Parallel Algorithms for Elastic Systems using Multipliers-Free Domain Decomposition Methods

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Summary. This is the third article of the Minisymposium "The Multipliers-free Domain Decomposition Methods for Symmetric and Non-symmetric Matrices". In it, we apply the Multipliers-Free Domain Decomposition Method (MF-DDM) to isotropic problems of Static Elasticity. The purpose of the paper is three-fold: to exhibit the applicability of such methods to systems of equations, confirm some of the advantages that are concomitant to MF-DDMs and to establish simple procedures for developing effective codes for parallel-processing problems of elasticity. Firstly, by means a FEM formulation for Static Elasticity we derive the system-matrix associated with this kind of problems. Once this matrix is available, code development for parallel processing only requires a straightforward application of the MF-DDM matrix formulas that unify non-overlapping DDMs and were presented in paper 1 of this Minisymposium: "An Overview". The general procedures that were explained in Paper 2, "Implementation Issues", are then applied for developing the parallel-processing codes. Such procedures can be applied straightforwardly whenever the matrix-system is available. In conclusion, this paper supplies a simple manner of developing efficient parallel codes for static elasticity, thereby demonstrating the applicability of MF-DDMs to systems of partial differential equations, and also corroborates in this particular case some of the many advantages that are concomitant to MF-DDMs.

1 Introduction

This is the third article of the Minisymposium that in the framework of DD19 was devoted to present and discuss a new methodology known as the *"Multipliers-free domain decomposition methods: MF-DDMs"*, which is equally applicable to symmetric and non-symmetric matrices. Such methodology is based on a direct approach, without recourse to Lagrange multipliers, boundary conditions. Then, a weak formulation is obtained weighting Eq.2 with a vector-valued function $\underline{\psi}$ and integrating by parts:

$$A(\overleftrightarrow{\underline{u}}, \underline{\psi}) = \int_{\overleftrightarrow{\Omega}} \{(\lambda+\mu)(\nabla \bullet \underline{\psi})(\nabla \cdot \overleftrightarrow{\underline{u}}) + \mu \nabla \underline{\psi} : \nabla \overleftrightarrow{\underline{u}}\} d\underline{x} = \int_{\overleftrightarrow{\Omega}} \underline{\psi} \bullet \underline{f}_{\Omega} d\underline{x} \quad (3)$$

Next, a partition of the problem-domain is introduced, whose internal nodes are $\underline{x}_p$, $p = 1, ..., N$. Sometimes this partition will be referred to as the *'fine partition'* because later one more partition, the *'coarse partition'*, will be introduced. With each node $\underline{x}_p$ we associate a $3D$-vector-valued test function to be denoted by $\underline{\psi}^{pi}$, $i = 1, 2, 3$. Furthermore, let $\left(\underline{\psi}^{pi}\right)_j$ be the $j - th$ component of $\underline{\psi}^{pi}$. Then, we choose

$$\left(\underline{\psi}^{pi}(\underline{x})\right)_j = \ell_p(\underline{x})\,\delta_{ij} \quad (4)$$

Here, $\ell_p(\underline{x})$ is the Lagrange linear interpolate that is characterized by being a piecewise-linear scalar function with the property that

$$\ell_p\left(\underline{x}_q\right) = \delta_{pq}, \; p, q = 1, ..., N \quad (5)$$

We observe that Eq.5 implies that all the test functions vanish at the nodes located on the domain-external-boundary, $\partial \overleftrightarrow{\Omega}$.
The set of base functions is taken to be the same as the set of test functions. Thus, we define the approximate solution of our boundary-value problem to be:

$$\overleftrightarrow{\underline{u}}(\underline{x}) \equiv \sum_{p=1}^{N} \sum_{i=1}^{3} \widehat{u}_{pi}\, \underline{\psi}^{pi}(\underline{x}) \quad (6)$$

We also define:

$$\widehat{f}_{pi} \equiv \int_{\overleftrightarrow{\Omega}} \underline{\psi}^{pi} \bullet \underline{f}_{\Omega} d\underline{x}, \; p = 1, ..., N \text{ and } i = 1, 2, 3 \quad (7)$$

Eqs.6 and 3 together imply that

$$\sum_{q=1}^{N} \sum_{j=1}^{3} \widehat{A}_{qj,pi} \widehat{u}_{qj} = \widehat{f}_{pi} \quad (8)$$

Here:

$$\widehat{A}_{qj,pi} \equiv \int_{\overleftrightarrow{\Omega}} \{(\lambda+\mu)(\nabla \cdot \underline{\psi}^{qj})(\nabla \cdot \underline{\psi}^{pi}) + \mu \nabla \underline{\psi}^{qj} : \nabla \underline{\psi}^{pi}\} d\underline{x} \quad (9)$$

Using Eq.4, it is seen that

$$\underline{u} \bullet \underline{w} \equiv \sum_{\alpha \in Z(p)} \sum_{p=1}^{N} \sum_{i=1}^{3} \frac{u(p,i,\alpha)\, w(p,i,\alpha)}{m(p)}, \ \forall \underline{u}, \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{14}$$

Here $Z(p) \subset \{1, ..., E\}$ is defined by the condition that $(p, \alpha) \in \bar{\Omega}$ when $\alpha \in Z(p)$. When $\widehat{\underline{u}}, \widehat{\underline{w}} \in \tilde{D}(\Omega)$ it can be seen that

$$\widehat{\underline{u}} \bullet \widehat{\underline{w}} = \tau\left(\widehat{\underline{u}}\right) \bullet \tau\left(\widehat{\underline{w}}\right) \tag{15}$$

The *average matrix*, $\underline{\underline{a}}$, is the orthogonal projection, with respect to the Euclidean Inner product, on the subspace $\tilde{D}_{12}(\bar{\Omega})$ of continuous vectors. Its explicit expression is

$$a_{(p,i,\alpha)(q,j,\beta)} = \frac{1}{m(p)} \delta_{pq} \delta_{ij} \tag{16}$$

while the *jump matrix*, is $\underline{\underline{j}} \equiv \underline{\underline{I}} - \underline{\underline{a}}$. Here, $\underline{\underline{I}}$ is the identity matrix.

4 The matrices $\underline{\underline{A}}^t$ and $\underline{\underline{A}}$

For $\gamma = 1, ..., E$, we define the linear transformations $\underline{\underline{A}}^{\gamma} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ by

$$A^{\gamma}_{(q,j,\beta)(p,i,\alpha)} \equiv \delta_{\alpha\gamma} \delta_{\beta\gamma} \int_{\vec{\Omega}_{\gamma}} \left\{ (\lambda + \mu) \frac{\partial \ell_p}{\partial x_i} \frac{\partial \ell_q}{\partial x_j} + \mu \sum_{r=1}^{3} \frac{\partial \ell_p}{\partial x_r} \frac{\partial \ell_q}{\partial x_r} \delta_{ij} \right\} d\underline{x} \tag{17}$$

Furthermore, $\underline{\underline{A}}^t : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ is defined by $\underline{\underline{A}}^t \equiv \sum_{\alpha=1}^{E} \underline{\underline{A}}^{\alpha}$. Next, we define the matrix $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$. To this end we choose a set $\Omega^{\pi} \subset \Omega^{\Gamma} \subset \Omega$ and define the set of *'primal nodes'*, $\pi \subset \Gamma \subset \bar{\Omega}$, to be defined by the condition that a *derived node*, (p, α), belongs to π when $p \in \Omega^{\pi}$. Then, the *'dual-primal vector subspace'*, $\tilde{D}^{DP}(\bar{\Omega}) \subset \tilde{D}(\bar{\Omega})$, is defined by requiring that its vector-members be continuous in the set π of *primal nodes*. Then, the transformation $\underline{\underline{a}}^{\pi} : \tilde{D}(\bar{\Omega}) \rightarrow \tilde{D}(\bar{\Omega})$ is defined to be the projection of $\tilde{D}(\bar{\Omega})$ on $\tilde{D}^{DP}(\bar{\Omega})$. It can be seen that [ ]:

$$a^{\pi}_{(p,i,\alpha)(q,j,\beta)} = \left\{ \frac{1}{m(p)} \delta_{pq} \delta^{\pi}_{pq} + \delta_{\alpha\beta} \delta_{pq} \left(1 - \delta^{\pi}_{pq}\right) \right\} \delta_{ij} \tag{18}$$

Here, the symbol δ^{π}_{pq} defined by

$$\delta^{\pi}_{pq} \equiv \begin{cases} 1, & \text{if } p, q \in \Omega^{\pi} \\ 0, & \text{if } p \text{ or } q \notin \Omega^{\pi} \end{cases} \tag{19}$$

Then, the matrix $\underline{\underline{A}}$ is defined by $\underline{\underline{A}} \equiv \underline{\underline{a}}^{\pi} \underline{\underline{A}} \underline{\underline{a}}^{\pi}$. It will be assumed in what follows that π, the set of *primal nodes*, is such that $\underline{\underline{A}} : \tilde{D}^{DP}(\bar{\Omega}) \rightarrow \tilde{D}^{DP}(\bar{\Omega})$

$$\textit{The Neumann} - \textit{Neumann method}: \underline{\underline{aS}}^{-1}\underline{\underline{aS}}\underline{u}_{\Delta} = \underline{\underline{aS}}^{-1}\underline{f}_{\Delta 2} \textit{ and } \underline{\underline{j}}\underline{u}_{\Delta} = 0 \tag{28}$$

$$\textit{The non} - \textit{preconditioned FETI}: \underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{u}_{\Delta} = -\underline{\underline{S}}^{-1}\underline{\underline{j}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \textit{ and } \underline{\underline{aS}}\underline{u}_{\Delta} = 0 \tag{29}$$

$$\textit{The preconditioned FETI}: \underline{\underline{S}}^{-1}\underline{\underline{jSj}}\underline{u}_{\Delta} = -\underline{\underline{S}}^{-1}\underline{\underline{jSj}}\underline{\underline{S}}^{-1}\underline{f}_{\Delta 2} \textit{ and } \underline{\underline{aS}}\underline{u}_{\Delta} = 0 \tag{30}$$

To implement these algorithms in parallel, we simply apply the procedures of paper 2, on *implementation issues.*

5 Conclusions

By means a FEM formulation for Static Elasticity we derived the matrix system corresponding to it and, by a straightforward application of the MF-DDM, a simple procedure for developing fully parallelizable computational codes for such problems was obtained. The computational codes so derived are very robust; in particular, using object-oriented programming techniques codes that are applicable to anisotropic materials are easily constructed. Thereby, this paper confirms the applicability of the general MF-DDM matrix formulas and procedures to systems of partial differential equations, as well as other attractive features of the MF-DDMs.

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