}} \cdot \underline{\hat{w}} = \underline{u} \cdot \tau(\underline{\hat{w}}), \quad \forall \underline{\hat{w}} \in \tilde{D}(\Omega) \quad (6.7)$$

Then

$$\underline{u} = \underline{\hat{m}}^{-1} \tau(\underline{\hat{u}}) = \tau(\underline{\hat{m}}^{-1} \underline{\hat{u}}) \quad (6.8)$$

VII. VECTOR SUBSPACES: THE AVERAGE AND JUMP MATRICES

The matrices $\underline{\underline{a}} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ and $\underline{\underline{j}} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ are the projections on the subspace $\tilde{D}(\bar{\Omega})$ of continuous vectors and on its orthogonal complement, respectively. They satisfy:

$$\underline{\underline{j}} = \underline{\underline{I}} - \underline{\underline{a}} \quad \text{and} \quad \underline{\underline{I}} = \underline{\underline{a}} + \underline{\underline{j}} \quad (7.1)$$

Here, $\underline{\underline{I}}$ is the identity matrix and the projection on $\tilde{D}(\bar{\Omega})$ is taken with respect to the Euclidean inner product. The matrices $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are referred to as the “average” and the “jump” matrices. The following properties should be noticed: $\underline{\underline{a}}$ and $\underline{\underline{j}}$ are both symmetric, non-negative, and idempotent. Furthermore,

$$\underline{\underline{a}} \underline{\underline{j}} = \underline{\underline{j}} \underline{\underline{a}} = \underline{\underline{0}} \quad \text{and} \quad \underline{\underline{j}} \tilde{D}(\bar{\Omega}) = \{0\} \quad (7.2)$$

The construction of the matrix $\underline{\underline{a}}$ is relatively simple [19]. Writing

$$\underline{\underline{a}} \equiv (a_{(p,\alpha)(q,\beta)}) \quad (7.3)$$

Then,

$$a_{(p,\alpha)(q,\beta)} = \frac{1}{m(p)} \delta_{pq}, \quad \forall (p,\alpha) \in \bar{\Omega} \quad \text{and} \quad \forall (q,\beta) \in \bar{\Omega} \quad (7.4)$$

The following expression permits computing the action of $\underline{\underline{j}}$ on any vector

$$\underline{\underline{j}} \underline{u} = \underline{u} - \underline{\underline{a}} \underline{u}, \quad \forall \underline{u} \in \bar{\Omega} \quad (7.5)$$

The following subspaces of $\tilde{D}(\bar{\Omega})$ are also defined:

$$\left. \begin{aligned} \tilde{D}_{11}(\bar{\Omega}) &\equiv \underline{\underline{j}} \tilde{D}(\bar{\Omega}) \subset \tilde{D}(\Gamma) \\ \tilde{D}_{12}(\bar{\Omega}) &\equiv \tilde{D}(\bar{\Omega}) = \underline{\underline{a}} \tilde{D}(\bar{\Omega}) \end{aligned} \right\} \quad \text{and} \quad \left\{ \begin{aligned} \tilde{D}_{11}(\Gamma) &\equiv \underline{\underline{j}} \tilde{D}(\Gamma) = \tilde{D}_{11}(\bar{\Omega}) \\ \tilde{D}_{12}(\Gamma) &\equiv \underline{\underline{a}} \tilde{D}(\Gamma) \end{aligned} \right. \quad (7.6)$$

VIII. THE DUAL-PRIMAL SUBSPACE

The dual-primal space $\tilde{D}^{\text{DP}}(\bar{\Omega})$ is the subspace of $\tilde{D}(\bar{\Omega})$ whose elements are continuous at every node belonging to π . For each node $k \in \Omega = \{1, \dots, d\}$, we define the local “jump-matrix at k ,” to be

$$\underline{j}^k \equiv \left(j_{(i,\alpha)(j,\beta)}^k \right) \quad (8.1)$$

where:

$$j_{(i,\alpha)(j,\beta)}^k \equiv \left(\delta_{\alpha\beta} - \frac{1}{m(k)} \right) \delta_{ik} \delta_{jk} \quad (8.2)$$

The “dual-primal” jump matrix is defined to be

$$\underline{j}^\pi \equiv \sum_{k \in \Omega^\pi} \underline{j}^k \quad (8.3)$$

Here, Ω^π is the set of primal nodes. Introducing the symbol δ_{ij}^π , defined by

$$\delta_{ij}^\pi \equiv \begin{cases} 1, & \text{if } i, j \in \Omega^\pi \\ 0, & \text{if } i \text{ or } j \notin \Omega^\pi \end{cases} \quad (8.4)$$

It is seen that

$$j_{(i,\alpha)(j,\beta)}^\pi = \left(\delta_{\alpha\beta} - \frac{1}{m(i)} \right) \delta_{ij} \delta_{ij}^\pi \quad (8.5)$$

The matrix $\underline{a}^\pi : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ is defined as

$$\underline{a}^\pi \equiv \underline{I} - \underline{j}^\pi \quad (8.6)$$

Therefore,

$$a_{(i,\alpha)(j,\beta)}^\pi = \frac{1}{m(i)} \delta_{ij} \delta_{ij}^\pi + \delta_{\alpha\beta} \delta_{ij} (1 - \delta_{ij}^\pi) \quad (8.7)$$

In words, this equation says that \underline{a}^π equals the identity matrix at every derived node except when the node belongs to the set $\pi \subset \bar{\Gamma}$ of primal nodes, in which case it equals the average matrix as given by Eq. (6.4). The primal jump operator \underline{j}^π , on the other hand, vanishes everywhere except at primal nodes, where it equals the jump operator.

The “dual-primal” space, $\tilde{D}^{\text{DP}}(\bar{\Omega})$, satisfies

$$\tilde{D}^{\text{DP}}(\bar{\Omega}) \equiv \underline{a}^\pi \tilde{D}(\bar{\Omega}) = \underline{a}^\pi \tilde{D}(\Pi) + \tilde{D}(\Delta) \quad (8.8)$$

So, $\underline{a}^\pi : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ is the projection matrix on the dual-primal subspace $\tilde{D}^{\text{DP}}(\bar{\Omega})$. In particular, $\tilde{D}^{\text{DP}}(\bar{\Omega}) = \tilde{D}(\bar{\Omega})$ when $\pi = \emptyset$. Furthermore, we adopt the notations

$$\tilde{D}^{\text{DP}}(\Pi) \equiv \underline{a}^\pi \tilde{D}(\Pi) \subset \underline{a} \tilde{D}(\bar{\Omega}) \subset \underline{a}^\pi \tilde{D}(\bar{\Omega}) \quad \text{and} \quad \tilde{D}^{\text{DP}}(\Delta) \equiv \underline{a}^\pi \tilde{D}(\Delta) = \tilde{D}(\Delta) \quad (8.9)$$

together with

$$\tilde{D}_{11}^{\text{DP}}(\bar{\Omega}) \equiv \underline{j} \tilde{D}^{\text{DP}}(\bar{\Omega}) \subset \tilde{D}^{\text{DP}}(\bar{\Omega}) \quad \text{and} \quad \tilde{D}_{12}^{\text{DP}}(\bar{\Omega}) \equiv \underline{a} \tilde{D}^{\text{DP}}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega}) \quad (8.10)$$

To prove that $\underline{j} \tilde{D}^{\text{DP}}(\bar{\Omega}) \subset \tilde{D}^{\text{DP}}(\bar{\Omega})$, given $\underline{w} \in \tilde{D}^{\text{DP}}(\bar{\Omega})$ we compute the projection of $\underline{j} \underline{w}$ on $\tilde{D}^{\text{DP}}(\bar{\Omega})$:

$$\underline{a}^{\pi} \underline{j} \underline{w} = (\underline{I} - \underline{j}^{\pi}) \underline{j} \underline{w} = \underline{j} \underline{w} - \underline{j}^{\pi} \underline{w} = \underline{j} \underline{w} \quad (8.11)$$

IX. THE MF-DP FORMULATION FOR NONSYMMETRIC MATRICES

In the remaining of this article, several matrices will be considered.

$$\hat{\underline{A}} : \tilde{D}(\Omega) \rightarrow \tilde{D}(\Omega), \quad \underline{A}^t : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega}) \quad \text{and} \quad \underline{A} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega}) \quad (9.1)$$

The matrices $\hat{\underline{A}}$, \underline{A}^t , and \underline{A} will be referred to as the “original matrix,” “total matrix,” and the “dual-primal matrix,” respectively. They satisfy the relation

$$\underline{A} = \underline{a}^{\pi} \underline{A}^t \underline{a}^{\pi} \quad (9.2)$$

And, we will use the notation:

$$\hat{\underline{A}} \equiv (\hat{A}_{pq}), \quad \text{where } p, q \in \Omega \quad (9.3)$$

The developments presented in Sections IV to VIII are very similar to those of [19]. However, in [19] the matrix $\hat{\underline{A}} : \tilde{D}(\Omega) \rightarrow \tilde{D}(\Omega)$ was assumed to be symmetric and positive definite, while here that assumption is dropped. Therefore, the arguments presented in what follows differ in many respects from those given in [19].

In particular, in this article the following concepts play an important role. Given any subset $\bar{X} \subset \bar{\Omega}$ we define

$$\tilde{D}^{\text{DP}}(\bar{X}) \equiv \underline{a}^{\pi} \tilde{D}(\bar{X}) \quad (9.4)$$

Let $\bar{X} \subset \bar{\Omega}$ be any subset of $\bar{\Omega}$ and write $E \equiv \tilde{D}^{\text{DP}}(\bar{X}) \subset \tilde{D}^{\text{DP}}(\bar{\Omega})$. We say that the dual-primal matrix \underline{A} is “well posed everywhere” when, for every $\bar{X} \subset \bar{\Omega}$, $\text{proj}_E \underline{A} : \tilde{D}^{\text{DP}}(\bar{X}) \rightarrow \tilde{D}^{\text{DP}}(\bar{X})$ is a bijection.

Assume $\underline{A} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ is well posed everywhere and define the matrix $\underline{C} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{X})$ by $\underline{C} \equiv \text{proj}_E \underline{A} \text{proj}_E$. Then, we define the matrix $\underline{C}^{-1} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ by

$$\underline{C}^{-1} \equiv \underline{B}^{-1} \text{proj}_E \quad (9.5)$$

Here, \underline{B}^{-1} is the inverse of $\underline{B} \equiv \text{proj}_E \underline{A} : \tilde{D}^{\text{DP}}(\bar{X}) \rightarrow \tilde{D}^{\text{DP}}(\bar{X})$, which in turn is well defined because $\underline{A} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ is well posed everywhere.

Using the notation of Eq. (9.3), it will be assumed throughout the article that:

1.

$$\widehat{A}_{pq} = 0, \text{ whenever } p \in \Omega^I \cap \Omega_\alpha, q \in \Omega^I \cap \Omega_\beta, \text{ and } \alpha \neq \beta \quad (9.6)$$

2. The matrix $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ satisfies the condition:

$$\widehat{w} \cdot \widehat{\underline{\underline{A}}\underline{\underline{u}}} = \tau(\widehat{w}) \cdot \underline{\underline{A}}^t \tau(\widehat{u}), \forall \widehat{u}, \widehat{w} \in \tilde{D}(\Omega) \quad (9.7)$$

where $\tau : \tilde{D}(\Omega) \rightarrow \tilde{D}(\bar{\Omega})$ is the natural immersion of $\tilde{D}(\Omega)$ into $\tilde{D}(\bar{\Omega})$. It should be observed that this condition does not determine $\underline{\underline{A}}^t$ uniquely;

3. For each $\alpha \in \{1, \dots, E\}$ there is defined a matrix $\underline{\underline{A}}^\alpha : \tilde{D}(\bar{\Omega}_\alpha) \rightarrow \tilde{D}(\bar{\Omega}_\alpha)$ such that

$$\underline{\underline{A}}^t = \sum_{\alpha=1}^E \underline{\underline{A}}^\alpha \quad (9.8)$$

A convenient procedure for constructing a matrix $\underline{\underline{A}}^t$, when $\widehat{\underline{\underline{A}}}$ is given, fulfilling the above conditions was presented in [19], proving thereby that there is always at least one such a matrix; and

4. The matrix $\underline{\underline{A}} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$, defined by Eq. (9.2), is well posed everywhere.

Now, we recall Eq. (8.8):

$$\tilde{D}^{\text{DP}}(\bar{\Omega}) = \underline{\underline{q}}^\pi \tilde{D}(\Pi) + \tilde{D}(\Delta) \quad (9.9)$$

and observe that $\underline{\underline{q}}^\pi \tilde{D}(\Pi) \subset \tilde{D}(\Pi)$ and $\tilde{D}(\Delta)$ are orthogonal complements, relative to $\tilde{D}^{\text{DP}}(\bar{\Omega})$. Taking $E \equiv \tilde{D}^{\text{DP}}(\Pi)$ and $F \equiv \tilde{D}^{\text{DP}}(\Delta)$, we adopt the notation

$$\begin{aligned} \underline{\underline{A}}_{\Pi\Pi} &\equiv \text{proj}_E \underline{\underline{A}} \text{proj}_E; \underline{\underline{A}}_{\Pi\Delta} \equiv \text{proj}_E \underline{\underline{A}} \text{proj}_F; \\ \underline{\underline{A}}_{\Delta\Pi} &\equiv \text{proj}_F \underline{\underline{A}} \text{proj}_E; \underline{\underline{A}}_{\Delta\Delta} \equiv \text{proj}_F \underline{\underline{A}} \text{proj}_F; \end{aligned} \quad (9.10)$$

This permits us writing the matrix $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ as

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \quad (9.11)$$

Furthermore, the matrices $\underline{\underline{A}}^{-1} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ and $\underline{\underline{A}}_{\Pi\Pi}^{-1} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ will be used in the sense of the definition of Eq. (9.5). We observe that the actions, on any vector $\underline{v} \in \tilde{D}(\bar{\Omega})$, of the matrices occurring in Eq. (9.11) are given by:

$$\begin{aligned} \underline{\underline{A}}_{\Pi\Pi} \underline{v} &= \underline{\underline{q}}(\underline{\underline{A}}^t \underline{v}_\Pi)_\Pi, & \underline{\underline{A}}_{\Pi\Delta} \underline{v} &= \underline{\underline{q}}(\underline{\underline{A}}^t \underline{v}_\Delta)_\Pi \\ \underline{\underline{A}}_{\Delta\Pi} \underline{v} &= (\underline{\underline{A}}^t \underline{v}_\Pi)_\Delta, & \underline{\underline{A}}_{\Delta\Delta} \underline{v} &= (\underline{\underline{A}}^t \underline{v}_\Delta)_\Delta \end{aligned} \quad (9.12)$$

Here, the facts that $\tilde{D}^{\text{DP}}(\Pi) = \underline{\underline{q}}^\pi \tilde{D}(\Pi) = \underline{\underline{q}} \tilde{D}(\Pi)$ and $\tilde{D}^{\text{DP}}(\Delta) = \tilde{D}(\Delta)$ have been used.

Definition 9.1. Let $\widehat{\underline{f}} \in \tilde{D}(\Omega)$. Then, the “original problem” consists in searching for a function $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ that satisfies

$$\widehat{\underline{A}}\widehat{\underline{u}} = \widehat{\underline{f}} \quad (9.13)$$

This problem is assumed to possess a unique solution. The “dual-primal formulation” consists in searching for a function $\underline{\tilde{u}} \in \tilde{D}^{\text{DP}}(\bar{\Omega})$ that satisfies

$$\underline{\underline{a}}\underline{\underline{A}}\underline{\tilde{u}} = \underline{\tilde{f}} \quad \text{and} \quad \underline{\underline{j}}\underline{\tilde{u}} = 0 \quad (9.14)$$

where $\underline{\tilde{f}} \in \bar{D}(\bar{\Omega}) = \tilde{D}_{12}(\bar{\Omega}) \subset \tilde{D}^{\text{DP}}(\bar{\Omega})$ is given by

$$\underline{\tilde{f}} \equiv \begin{pmatrix} \underline{\tilde{f}}_{\Pi} \\ \underline{\tilde{f}}_{\Delta} \end{pmatrix} \equiv \underline{\underline{m}}^{-1} \tau(\widehat{\underline{f}}) \quad \text{and} \quad \underline{\tilde{f}}_{\Delta} = \underline{\tilde{f}}_{\Delta 2} \quad (9.15)$$

Theorem 9.1. A function $\underline{\tilde{u}} \in \tilde{D}^{\text{DP}}(\bar{\Omega})$ is the solution of the dual-primal formulation if and only if

$$\widehat{\underline{u}} \equiv \tau^{-1}(\underline{\tilde{u}}) \quad (9.16)$$

is the solution of the original problem.

Proof. Because we have

1. If $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ is solution of the original problem, then $\underline{\tilde{u}} \equiv \tau(\widehat{\underline{u}}) \in \bar{D}(\bar{\Omega}) \subset \tilde{D}^{\text{DP}}(\bar{\Omega})$ fulfills Eq. (9.14);
2. Conversely, Eq. (9.14) implies $\underline{\tilde{u}} \in \bar{D}(\bar{\Omega})$, so that τ^{-1} is well defined. Taking $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ given by Eq. (9.16), it is seen that $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ fulfills Eq. (9.13).

■

Corollary 9.1. The dual-primal formulation possesses a unique solution.

Proof. It is clear by virtue of Theorem 8.1 and the corresponding property of the original problem. ■

It is straightforward to see that the solution of the dual-primal formulation is independent of the choice of the set of primal nodes $\pi \subset \bar{\Omega}$. In the particular case, when primal nodes are not used $\pi = \emptyset$ and the dual-primal formulation reduces to: “Find a function $\underline{\tilde{u}} \in \tilde{D}(\Omega)$ such that

$$\underline{\underline{a}}\underline{\underline{A}}'\underline{\tilde{u}} = \underline{\tilde{f}} \quad \text{and} \quad \underline{\underline{j}}\underline{\tilde{u}} = 0 \quad (9.17)$$

X. THE SCHUR-COMPLEMENT FORMULATIONS

The matrices $\underline{\underline{L}} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ and $\underline{\underline{R}} : \tilde{D}^{\text{DP}}(\bar{\Omega}) \rightarrow \tilde{D}^{\text{DP}}(\bar{\Omega})$ are introduced next:

$$\underline{\underline{L}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi\Pi} & \underline{\underline{A}}_{\Pi\Delta} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \underline{\underline{R}} \equiv \begin{pmatrix} 0 & 0 \\ \underline{\underline{A}}_{\Delta\Pi} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} \quad (10.1)$$

Then, Eq. (9.11) implies

$$\underline{\underline{A}} = \underline{\underline{L}} + \underline{\underline{R}} \quad (10.2)$$

In view of Eq. (9.10), the range of $\underline{\underline{L}}$ is contained in $\tilde{D}^{\text{DP}}(\Pi) \subset \underline{\underline{a}}\tilde{D}(\bar{\Omega})$ and therefore

$$\underline{\underline{a}}\underline{\underline{L}} = \underline{\underline{L}} \quad (10.3)$$

Equation (9.14) can now be written as

$$(\underline{\underline{L}} + \underline{\underline{a}}\underline{\underline{R}})\underline{\underline{u}} = \underline{\underline{f}} \quad \text{and} \quad \underline{\underline{j}}\underline{\underline{u}} = 0 \quad (10.4)$$

We observe that the ranges of $\underline{\underline{L}}$ and $\underline{\underline{a}}\underline{\underline{R}}$ are linearly independent and Eq. (10.4) is fulfilled, if and only if,

$$\underline{\underline{L}}\underline{\underline{u}} = \underline{\underline{f}}_{\Pi}, \quad \underline{\underline{a}}\underline{\underline{R}}\underline{\underline{u}} = \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{\underline{u}} = 0 \quad (10.5)$$

This because $\underline{\underline{f}} = \underline{\underline{f}}_{\Pi} + \underline{\underline{f}}_{\Delta 2}$, since $\underline{\underline{f}}_{\Delta 1} = 0$, by virtue of Eq. (8.15). It is advantageous to transform the problem of Eq. (10.5), by subtracting the auxiliary vector (see [19]):

$$\underline{\underline{u}}_P \equiv \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{f}}_{\Pi} \quad (10.6)$$

We notice that Eq. (10.6) implies

$$(\underline{\underline{u}}_P)_{\Delta} = 0 \quad (10.7)$$

Therefore, $\underline{\underline{j}}\underline{\underline{u}}_P = 0$. Defining $\underline{\underline{u}} \equiv \underline{\underline{u}} - \underline{\underline{u}}_P$, then Eq. (10.5) becomes

$$\underline{\underline{L}}\underline{\underline{u}} = 0, \quad \underline{\underline{a}}\underline{\underline{R}}\underline{\underline{u}} = \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{\underline{u}} = 0 \quad (10.8)$$

Here, $\underline{\underline{f}}_{\Delta 2} \in \tilde{D}_{12}(\Delta)$ is defined by

$$\underline{\underline{f}}_{\Delta 2} \equiv \underline{\underline{f}}_{\Delta 2} - \underline{\underline{a}}\underline{\underline{A}}_{\Delta\Pi} \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{f}}_{\Pi} \quad (10.9)$$

The “dual-primal harmonic functions space,” is defined to be the null subspace of $\underline{\underline{L}}$; i.e.,

$$D \equiv \left\{ \underline{\underline{u}} \in \tilde{D}^{\text{DP}}(\bar{\Omega}) \mid \underline{\underline{L}}\underline{\underline{u}} = 0 \right\} \quad (10.10)$$

Hence, the problem of Eq. (10.8) can be stated as: find a harmonic vector (i.e., such that $\underline{\underline{u}} \in D$) that satisfies

$$\underline{\underline{a}}\underline{\underline{R}}\underline{\underline{u}} = \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{\underline{u}} = 0 \quad (10.11)$$

Some important properties are listed next.

- A. Dual-primal harmonic functions are characterized by their dual-values. Indeed, if $\underline{u} \in D$, is such that $\underline{\underline{L}}\underline{u} = 0$, then

$$\underline{u}_\Pi = -\underline{\underline{A}}_{\Pi\P}^{-1}\underline{\underline{A}}_{\Pi\Delta}\underline{u}_\Delta \quad (10.12)$$

- B. When $\underline{u} \in D$,

$$\underline{\underline{A}}\underline{u} = \underline{\underline{R}}\underline{u} = \underline{\underline{S}}\underline{u} \quad (10.13)$$

where $\underline{\underline{S}}$ is the “dual-primal Schur complement matrix,” defined by

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\P}\underline{\underline{A}}_{\Pi\P}^{-1}\underline{\underline{A}}_{\Pi\Delta} \quad (10.14)$$

- C. $\underline{\underline{S}} : \tilde{D}(\Delta) \rightarrow \tilde{D}(\Delta)$ possesses an inverse that will be denoted by $\underline{\underline{S}}^{-1} : \tilde{D}(\Delta) \rightarrow \tilde{D}(\Delta)$.

Here, the equality $\tilde{D}(\Delta) = \tilde{D}^{\text{DP}}(\Delta)$ is recalled.

- D. When the dual-primal matrix $\underline{\underline{A}}$ is well posed everywhere:

$$\underline{\underline{j}}\underline{\underline{S}} : \tilde{D}_{11}(\Delta) \rightarrow \tilde{D}_{11}(\Delta) \text{ is bijective} \quad (10.15)$$

since $\underline{\underline{j}}\underline{\underline{S}} : \tilde{D}_{11}(\Delta) \rightarrow \tilde{D}_{11}(\Delta)$ and $\underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}} : \tilde{D}_{11}(\Delta) \rightarrow \tilde{D}_{11}(\Delta)$ are equal.

Theorem 10.1. Let $\underline{u} \equiv (\underline{u}_\Pi + \underline{u}_\Delta) \in D^{\text{DP}}$. Then, \underline{u} is solution of Eq. (10.8), if and only if

$$\underline{\underline{a}}\underline{\underline{S}}\underline{u}_\Delta = \underline{f}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}}\underline{\underline{u}}_\Delta = 0 \quad (10.16)$$

Proof. Because when $\underline{u} \in D$, Eq. (10.8) reduces to

$$\underline{\underline{a}}\underline{\underline{A}}\underline{u} = \underline{\underline{a}}\underline{\underline{S}}\underline{u}_\Delta \quad \text{and} \quad \underline{\underline{j}}\underline{u} = \underline{\underline{j}}(\underline{u}_\Pi + \underline{u}_\Delta) = \underline{\underline{j}}\underline{u}_\Delta \quad (10.17)$$

In what follows, these properties will be used to derive a wide variety of non-overlapping domain decomposition methods, which permit obtaining the dual-values, $\underline{u}_\Delta \in \tilde{D}(\Delta)$. Once \underline{u}_Δ is known, $\underline{u}_\Pi \in \tilde{D}^{\text{DP}}(\Pi)$ is obtained by means of Eq. (10.12). ■

XI. MULTIPLIERS-FREE METHOD FOR NONSYMMETRIC MATRICES

Let be

$$\begin{cases} \tilde{D}_{11}(\Delta) \equiv \underline{\underline{j}}\tilde{D}(\Delta) & \text{and} & \tilde{D}_{12}(\Delta) \equiv \underline{\underline{a}}\tilde{D}(\Delta) \\ \tilde{D}_{21}(\Delta) \equiv \{\underline{u} \in \tilde{D}(\Delta) | \underline{\underline{j}}\underline{\underline{S}}\underline{u} = 0\} & \text{and} & \tilde{D}_{22}(\Delta) \equiv \{\underline{u} \in \tilde{D}(\Delta) | \underline{\underline{a}}\underline{\underline{S}}\underline{u} = 0\} \end{cases} \quad (11.1)$$

We observe that the relations

$$\tilde{D}_{21}(\Delta) = \{\underline{u} \in \tilde{D}(\Delta) | \underline{\underline{S}}\underline{u} = \underline{\underline{a}}\underline{\underline{S}}\underline{u}\} \quad \text{and} \quad \tilde{D}_{22}(\Delta) = \{\underline{u} \in \tilde{D}(\Delta) | \underline{\underline{S}}\underline{u} = \underline{\underline{j}}\underline{\underline{S}}\underline{u}\} \quad (11.2)$$

are satisfied.

Next, we show that each pair $\{\tilde{D}_{11}(\Delta), \tilde{D}_{12}(\Delta)\}$ and $\{\tilde{D}_{21}(\Delta), \tilde{D}_{22}(\Delta)\}$ is a *coordinate system* for $\tilde{D}(\Delta)$, in the sense of Definition 3.1; i.e.,

$$\tilde{D}(\Delta) = \tilde{D}_{11}(\Delta) \oplus \tilde{D}_{12}(\Delta) \quad \text{and} \quad \tilde{D}(\Delta) = \tilde{D}_{21}(\Delta) \oplus \tilde{D}_{22}(\Delta) \quad (11.3)$$

The first of these relations can be easily shown using the fact that $\underline{v} = \underline{j}\underline{v} + \underline{a}\underline{v}$. To prove the second one, notice that $\underline{v} \in \tilde{D}_{21}(\Delta) \cap \tilde{D}_{22}(\Delta)$ implies $\underline{S}\underline{v} = \underline{j}\underline{S}\underline{v} + \underline{a}\underline{S}\underline{v} = 0$ and this is tantamount to $\underline{v} = 0$; hence, $\tilde{D}_{21}(\Delta) \cap \tilde{D}_{22}(\Delta) = \{0\}$. On the other hand, given $\underline{v} \in \tilde{D}(\Delta)$ define

$$\underline{v}_{21} = \underline{S}^{-1}\underline{a}\underline{S}\underline{v} \quad \text{and} \quad \underline{v}_{22} = \underline{S}^{-1}\underline{j}\underline{S}\underline{v} \quad (11.4)$$

Then, it can be verified that $\underline{v}_{21} + \underline{v}_{22} = \underline{v}$, while $\underline{v}_{21} \in \tilde{D}_{21}(\Delta)$ and $\underline{v}_{22} \in \tilde{D}_{22}(\Delta)$. In view of the above, we define

$$\begin{cases} \sigma_{11} \equiv \underline{j}, \sigma_{12} \equiv \underline{a} \\ \sigma_{21} \equiv \underline{S}^{-1}\underline{a}\underline{S}, \sigma_{22} \equiv \underline{S}^{-1}\underline{j}\underline{S} \end{cases} \quad (11.5)$$

Next, we show that the couple $(\{\tilde{D}_{11}(\Delta), \tilde{D}_{12}(\Delta)\}, \{\tilde{D}_{21}(\Delta), \tilde{D}_{22}(\Delta)\})$ is a conjugate pair of coordinate systems of $\tilde{D}(\Delta)$, in the sense of Definition 3.2; i.e.,

$$\tilde{D}_{11}(\Delta) \cap \tilde{D}_{21}(\Delta) = \{0\} \quad \text{and} \quad \tilde{D}_{12}(\Delta) \cap \tilde{D}_{22}(\Delta) = \{0\} \quad (11.6)$$

Now, let $\underline{u}_\Delta \in \tilde{D}_{12}(\Delta) \cap \tilde{D}_{22}(\Delta)$, then

$$\underline{a}\underline{S}\underline{u}_\Delta = 0 \quad \text{and} \quad \underline{j}\underline{u}_\Delta = 0 \quad (11.7)$$

Because of Corollary 9.1, Eq. (11.7) implies that $\underline{u}_\Delta = 0$.

Let $\underline{u}_\Delta \in \tilde{D}_{11}(\Delta) \cap \tilde{D}_{21}(\Delta)$, then

$$\underline{j}\underline{S}\underline{u}_\Delta = 0 \quad \text{and} \quad \underline{a}\underline{u}_\Delta = 0 \quad (11.8)$$

Now, $\underline{u}_\Delta \in \tilde{D}_{11}(\Delta)$ since $\underline{a}\underline{u}_\Delta = 0$ and by assumption $\underline{j}\underline{S} : \tilde{D}_{11}(\Delta) \rightarrow \tilde{D}_{11}(\Delta)$ is a bijection; hence, $\underline{u}_\Delta = 0$. Therefore, $(\{\tilde{D}_{11}(\Delta), \tilde{D}_{12}(\Delta)\}, \{\tilde{D}_{21}(\Delta), \tilde{D}_{22}(\Delta)\})$ is indeed a conjugate pair of coordinate systems of $\tilde{D}(\Delta)$.

Problem 1. The first problem to be considered is obtained taking $i = 1$ and $j = 2$ (and, consequently $\alpha = 2, \beta = 1$) in Problem A of Section II: “Given $g \in D_{21}$, find $u \in D_{12}$ such that $\sigma_{21}u = g$.”

We consider two formulations of this problem:

Formulation 1a. “Given $\underline{g} \in \tilde{D}(\Delta)$, which has the property that

$$\underline{j}\underline{S}\underline{g} = 0 \quad (11.9)$$

find a $\underline{u} \in \tilde{D}(\Delta)$ such that

$$\underline{S}^{-1}\underline{a}\underline{S}\underline{u} = \underline{g} \quad \text{and} \quad \underline{j}\underline{u} = 0 \quad (11.10)$$

Formulation 1b (the Schur-Complement Method). “Given $\underline{f} \in \tilde{D}(\Delta)$, which has the property that

$$\underline{j}\underline{f} = 0 \quad (11.11)$$

find a $\underline{u} \in \tilde{D}(\Delta)$ such that

$$\underline{a}\underline{S}\underline{u} = \underline{f} \quad \text{and} \quad \underline{j}\underline{u} = 0 \quad (11.12)$$

Lemma 11.1. Assume $\underline{g} \in \tilde{D}(\Delta)$ and $\underline{f} \in \tilde{D}(\Delta)$, in Formulations 1a and 1b, are related by

$$\underline{f} = \underline{a}\underline{S}\underline{g} \quad (11.13)$$

Or equivalently, by

$$\underline{f} = \underline{S}\underline{g}; \quad \text{i.e.,} \quad \underline{g} = \underline{S}^{-1}\underline{f} \quad (11.14)$$

Then, for any $\underline{u} \in \tilde{D}(\Delta)$, the following statements are equivalent:

- i. \underline{u} is solution of Problem 1;
- ii. \underline{u} satisfies Formulation 1a; and
- iii. \underline{u} satisfies Formulation 1b.

Proof. The equivalence between i and ii, follows from the above definitions and the use of Eqs. (11.9) and (11.10). Next, we observe that Eq. (11.12) is obtained when the first of the relations occurring in Eq. (11.10) is multiplied by \underline{S} . This establishes the equivalence between Formulations 1a and 1b, since \underline{S} is non singular.

In what follows it will be assumed that $\underline{g} \in \tilde{D}(\Delta)$ and $\underline{f} \in \tilde{D}(\Delta)$ are related by Eq. (11.14), in which case the condition $\underline{f} \in \tilde{D}_{12}(\Delta)$ is equivalent to $\underline{g} \in \tilde{D}_{21}(\Delta)$. Formulation 1b will be referred as the Schur Complement Method for nonsymmetric matrices. ■

A. The Neumann–Neumann Method for Nonsymmetric Matrices

A new version of the Neumann–Neumann method, applicable to nonsymmetric matrices, will be derived applying the preconditioned algorithm of section II to Problem 1.

In view of Eq. (11.5), Eq. (3.9) implies:

$$\underline{a}\underline{S}^{-1}\underline{a}\underline{S}\underline{u} = \underline{a}\underline{S}^{-1}\underline{f} \quad \text{and} \quad \underline{j}\underline{u} = 0 \quad (11.15)$$

Since

$$\underline{a}\underline{g} = \underline{a}\underline{S}^{-1}\underline{f} \quad (11.16)$$

Problem 2. The second problem to be considered is obtained taking $i = 1$ and $j = 2$ (and, consequently $\alpha = 1$, $\beta = 2$) in Problem A: “Given $\underline{g} \in \tilde{D}_{11}(\Delta)$, find $\underline{u} \in \tilde{D}_{22}(\Delta)$ such that $\sigma_{11}\underline{u} = \underline{g}$.”

Next, we consider two formulations of this problem:

Formulation 2a. Given $\underline{g} \in \tilde{D}(\Delta)$, which has the property that

$$\underline{a}\underline{g} = 0 \quad (11.17)$$

find a $\underline{u} \in \tilde{D}$ such that

$$\underline{j}\underline{u} = \underline{g} \quad \text{and} \quad \underline{a}\underline{S}\underline{u} = 0 \quad (11.18)$$

Formulation 2b (the FETI- Method). Given $\underline{f} \in \tilde{D}(\Delta)$, which has the property that

$$\underline{a}\underline{S}^{-1}\underline{f} = 0 \quad (11.19)$$

find a $\underline{u} \in \tilde{D}(\Delta)$ such that

$$\underline{S}\underline{j}\underline{u} = \underline{f} \quad \text{and} \quad \underline{a}\underline{S}\underline{u} = 0 \quad (11.20)$$

The FETI-MF-DP of Eq. (1.1), can be derived multiplying Eq. (11.20) by \underline{S} to the minus two.

Lemma 11.2. *As said before, we assume that $\underline{g} \in \tilde{D}(\Delta)$ and $\underline{f} \in \tilde{D}(\Delta)$ are related by*

$$\underline{f} = \underline{S}\underline{g}; \quad \text{i.e.,} \quad \underline{g} = \underline{S}^{-1}\underline{f} \quad (11.21)$$

Then, for any $\underline{u} \in \tilde{D}(\Delta)$, the following statements are equivalent:

- i. \underline{u} is solution of Problem 2;*
- ii. \underline{u} satisfies Formulation 2a; and*
- iii. \underline{u} satisfies Formulation 2b.*

Proof. First, we notice that Eqs. (11.17) and (11.19) are equivalent when $\underline{g} \in \tilde{D}(\Delta)$ and $\underline{f} \in \tilde{D}(\Delta)$ are related by Eq. (11.21). Then, the equivalence between *i* and *ii*, follows from Eqs. (11.1) and (11.5). Next, we observe that Eq. (11.20) is obtained when the first of the relations occurring in Eq. (11.18) is multiplied by \underline{S} . This establishes the equivalence between Formulations 2a and 2b, since \underline{S} is non singular. ■

Before leaving this Section we observe that the summary of formulas presented in Eqs. (1.1) and (1.2) of the Introduction, correspond to Eqs.(11.12), (11.15), (11.20) and (11.22).

B. The Preconditioned FETI Method for Nonsymmetric Matrices

A new version of the preconditioned-FETI method, applicable to nonsymmetric matrices, is here derived applying the matrix $\underline{S}^{-1}\underline{j}$ to the first equation in Eq. (11.20):

$$\underline{S}^{-1}\underline{j}\underline{S}\underline{j}\underline{u} = \underline{S}^{-1}\underline{j}\underline{f} \quad \text{and} \quad \underline{a}\underline{S}\underline{u} = 0 \quad (11.22)$$

We recall that here $\underline{f} \in \tilde{D}_{22}(\Delta)$.

XII. NUMERICAL PROCEDURES AND RESULTS

The developments of this Section are done in $\tilde{D}^{\text{DP}}(\bar{\Omega})$, the extended dual-primal space of vectors, whose members are generally discontinuous; i.e., all vectors considered here belong to $\tilde{D}(\bar{\Omega})$. We recall that in Eq. (8.11)

$$\begin{cases} \underline{\underline{A}}_{\Pi\Pi} : \tilde{D}^{\text{DP}}(\Pi) \rightarrow \tilde{D}^{\text{DP}}(\Pi), & \underline{\underline{A}}_{\Pi\Delta} : \tilde{D}(\Delta) \rightarrow \tilde{D}^{\text{DP}}(\Pi) \\ \underline{\underline{A}}_{\Delta\Pi} : \tilde{D}^{\text{DP}}(\Pi) \rightarrow \tilde{D}(\Delta), & \underline{\underline{A}}_{\Delta\Delta} : \tilde{D}(\Delta) \rightarrow \tilde{D}(\Delta) \end{cases} \quad (12.1)$$

Also, $\tilde{D}(\Delta) \subset \tilde{D}^{\text{DP}}(\bar{\Omega})$ is the linear space of vectors of $\tilde{D}(\bar{\Omega})$ that vanish at every derived node that is not a dual node, whereas $\tilde{D}^{\text{DP}}(\Pi) \subset \tilde{D}(\Pi)$ is the linear space of vectors that are continuous at the nodes of Π and vanish at every vector of $\tilde{D}(\Delta)$. Furthermore,

$$\tilde{D}^{\text{DP}}(\bar{\Omega}) = \tilde{D}^{\text{DP}}(\Pi) \oplus \tilde{D}(\Delta) \quad (12.2)$$

We now define $\Sigma \equiv I \cup \Delta$ and, in a similar fashion to that of Eq. (8.11), shall write

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Sigma\Sigma} & \underline{\underline{A}}_{\Sigma\pi} \\ \underline{\underline{A}}_{\pi\Sigma} & \underline{\underline{A}}_{\pi\pi} \end{pmatrix} \quad (12.3)$$

where

$$\begin{cases} \underline{\underline{A}}_{\Sigma\Sigma} : \tilde{D}^{\text{DP}}(\Sigma) \rightarrow \tilde{D}^{\text{DP}}(\Sigma), & \underline{\underline{A}}_{\Sigma\pi} : \tilde{D}^{\text{DP}}(\pi) \rightarrow \tilde{D}^{\text{DP}}(\Sigma) \\ \underline{\underline{A}}_{\pi\Sigma} : \tilde{D}^{\text{DP}}(\Sigma) \rightarrow \tilde{D}^{\text{DP}}(\pi), & \underline{\underline{A}}_{\pi\pi} : \tilde{D}^{\text{DP}}(\pi) \rightarrow \tilde{D}^{\text{DP}}(\pi) \end{cases} \quad (12.4)$$

Using Eqs. (8.8), it can be verified that

$$\underline{\underline{A}}_{\Sigma\Sigma} \equiv \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\Delta} \\ \underline{\underline{A}}_{\Delta I} & \underline{\underline{A}}_{\Delta\Delta} \end{pmatrix} = \sum_{\alpha=1}^E \begin{pmatrix} \underline{\underline{A}}_{II}^{\alpha} & \underline{\underline{A}}_{I\Delta}^{\alpha} \\ \underline{\underline{A}}_{\Delta I}^{\alpha} & \underline{\underline{A}}_{\Delta\Delta}^{\alpha} \end{pmatrix} \quad (12.5)$$

According to our previous results, the numerical application of the multipliers-free domain-decomposition methods requires the use of the following formulas:

$$\text{For the Schur complement MF-DP : } \underline{\underline{a}} \underline{\underline{S}} \underline{\underline{u}}_{\Delta} = \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}} \underline{\underline{u}}_{\Delta} = 0 \quad (12.6)$$

$$\text{For Neumann-Neumann MF-DP : } \underline{\underline{a}} \underline{\underline{S}}^{-1} \underline{\underline{a}} \underline{\underline{S}} \underline{\underline{u}}_{\Delta} = \underline{\underline{a}} \underline{\underline{S}}^{-1} \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{j}} \underline{\underline{u}}_{\Delta} = 0 \quad (12.7)$$

$$\text{For non-preconditioned FETI-MF-DP : } \underline{\underline{S}}^{-1} \underline{\underline{j}} \underline{\underline{u}} = -\underline{\underline{S}}^{-1} \underline{\underline{j}} \underline{\underline{S}}^{-1} \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{a}} \underline{\underline{S}} \underline{\underline{u}} = 0 \quad (12.8)$$

$$\text{For preconditioned FETI-MF-DP : } \underline{\underline{S}}^{-1} \underline{\underline{j}} \underline{\underline{S}} \underline{\underline{j}} \underline{\underline{u}} = -\underline{\underline{S}}^{-1} \underline{\underline{j}} \underline{\underline{S}} \underline{\underline{j}} \underline{\underline{S}}^{-1} \underline{\underline{f}}_{\Delta 2} \quad \text{and} \quad \underline{\underline{a}} \underline{\underline{S}} \underline{\underline{u}} = 0 \quad (12.9)$$

So, when iterating we need to have codes for computing the action of the following matrices $\underline{\underline{a}}$, $\underline{\underline{j}}$, $\underline{\underline{S}}$, and $\underline{\underline{S}}^{-1}$. The actions $\underline{\underline{a}} \underline{\underline{u}}$ and $\underline{\underline{j}} \underline{\underline{u}}$ of the *average* and *jump matrices* on any vector $\underline{\underline{u}} \in \tilde{D}(\bar{\Omega})$, which are given by Eqs. (6.4) and (6.5), are easy to compute so that their parallelization is not an issue.

A. Computation of $\underline{\underline{S}}$ and $\underline{\underline{S}}^{-1}$

As for the action of $\underline{\underline{S}}$, recall that

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta\Pi} \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{\underline{A}}_{\Pi\Delta} \quad (12.10)$$

Here, only the action of $\underline{\underline{A}}_{\Pi\Pi}^{-1}$ requires further explanation. Given $\underline{w} \in \tilde{D}(\Pi)$, let $\underline{v} \in \tilde{D}(\Pi)$ be such that $\underline{v} = \underline{v}_I + \underline{v}_\pi \equiv \underline{\underline{A}}_{\Pi\Pi}^{-1} \underline{w}$. Then, since

$$\underline{\underline{A}}_{\Pi\Pi} \underline{v} \equiv \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\pi} \\ \underline{\underline{A}}_{\pi I} & \underline{\underline{A}}_{\pi\pi} \end{pmatrix} \begin{pmatrix} \underline{v}_I \\ \underline{v}_\pi \end{pmatrix} = \begin{pmatrix} \underline{w}_I \\ \underline{w}_\pi \end{pmatrix} \quad (12.11)$$

Then, $\underline{v}_\pi \in \tilde{D}(\pi)$ is the solution of

$$\left(\underline{\underline{A}}_{\pi\pi} - \underline{\underline{A}}_{\pi I} \underline{\underline{A}}_{II}^{-1} \underline{\underline{A}}_{I\pi} \right) \underline{v}_\pi \equiv \underline{\underline{S}}_\pi \underline{v}_\pi = \underline{w}_\pi - \underline{\underline{A}}_{\pi I} \underline{\underline{A}}_{II}^{-1} \underline{w}_I \quad (12.12)$$

While

$$\underline{v}_I = \underline{\underline{A}}_{II}^{-1} (\underline{w}_I - \underline{\underline{A}}_{I\pi} \underline{v}_\pi) \quad (12.13)$$

This last problem [Eq. (12.12)] can be treated by two separate techniques: The first involves the explicit computation of the matrix $\underline{\underline{S}}_\pi$ (almost always banded) and its LU factorization whereas the second involves an iterative approach formulated in the vector space $\tilde{D}(\pi)$ whose dimension is much smaller. It should be pointed out that this second approach can be carried out in parallel. Once the vector \underline{v}_π has been obtained, \underline{v}_I is computed in parallel from the fact that

$$\underline{\underline{A}}_{II}^{-1} = \sum_{\alpha=1}^E \underline{\underline{A}}_{II}^{(\alpha)-1} \quad (12.14)$$

To obtain the action of $\underline{\underline{S}}^{-1} \underline{u}_\Delta$ for some $\underline{u}_\Delta \in \tilde{D}(\Delta)$, set $\underline{w}_\Delta \equiv \underline{\underline{S}}^{-1} \underline{u}_\Delta$ and write $\underline{w} \equiv \underline{w}_\pi + \underline{w}_\Sigma$ for the dual-primal harmonic extension of \underline{w}_Δ . Then,

$$\begin{pmatrix} \underline{\underline{A}}_{\Sigma\Sigma} & \underline{\underline{A}}_{\Sigma\pi} \\ \underline{\underline{A}}_{\pi\Sigma} & \underline{\underline{A}}_{\pi\pi} \end{pmatrix} \begin{pmatrix} \underline{w}_\Sigma \\ \underline{w}_\pi \end{pmatrix} = \begin{pmatrix} \underline{u}_\Delta \\ 0 \end{pmatrix} \quad (12.15)$$

Using Eqs. (12.5) and (12.15), it can be seen that

$$\left(\underline{\underline{A}}_{\pi\pi} - \underline{\underline{A}}_{\pi\Sigma} \underline{\underline{A}}_{\Sigma\Sigma}^{-1} \underline{\underline{A}}_{\Sigma\pi} \right) \underline{w}_\pi \equiv \underline{\underline{S}}'_\pi \underline{w}_\pi = -\underline{\underline{A}}_{\pi\Sigma} \underline{\underline{A}}_{\Sigma\Sigma}^{-1} \underline{u}_\Delta = -\underline{\underline{A}}_{\pi\Sigma} (\underline{\underline{A}}_{\Delta\Delta} - \underline{\underline{A}}_{\Delta I} \underline{\underline{A}}_{II}^{-1} \underline{\underline{A}}_{I\Delta}) \underline{u}_\Delta \quad (12.16)$$

Again, there are two approaches to the solution [similar to (12.12) of this equation]. Namely, explicitly compute the banded, small-dimensional matrix $\underline{\underline{S}}'_\pi$ and its LU factorization or solve (12.16) iteratively in parallel since $\underline{\underline{A}}_{\Sigma\Sigma}^{-1}$ is the sum of local inverses. Again, once \underline{w}_π has been obtained, we have:

$$\underline{w}_\Sigma = \underline{\underline{A}}_{\Sigma\Sigma}^{-1} (\underline{u}_\Delta - \underline{\underline{A}}_{\Sigma\pi} \underline{w}_\pi) \quad (12.17)$$

which completes the parallel computation of $\underline{w}_\Delta = (\underline{w}_\Sigma)_\Delta = \underline{\underline{S}}^{-1} \underline{u}_\Delta$.

B. Numerical Procedure

The uniformity of the formulas of Eqs. (12.6)–(12.9) allows the development of very robust codes, since the developments stem from the original matrix independently of the problem that motivated it. In this manner, for example, the same code was applied to treat 2D and 3D problems; the only routine of the code that had to be changed, when going from one class of problems to the other, was that defining the geometry and that is a very small part of it.

The numerical procedure requires the computation of the dual values on the internal boundary. Either the CGM (symmetric case) or GMRES (nonsymmetric case) algorithm is implemented [29]. During the evaluation of the algorithm, the application of \underline{S} , \underline{a} , and \underline{S}^{-1} are generally needed, which, as explained above, can be achieved in parallel through the assignment of different processors to distinct subdomains. In the calculations for \underline{S} and \underline{S}^{-1} , first the values at the primal variables are obtained by either of the approaches outlined in the previous section and then the dual values. It can be seen that for each subdomain α , the applications of the matrices:

$$\underline{A}^{\alpha}, \underline{A}_{II}^{(\alpha)-1}, \underline{A}_{\Sigma\Sigma}^{(\alpha)-1} \quad (12.18)$$

are the only calculations required. The ability to generate robust codes stems in part from the fact that these are the only main routines required for the subdomains.

C. Numerical Results

The problems implemented have the form:

$$-a\nabla^2 u + \underline{b} \cdot \nabla u + cu = f(x) \quad x \in \Omega \quad u = g(x) \quad x \in \partial\Omega \quad \Omega = \prod_{i=1}^d (\alpha_i, \beta_i) \quad (12.19)$$

where $a, c > 0$ are constants, while $\underline{b} = (b_1, \dots, b_{\dim})$ is a constant vector and $\dim = 1, 2, 3$. The family of subdomains $\{\Omega_1, \dots, \Omega_E\}$ is assumed to be a partition of the set $\Omega \equiv \{1, \dots, d\}$ of original nodes (this count does not include the nodes that lie on the external boundary). In the applications we present, d is equal to the number of degrees of freedom (dof), because we use linear functions and only one of them is associated with each original node (see, Table I).

The matrices treated were obtained by discretization of two cases, in two and three dimensions, of the above boundary value problem with $a = 1$. The choice $\underline{b} = (1, 1)$ or $\underline{b} = (1, 1, 1)$ with $c = 0$ yields a nonsymmetric matrix. Choosing $c = 1$ and $\underline{b} = 0$ a symmetric matrix was obtained that was also treated for comparison purposes. Discretization is accomplished using central finite differences and the original problem is then to solve:

$$\widehat{A} \cdot \widehat{u} = \widehat{f}. \quad (12.20)$$

In each domain, Ω_{α} , the local matrix $A_{(i,\alpha),(j,\alpha)}^{\alpha}$ is defined as in [19] as:

$$A_{(i,\alpha),(j,\alpha)}^{\alpha} = \frac{1}{m(i,j)} \widehat{A}_{ij} \quad (12.21)$$

TABLE I. Iteration table (2-D).

Vertices	Subdomains	dof	Primals	Symmetric case		Nonsymmetric case	
				N-N	FETI	N-N	FETI
2	4	9	1	2	1	2	1
4	16	225	9	7	7	9	8
6	36	1225	25	9	9	13	12
8	64	3969	49	10	10	15	14
10	100	9801	81	11	11	17	15
12	144	20,449	121	12	11	18	17
14	196	38,025	169	12	12	19	18
16	256	65,025	225	13	12	20	19
18	324	1,04,329	289	13	13	21	19
20	400	1,59,201	361	13	13	21	20
22	484	2,33,289	441	13	14	22	20
24	576	3,30,625	529	14	14	23	21
26	676	4,55,625	625	14	14	23	22
28	784	6,13,089	729	14	14	23	22
30	900	8,08,201	841	15	14	24	22

where $m(i, j)$ is the minimum of the multiplicities of i and j . The total matrix A^t then satisfies the criteria of (7.4) and (7.5):

$$A^t = \sum_{\alpha=1}^E A^\alpha$$

$$\widehat{w} \cdot \widehat{A} \cdot \widehat{u} = \tau(\widehat{w}) \cdot A^t \cdot \tau(\widehat{u}). \quad (12.22)$$

The DQGMRES algorithm [29] was implemented for the iterative solution of the nonsymmetric problems (12.19).

D. Numerical Tables

The results of the numerical experiments are displayed in two tables. Only the results for the preconditioned algorithms, Neumann–Neumann MF-DP, and preconditioned FETI-MF-DP, are presented.

In Table I, the number of iterations required by the MF-DP algorithms in 2D for convergence are reported. When the efficiency of such algorithms for nonsymmetric and symmetric matrices are compared, it is observed that they are of the same order. Indeed, for the Neumann–Neumann algorithm the number of iterations in the symmetric case is 62.5% of that required in the nonsymmetric case treated, whereas for the preconditioned FETI-MF-DP such a percentage is 64%.

In Table II, the results that were obtained for the 3D problems are reported. This table is organized in a similar fashion to that of Table I. Again, the efficiency of the MF-DP algorithms for the nonsymmetric and symmetric matrices treated are of the same order: 53% and 58% for Neumann–Neumann and preconditioned FETI, respectively.

XIII. CONCLUSIONS

The MF-DP approach to domain decomposition methods previously developed [16–19] has been successfully extended to nonsymmetric matrices. In the numerical experiments in 2D and 3D

TABLE II. Iteration table (3-D).

Vertices	Subdomains	dof	Primals	Symmetric case		Nonsymmetric case	
				N-N	FETI	N-N	FETI
2	8	27	7	3	2	3	2
8	27	512	80	5	4	7	6
27	64	3375	351	6	5	9	7
64	125	13,824	1024	7	6	11	9
125	216	42,875	2375	7	7	13	10
216	343	1,10,592	4752	8	7	14	11
343	512	2,50,047	8575	8	7	15	12
512	8019	5,12,000	14,336	8	8	16	13
8019	1000	9,70,299	22,599	8	8	16	14

carried out thus far, the MF-DP algorithms for nonsymmetric matrices exhibit an efficiency of the same order as state-of-the-art algorithms for symmetric matrices, such as BDDC, FETI-DP, and MF-DP (for symmetric matrices the number of iterations for convergence is between 53% and 62.5% of that required for nonsymmetric matrices).

The extension of the MF-DP approach to nonsymmetric matrices was accomplished by means of the general abstract scheme that yields the very broad class of preconditioned DDM algorithms of Section III; i.e., the new MF-DP algorithms for nonsymmetric matrices belongs to such a class. Another concept that is fundamental for developing the extension of the MF-DP approach to nonsymmetric matrices is that of “well posed everywhere,” introduced for the dual-primal matrix, in Section IX. To obtain the desired results, this concept replaces the assumption that the dual-primal matrix is positive definite, which is used when developing the theory for matrices possessing such a property. Indeed, it is easy to see that the dual-primal matrix is well posed everywhere, in the sense of Section IX, whenever it is positive definite. Thus, the concept of well posed everywhere is indeed a generalization of the concept of positive definite, in this respect; it is a key concept that permits applying the general scheme of Section 3 in the developments of the present article. Except for these concepts of Sections III and IX, the arguments of the theory for nonsymmetric matrices are very similar to those used when developing the theory of the MF-DP methods for symmetric matrices.

To finish, we recall that the algorithms for nonsymmetric matrices presented in this article share many of the properties enjoyed by their symmetric counterparts; namely:

- In the case of nonsymmetric matrices, the numerical efficiency of the preconditioned algorithms is of the same order as state-of-the-art DDMs algorithms for symmetric matrices. We are not aware of other algorithms for nonsymmetric matrices with this property. Furthermore, their computational properties are very good.
- The unifying, explicit matrix formulas given in Eqs. (1.1) and (1.2), possess several attractive features worth noticing, among them: once the original matrix is given, they are uniquely determined and are equally applicable to a single linear differential equation or to a system of such equations.
- Code development is simplified.
- Very robust codes are obtained; for example, a code has been developed that has been applied in 2D and 3D problems (such a code was used to obtain the numerical results reported in Section XII of this article), something that is not possible when standard approaches are used.
- The MF-DP algorithms are 100% parallelizable, as it is shown in Section XII.

We express our gratitude to Antonio Carrillo L. for valuable assistance when performing the parallel computations of the numerical examples.

References

1. Ch. Farhat and F. Roux, A method of finite element tearing and interconnecting and its parallel solution algorithm, *Int J Numer Methods Eng* 32 (1991), 1205–1227.
2. J. Mandel and R. Tezaur, Convergence of a substructuring method with Lagrange multipliers, *Numer Math* 73 (1996), 473–487.
3. C. Farhat, M. Lesoinne, P. LeTallec, K. Pierson, and D. Rixen, FETI–DP a dual-primal unified FETI method, Part I: a faster alternative to the two-level FETI method, *Int J Numer Methods Eng* 50 (2001), 1523–1544.
4. J. Mandel and R. Tezaur, On the convergence of a dual-primal substructuring method, *Numer Math* 88 (2001), 543–558.
5. J. Mandel, Balancing domain decomposition, *Commun Numer Methods Eng* 1 (1993), 233–241.
6. J. Mandel and M. Brezina, Balancing domain decomposition for problems with large jumps in coefficients, *Math Comput* 65 (1996), 1387–1401.
7. Y. H. De Roeck and P. LeTallec, Analysis and test of a local domain decomposition preconditioner, R. Glowinski, Y. Kuznetsov, G. Meurant, J. Périaux, and O. Widlund, editors, *Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, PA, 1991.
8. J. F. Bougat, R. Glowinski, P. LeTallec, and M. Vidrascu, Variational formulation and algorithm for trace operator in domain decomposition calculations, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, editors, *Second International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, PA, 1989.
9. R. Glowinski and M. F. Wheeler, Domain decomposition and mixed finite element methods for elliptic problems, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, editors, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, SIAM, Philadelphia, PA, 1988.
10. C. Dohrman, A preconditioner for substructuring based on constrained energy minimization, *SIAM J Sci Comput* 25 (2003), 246–258.
11. J. Mandel and C. Dohrmann, Convergence of a balancing domain decomposition by constraints and energy minimization, *Numer Linear Algebra Appl* 10 (2003), 639–659.
12. A. Toselli and O. Widlund, *Domain decomposition methods—algorithms and theory*, Springer series in computational mathematics, Springer-Verlag, Berlin, 2005, p. 450.
13. J. Li and O. Widlund, FETI-DP, BDDC, and block Cholesky methods, *Int J Numer Methods Eng* 66 (2006), 250–271.
14. J. Mandel, C. Dormann, and R. Tezaur, An algebraic theory for primal and dual substructuring methods by constraints, *Appl Numer Math* 54 (2005), 167–193.
15. A. Quarteroni and A. Valli, *Domain decomposition methods for partial differential equations*, Numerical mathematics and scientific computation, Oxford Science Publications, Clarendon Press-Oxford, 1999.
16. I. Herrera, Theory of differential equations in discontinuous piecewise-defined-functions, *Numer Methods Partial Differential Eq* 23 (2007), 597–639.
17. I. Herrera, New formulation of iterative substructuring methods without Lagrange multipliers: Neumann–Neumann and FETI, *Numer Methods Partial Differential Eq* 24 (2008), 845–878.
18. I. Herrera and R. Yates, Unified multipliers-free theory of dual primal domain decomposition methods, *Numer Methods Partial Differential Eq* 25 (2009), 552–581.

19. I. Herrera and R. A. Yates, The multipliers-free domain decomposition methods, *Numer Methods Partial Differential Eq* 26 (2010), 874–905.
20. DDM Organization, Proceedings of 19th International Conferences on Domain Decomposition Methods (1988–2009). Available at www.ddm.org.
21. I. Herrera, Unified approach to numerical methods. Part 1. Green's formulas for operators in discontinuous fields, *J Numer Methods Partial Differential Eq* 1 (1985), 12–37.
22. I. Herrera, Unified approach to numerical methods. Part 2. Finite elements, boundary methods and its coupling, *J Numer Methods Partial Differential Eq* 1 (1985), 159–186.
23. I. Herrera, L. Chargo, and G. y Alduncin, Unified approach to numerical methods. Part 3. finite differences and ordinary differential equations, *J Numer Methods Partial Differential Eq* 1 (1985), 241–258.
24. I. Herrera, Boundary methods. An algebraic theory, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1984.
25. I. Herrera, Localized adjoint method: A new discretization methodology, W. E. Fitzgibbon and M. F. Wheeler, editors, *Computational methods in geosciences*, SIAM, 1990, pp. 45–51.
26. I. Herrera, R. E. Ewing, M. A. Celia, and T. y Russell, Eulerian-Lagrangian localized adjoint method: the theoretical framework, *J Numer Methods Partial Differential Eq* 9 (1993), 431–457.
27. J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. 1, Springer-Verlag, New York, Heidelberg, Berlin, 1972, p. 357.
28. S. Brenner and L. Sung, BDDC and FETI-DP without matrices or vectors, *Comput Methods Appl Mech Eng* 196 (2007), 1429–1435.
29. Y. Saad, Iterative methods for sparse linear systems, 2000, p. 447.