The Multipliers-Free Dual-Primal Domain Decomposition Methods for Symmetric and Non-symmetric Matrices: An Overview

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Summary. Non-overlapping domain decomposition methods are usually treated as a constrained optimization problem. When research in domain decomposition methods begun, this artifice no doubt was a natural and very convenient approach since it avoided dealing with the complications one has to face when a formulation in fully-discontinuous functions is introduced. However, this second option, of formulating such methods in a direct manner in spaces of discontinuous functions had remained unexplored. Recently, work to fill this gap has been carried out and its results show that the standard approach is not free of shortcomings. Indeed, the methodology obtained when a more direct approach is followed –the multipliers-free method- exhibit many advantages; among them, greater conceptual unity, simpler and more robust codes, and better parallelization properties. The present paper is part of the Minisymposium devoted to present the results of this line of research and in it an overview of this new approach is given, including its extension to non-symmetric matrices.

1 Introduction

Nowadays it is standard to treat preconditioned non-overlapping domain decomposition methods with recourse to Lagrange-multipliers using a saddle point formulation [1] [2]. This artifice, no doubt, has been quite valuable since it has permitted dealing with a large variety of problems; however, it is an indirect approach that bypasses many details of the problems considered and it is not free of shortcomings, as it is here discussed.

The Neumann-Neumann and preconditioned-FETI, and similar algorithms, lead directly to incorporate discontinuous functions in their formulations. Indeed, Neumann problems formulated in each one of the subdomains of a domain decomposition, in such a manner that the normal derivatives are continuous across the internal boundary, yield solutions that are discontinuous there [3]. Therefore, two approaches are feasible when formulating them:

1. Avoid the introduction of discontinuous functions, keeping the treatment within spaces of continuous functions, and deal with the problem as one of constrained optimization by means of a Lagrange multipliers formulation; or
2. Enlarge the functions-space considered to one whose members are generally discontinuous and contains the continuous-functions-space as a linear subspace.

When research on non-overlapping methods began, the first one of the above options was followed. At that stage, this no doubt was a natural and very convenient approach since it avoided dealing with the complications one has to face when a fully-discontinuous formulation is introduced. However, the second option had remained unexplored in spite of the many years elapsed since DDM research began.

Recently, work to fill this gap has been carried out [4]-[8] and the purpose of this article and of the whole mini-symposium is to summarize its results. The general methodology is here explained while numerical and computational issues are discussed in a parallel paper of these Proceedings [9]. The results of this line of research have shown that this more direct approach indeed yields many advantages. Among them:

1. A framework applicable to symmetric and non-symmetric matrices. As a matter of fact, it yields domain decomposition methods that work nearly as efficiently for non-symmetric matrices as they do for symmetric ones;
2. Explicit matrix formulas that unify the different methods. Indeed the very same formulas are applicable in both the symmetric and non-symmetric cases;
3. Simplified code development;
4. Very robust codes. For example, a code has been developed that can be applied in 2-D and 3-D problems [8], something that is not possible when other approaches are used;
5. 100% parallelizable algorithms;
6. In the case of symmetric matrices, the numerical efficiency of the preconditioned algorithms is at least as good as other state-of-the-art DDMs. However, the computational properties of the multipliers-free DDMs are superior;
7. For non-symmetric matrices there is little to compare with, since the literature on DDMs for non-symmetric matrices is relatively limited (see [2],[10]-[14], for some background material on this subject). However, as already mentioned, the multipliers-free methods work nearly as efficiently for non-symmetric matrices as they do for symmetric ones; and
8. The procedures are equally applicable to a single differential equation or to a system of such equations, when they are linear.

It is also worth mentioning that the *jump matrix*, $\underline{\underline{j}}$, introduced in previous papers, is probably the optimal choice of the matrix $\underline{\underline{B}}$ used to specify the continuity constrain in standard formulations [6][7].
The non-overlapping domain decomposition methods are summarized in four formulas that apply to both symmetric and non-symmetric matrices [7][8]. The non-preconditioned algorithms are given by

$$\begin{cases} \underline{\underline{aS}}\,\underline{u} = \underline{f}_{\Delta 2} \text{ and } \underline{\underline{j}}\,\underline{u} = 0; \ \textit{Schur Complement} \\ \underline{\underline{S}}^{-1}\underline{\underline{j}}\,\underline{u} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \text{ and } \underline{\underline{aS}}\,\underline{u} = 0; \ \textit{FETI} \end{cases} \quad (1)$$

While the preconditioned algorithms are given by

$$\begin{cases} \underline{\underline{aS}}^{-1}\underline{\underline{aS}}\,\underline{\underline{j}}\,\underline{u}_{\Delta} = \underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2}; \ \textit{Neumann} - \textit{Neumann} \\ \underline{\underline{S}}^{-1}\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{u} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\,\underline{\underline{S}}\,\underline{\underline{j}}\,\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2}; \ \textit{FETI} \end{cases} \quad (2)$$

Above, the matrix $\underline{\underline{S}}$ is the *dual-primal Schur-complement matrix* defined in [7]. These formulas hold when $\underline{\underline{S}}$ is positive definite; otherwise $\underline{\underline{S}}$ must be replaced by $\underline{\underline{M}}$, a generalization of the *dual-primal Schur-complement matrix* [7] that, for brevity, will not be discussed here. Similar formulas hold for the Robin methods.
Although the use of matrix notation in the above formulas is useful because it permits expressing them in a very compact and synthetic form, the computation of such matrices is not required when applying them; indeed, it is only their actions on vectors that needs to be computed and this can be done 100% in parallel, as is shown in a parallel paper of this Mini-symposium [9] (see also [7]. It should be mentioned that Eqs.1 and 2 are uniquely defined once the original matrix, which corresponds to the original continuous problem, is specified. This is an important property because it permits organizing the computations in a manner that is independent of the problem considered; due to this fact, the codes so obtained are extremely robust when they are properly developed.

2 The Multipliers-free Formulation

Starting from the matrix formulation of the *'original'* continuous problem, let $\Omega \equiv \{1, ..., N\}$ be the set of *original nodes*, excluding those that lie on the external boundary of the domain of problem definition. Given a non-disjoint domain decomposition, let $\{\Omega_1, ..., \Omega_E\}$ with $\Omega_\alpha \subset \Omega$ be the corresponding subsets of *original nodes* lying in each subdomain. Generally the same original node may be in more than one subset Ω_α and the multiplicity $m(p)$ of an *original node*, p, is the number of different subdomains Ω_α in which p occurs. The set of *'derived nodes'* is defined to be $\bar{\Omega} \equiv \left\{ \underline{p} = (p, \alpha) \middle| \, p \in \Omega_\alpha \right\}$. Both the real-valued functions defined in Ω, as well as those defined in $\bar{\Omega}$, constitute vector spaces to be denoted by $\tilde{D}(\Omega)$ and $\tilde{D}(\bar{\Omega})$, respectively. A vector $\underline{\tilde{u}} \in$

$\tilde{D}(\bar{\Omega})$ such that $\underline{\tilde{u}}(p,\alpha)$ is independent of α is said to be *continuous* and we write $\tilde{D}_{12}(\bar{\Omega})$ for the set of such vectors. A *natural immersion*, $\tau : \tilde{D}(\Omega) \to \tilde{D}(\bar{\Omega})$, such that $\tau\tilde{D}(\Omega) = \tilde{D}_{12}(\bar{\Omega})$ (see [7]) is defined. If we identify $\tilde{D}(\Omega)$ with its *natural immersion*, then we have that

$$\tilde{D}(\Omega) = \tilde{D}_{12}(\bar{\Omega}) \subset \tilde{D}(\bar{\Omega}) \tag{3}$$

The vector-space $\tilde{D}(\bar{\Omega})$ becomes a finite-dimensional Hilbert-space when the *Euclidean inner product* is introduced (see [7]). The *'average matrix'*, $\underline{\underline{a}}{:}\tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$, is defined to be the orthogonal-projection on $\tilde{D}_{12}(\bar{\Omega})$, with respect to the Euclidean inner-product, while the *jump matrix* $\underline{\underline{j}} \equiv \underline{\underline{I}} - \underline{\underline{a}}$ is the orthogonal projection on the *zero-average subspace*, $\tilde{D}_{11}(\bar{\Omega})$, which is the orthogonal complement of $\tilde{D}_{12}(\bar{\Omega})$. A node (p,α) is *'internal'* when $m(p) = 1$ and it is a *'boundary node'* when $m(p) > 1$; the sets of *internal* and *boundary nodes* are represented by I and Γ, respectively. Observe that $\bar{\Omega} = I \cup \Gamma$.

The *'original problem'* is: "given $\underline{\widehat{f}} \in \tilde{D}(\Omega)$, find a $\underline{\widehat{u}} \in \tilde{D}(\Omega)$ such $\underline{\underline{\widehat{A}}}\underline{\widehat{u}} = \underline{\widehat{f}}$, $\underline{\widehat{f}} \in \tilde{D}(\Omega)$"

Here, $\underline{\underline{\widehat{A}}}{:}\tilde{D}(\Omega) \to \tilde{D}(\Omega)$ is the *'original matrix'* of the discrete formulation of the problem that has been obtained by FEM or some other discretization procedure. Furthermore, we choose a set of *'primal nodes'*, $\pi \subset \Gamma \subset \bar{\Omega}$, and define the *'dual-primal vector subspace'*, $\tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}(\bar{\Omega})$, by requiring that its vector-members be continuous in the set π of *primal nodes*. It can be seen that $\tilde{D}_{12}(\bar{\Omega}) \subset \tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}(\bar{\Omega})$. We define $\Pi \equiv \mathrm{I} \cup \pi$ and $\Delta \equiv \bar{\Omega} - \Pi$, so that

$$\begin{cases} \bar{\Omega} = \Delta \cup \Pi \\ \Delta \cap \Pi = \emptyset \end{cases} \tag{4}$$

In order to formulate the original problem in the extended space of vectors, $\tilde{D}(\bar{\Omega})$, without recourse to Lagrange multipliers, several matrices are introduced [7]. For each $\alpha \in \{1,...,E\}$ a matrix $\underline{\underline{A}}^{\alpha} : \tilde{D}(\bar{\Omega}_{\alpha}) \to \tilde{D}(\bar{\Omega}_{\alpha})$ is defined and in turn

$$\underline{\underline{A}}^{t} \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}^{\alpha} \tag{5}$$

This matrix, has the property that

$$\underline{\underline{m}}^{-1}\underline{\underline{A}}^{t}\underline{u} = \underline{\underline{\widehat{A}}}\tau^{-1}(\underline{u}),\ \forall \underline{u} \in \tilde{D}_{12}(\bar{\Omega}) \tag{6}$$

Here, $\underline{\underline{m}}$ is an auxiliary diagonal-matrix whose diagonal values are the nodes-multiplicities. The projection matrix on $\tilde{D}^{DP}(\bar{\Omega})$ is $\underline{\underline{a}}^{\pi} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ and $\underline{\underline{A}} : \tilde{D}^{DP}(\bar{\Omega}) \to \tilde{D}^{DP}(\bar{\Omega})$ is defined by

$$\underline{\underline{A}} \equiv \underline{\underline{a}}^{\pi} \underline{\underline{A}}^{t} \underline{\underline{a}}^{\pi} \tag{7}$$

Procedures for constructing such matrices are given in [7], [8]. Then, a vector $\widehat{\underline{u}} \in \tilde{D}(\Omega)$ is solution of the *original problem* of Eq.2, if and only if, $\underline{u} \equiv \tau \widehat{\underline{u}}$ satisfies

$$\underline{\underline{a}}\underline{\underline{A}}\underline{u} = \underline{\underline{m}}^{-1} \widehat{\underline{f}} \equiv \bar{\underline{f}} \ \ and \ \ \underline{\underline{j}}\underline{u} = 0 \tag{8}$$

Here, $\underline{f}_{\Delta 2} \equiv \left(\bar{\underline{f}}_{\Delta 2} - \underline{\underline{a}}\underline{\underline{A}}_{\Delta \Pi} \underline{\underline{A}}_{\Pi \Pi}^{-1} \bar{\underline{f}}_{\Pi} \right) \in \tilde{D}_{12}(\bar{\Omega})$.

3 Preconditioned and non-preconditioned Formulations

Some notations are introduced:

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi \Pi} \underline{\underline{A}}_{\Pi \Delta} \\ \underline{\underline{A}}_{\Delta \Pi} \underline{\underline{A}}_{\Delta \Delta} \end{pmatrix}, \underline{\underline{L}} \equiv \begin{pmatrix} \underline{\underline{A}}_{\Pi \Pi} \underline{\underline{A}}_{\Pi \Delta} \\ 0 \quad 0 \end{pmatrix} \ \ and \ \underline{\underline{R}} \equiv \begin{pmatrix} 0 \quad 0 \\ \underline{\underline{A}}_{\Delta \Pi} \underline{\underline{A}}_{\Delta \Delta} \end{pmatrix} \tag{9}$$

together with the *dual-primal Schur complement matrix*, $\underline{\underline{S}} : \tilde{D}(\Delta) \to \tilde{D}(\Delta)$:

$$\underline{\underline{S}} \equiv \underline{\underline{A}}_{\Delta \Delta} - \underline{\underline{A}}_{\Delta \Pi} \underline{\underline{A}}_{\Pi \Pi}^{-1} \underline{\underline{A}}_{\Pi \Delta} \tag{10}$$

In the above

$$\begin{cases} \underline{\underline{A}}_{\Pi \Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Pi), & \underline{\underline{A}}_{\Pi \Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Pi) \\ \underline{\underline{A}}_{\Delta \Pi} : \tilde{D}(\Pi) \to \tilde{D}(\Delta), & \underline{\underline{A}}_{\Delta \Delta} : \tilde{D}(\Delta) \to \tilde{D}(\Delta) \end{cases} \tag{11}$$

Then $\underline{u} \equiv \underline{u}_{\Delta} + \underline{u}_{\Pi}$ satisfies the formulation of Eq.8, if and only if,

$$\underline{u}_{\Pi} = -\underline{\underline{A}}_{\Pi \Pi}^{-1} \underline{\underline{A}}_{\Pi \Delta} \underline{u}_{\Delta} \tag{12}$$

while $\underline{u}_{\Delta} \in \tilde{D}(\Delta)$ fulfills any one of the following formulations:

$$\underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} = \underline{f}_{\Delta 2} \ and \ \underline{\underline{j}}\underline{u}_{\Delta} = 0, \ \text{The Schur} - \text{complement} \tag{13}$$

$$\underline{\underline{a}}\underline{\underline{S}}^{-1} \underline{\underline{a}}\underline{\underline{S}}\underline{u}_{\Delta} = \underline{\underline{a}}\underline{\underline{S}}^{-1} \underline{f}_{\Delta 2} \ and \ \underline{\underline{j}}\underline{u}_{\Delta} = 0; \ \textit{The Neumann} - \textit{Neumann} \tag{14}$$

$$\underline{\underline{S}}^{-1} \underline{\underline{j}}\underline{u} = -\underline{\underline{S}}^{-1} \underline{\underline{j}}\underline{\underline{S}}^{-1} \underline{f}_{\Delta 2} \ and \ \underline{\underline{a}}\underline{\underline{S}}\underline{u} = 0; \ \textit{The non} - \textit{preconditioned FETI} \tag{15}$$

$$\underline{\underline{S}}^{-1} \underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{u} = -\underline{\underline{S}}^{-1} \underline{\underline{j}}\underline{\underline{S}}\underline{\underline{j}}\underline{\underline{S}}^{-1} \underline{f}_{\Delta 2} \ and \ \underline{\underline{a}}\underline{\underline{S}}\underline{u} = 0; \ \textit{The preconditioned FETI} \tag{16}$$

4 Conclusions

As explained in the Introduction, there are two possible approaches to the treatment of preconditioned non-overlapping domain decomposition methods:

1. Avoid the introduction of discontinuous functions, keeping within spaces of continuous functions, and treat the problem as one of constrained optimization by means of a Lagrange multipliers formulation; or
2. Enlarge the functions-space considered to one whose members are generally discontinuous and contains the continuous-functions-space as a linear subspace.

Of these two options, the second one had remained unexplored. In the present article, a contribution to fill this gap, referred to as the Multipliers-free Domain Decomposition Method, is presented. This methodology exhibits many advantages of this more direct approach, which were described in the Introduction Section of this paper.
The applicability of the theory is considerably wide, but not unlimited since as any theory it can be applied only when the assumptions on which it is based are fulfilled. In this respect, when the matrices are symmetric such assumptions are essentially the same as those of standard approaches. In the case of non-symmetric matrices they are also natural and in the numerical examples treated the procedures introduced in this paper have worked very satisfactorily. It must be mentioned that indefinite problems yield indefinite matrices and then, instead of CGM, iterative methods for non-symmetric matrices have to be used (see, for example, [15]).

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