UNIFIED THEORY OF DIFFERENTIAL OPERATORS ACTING ON DISCONTINUOUS FUNCTIONS AND OF MATRICES ACTING ON DISCONTINUOUS VECTORS

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Abstract

For the application of discontinuous functions in the area of numerical methods for partial differential equations (NMPDE), there are mainly two approaches: Trefftz methods and discontinuous Galerkin (dG) methods. The Theory of Differential Equations in Discontinuous Piecewise-Defined-Functions, introduced and developed by Herrera [5,6], constitutes a unified framework for these procedures. On the other hand, nowadays the application of high performance computing to the solution of PDEs is progressing at a very swift pace. Among the new computational resources parallel computing is outstanding. In turn, the most effective means for applying parallel computing are domain decomposition methods (DDM). Since the 1980s parallel computing has received considerable attention by the NMPDE community and at present it is recognized that non-overlapping DDMs are most effective. Most of the work done up to recently for this latter kind of methods had been restricted mainly to symmetric and positive definite problems. However, recently Herrera [5-11] has introduced a new formulation in which symmetric and nonsymmetrical problems are handled in a unified manner, thereby producing a systematic non-overlapping and preconditioned DDM for non-symmetric matrices. These procedures are carried out in vector-spaces whose elements are discontinuous, using a unified theory of differential operators acting on discontinuous functions and of matrices acting on discontinuous vectors, to which we devote this paper.

Keywords: Trefftz method, discontinuous Galerkin; domain decomposition methods; Steklov-Poincaré operator; multipliers-free DDM; Lagrange multipliers.

1.- INTRODUCTION

Nowadays the application of high performance computing to the solution of PDEs is progressing at a very swift pace. Among the new computational resources parallel computing is outstanding. In turn, the most effective means for applying parallel computing are domain decomposition methods (DDM). Since the 1980s such methods have received considerable attention by the NMPDE community and at present it is recognized that non-overlapping DDMs are the most effective.

On the other hand, many ideas of *domain-decomposition-methods (DDMs)*, when they are approached after discretization, have been inspired by concepts stemming from DDMs formulations of partial differential equations before discretization and correspondingly there have been efforts by many researchers and scholars to establish links between the concepts of differential equations and the discrete systems derived from them [1,2]. However, the parallelism between discrete-matrix theories and partial differential equation (PDE) theories had been up to recently considerably limited. Perhaps the most important concept that originated in this manner is the Steklov-Poincaré operator at the discrete level.

A shortcoming, which is apparent when one examines the efforts that had been done up to now, is that non-overlapping DDM formulations have been usually compared with PDE formulations in continuous functions spaces. Taking into account that a very important goal sought by non-overlapping DDM formulations is disconnecting the partition subdomains from each other, in order to achieve independence in their processing, this does not seem to be the most appropriate PDE formulation to be used for such a comparison. Indeed, it would be better to use a PDE formulation in discontinuous functions.

The best known theories of PDE in discontinuous functions are *discontinuous Galerkin* (dG) and *Trefftz methods*. However, standard theories of dG methods do not address such formulations in a direct manner [3,4]. As for *Trefftz methods'* theoretical framework, -the *"General theory of differential equations in discontinuous piecewise-defined-functions"*, due to Herrera [5, 6]- it is based on a direct approach that is developed without resource to Lagrange multipliers and which is better suited for our purposes. Also, this latter theory is equally applicable to symmetric and non-symmetric problems.

Although in this article we draw extensively from previous work [5-11], the main goal here is to extend the *"General theory of differential equations in discontinuous piecewise-defined-functions"* to make it applicable to the algebraic systems that are

obtained after the PDEs have been discretized. Key elements in these developments are *Green-Herrera formulas*, which were introduced in 1985 [12, 13]. Among the results obtained in this manner an improved definition of the *Steklov-Poincaré* operator has especial interest.

The theory here presented, just as the theory from which it derives, is equally applicable to symmetric and non-symmetric matrices. Therefore, it constitutes a systematic non-overlapping and preconditioned domain decomposition methodology for non-symmetric matrices. The most commonly used domain decomposition methods are different versions of the *finite-element tearing and interconnecting (FETI)* and *the balancing domain decomposition (BDD)* [14-19]. It is generally accepted that FETI formulations correspond to discrete formulations of Dirichlet problems [2], while BDD formulations correspond to discrete formulations of Neumann problems. These methods were originally developed for symmetric positive problems, but some extensions for non-symmetric matrices have been recently developed [20-22].

The results reported in the present paper supply a new and systematic manner of interpreting discrete formulations as *boundary-value* problems. In particular, the improved definition of the Steklov-Poincaré operator at the discrete level (first given in [9,10]) implies that, contrary to what is generally believed, FETI is also a formulation a Neumann problem. In summary, FETI is a formulation of a Neumann problem using Lagrange multipliers while BDD is also a formulation of a Neumann problem, but without resource to Lagrange multipliers. A well-known and celebrated result by Mandel, Dohrmann and Tezaur [16], which shows that all but possibly two eigenvalues of the relevant operators of FETI-DP and BDDC are the same, constitutes an indirect confirmation of this fact.

Since FETI formulations do not correspond to Dirichlet boundary value problems one must conclude that, up to now, no truly Dirichlet preconditioned formulations have been reported in the literature. Therefore, there exists an important gap that

should be filled; it corresponds to two preconditioned non-overlapping methods. In [11], they have been obtained adopting a Dirichlet formulation of the problem that is addressed by domain decomposition methods and then approaching such a formulation in two different fashions: one with resource Lagrange multipliers and the other one without recourse to them. Furthermore, in [11], we have given explicit matrix formulas for each one of the corresponding non-overlapping-preconditioned algorithms.

2.- DISCONTINUOUS PIECEWISE-DEFINED FUNCTIONS

In this Section we draw from [6]. In what follows, $\Omega \subset R^n$ will be a *domain* and $\Pi \equiv \{\Omega_1,...,\Omega_E\}$ a *domain partition* of Ω. The notations $\partial\Omega$ and $\partial\Omega_\alpha$, $\alpha = 1,...,E$, are adopted for the boundaries of Ω and Ω_α, respectively. The *outer boundary* is $\partial\Omega$, while

$$\Gamma \equiv \left(\bigcup_{\alpha=1}^{E} \partial\Omega_\alpha \right) - \partial\Omega \tag{2.1}$$

is the *'internal boundary'*. It will be assumed that Γ has been oriented, so that positive and negative sides have been defined on Γ [6].

Given a family of linear spaces, $\{D(\Omega_1),...,D(\Omega_E)\}$, such that $D(\Omega_\alpha)$, for every $\alpha = 1,...,E$, is a linear space of functions defined (a.e.) in Ω_α, we consider the space

$$\hat{D}(\Omega) \equiv D(\Omega_1) \oplus ... \oplus D(\Omega_E) \tag{2.2}$$

Then, the elements of $\hat{D}(\Omega)$ are *'piecewise defined functions'*, $(w_1,...,w_E)$, with $w_\alpha \in D(\Omega_\alpha)$, $\alpha = 1,...,E$, which in general may be vector-valued. It will be assumed that the *trace* of each w_α, is defined on $\partial\Omega_\alpha$, so that two functions w_+ and w_- are defined on the positive and negative sides of Γ, respectively. This permits defining the *jump* and the *average of* w on Γ, by

$$[\![w]\!] \equiv w_+ - w_- \text{ and } \dot{w} \equiv \frac{1}{2}(w_+ + w_-) \tag{2.3}$$

Then, the following identities are fulfilled:

$$w_+ = \dot{w} + \frac{1}{2}[\![w]\!] \; and \; w_- = \dot{w} - \frac{1}{2}[\![w]\!] \tag{2.4}$$

Examples of such function-spaces are the *'Sobolev spaces of piecewise-defined functions'* discussed in [3]; namely:

$$\hat{H}^p(\Omega) \equiv H^p(\Omega_1) \oplus ... \oplus H^p(\Omega_E), \; p = 0,1,... \tag{2.5}$$

3. GREEN-HERRERA FORMUALS AND VARIATIONAL FORMULATIONS

In this Section we draw from [5, 6]. By the definition of formal adjoint, there exists a vector-valued bilinear function, $\underline{\mathcal{D}}(u,w)$, which satisfies

$$w\mathcal{L}u - u\mathcal{L}^* w \equiv \nabla \bullet \underline{\mathcal{D}}(u,w) \tag{3.1}$$

Integrating this equation over Ω, we get [3,4]

$$\int_\Omega w\mathcal{L}u dx - \int_\Omega u\mathcal{L}^* w dx = \int_{\partial\Omega} \underline{\mathcal{D}}(u,w) \bullet \underline{n} dx - \int_\Gamma [\![\underline{\mathcal{D}}(u,w)]\!] \bullet \underline{n} dx \tag{3.2}$$

We introduce bilinear functions $\mathcal{B}(u,w)$, $\mathcal{J}(u,w)$, $\mathcal{C}(w,u)$, and $\mathcal{K}(w,u)$, which satisfy

$$\begin{aligned} \underline{\mathcal{D}}(u,w) \bullet \underline{n} &= \mathcal{B}(u,w) - \mathcal{C}(w,u), \; on \; \partial\Omega \\ -[\![\underline{\mathcal{D}}(u,w)]\!] \bullet \underline{n} &= \mathcal{J}(u,w) - \mathcal{K}(w,u), \; on \; \Gamma \end{aligned} \tag{3.3}$$

For the case of continuous differential-operator-coefficients, and using a very compact notation, a suitable choice is

$$\mathcal{J}(u,w) \equiv -\underline{\mathcal{D}}\left([\![u]\!], \dot{w}\right) \; and \; \mathcal{K}(w,u) \equiv \underline{\mathcal{D}}\left(\dot{u}, [\![w]\!]\right), \; on \; \Gamma \tag{3.4}$$

In what follows it will be assumed that the *base* and *test* functions are in $\hat{D}_0(\Omega) \equiv \left\{ u \in \hat{D}(\Omega) \middle| \mathcal{B}(u,\bullet) = 0 \right\}$ and $\hat{D}_0^*(\Omega) \equiv \left\{ w \in \hat{D}(\Omega) \middle| \mathcal{C}(w,\bullet) = 0 \right\}$, respectively.

Eq.(3.2) then implies:

$$\int_\Omega w\mathcal{L}u dx - \int_\Gamma \mathcal{J}(u,w) dx = \int_\Omega u\mathcal{L}^* w dx - \int_\Gamma \mathcal{K}(w,u) dx, \forall u \in \hat{D}_0(\Omega) \; and \; \forall w \in \hat{D}_0^*(\Omega) \tag{3.5}$$

Eq.(3.5) is said to be a *'Green-Herrera formula'* when:

$$\begin{aligned} &\int_\Omega w\mathcal{L}u dx - \int_\Gamma \mathcal{J}(u,w) dx = 0, \forall w \in \hat{D}_0^*(\Omega) \Rightarrow \\ &\int_\Omega w\mathcal{L}u dx = 0 \; and \; \int_\Gamma \mathcal{J}(u,w) dx = 0, \forall w \in \hat{D}_0^*(\Omega) \end{aligned} \tag{3.6}$$

As an example, when $\mathcal{L}u \equiv -\nabla\bullet\left(\underline{\underline{a}}\nabla u\right)+\nabla\bullet\left(\underline{b}u\right)+cu$. In this case we can take:

$$\begin{aligned}\mathcal{J}(u,w) &\equiv \dot{w}\left[\!\left[\underline{a}_n\bullet\nabla u\right]\!\right]-\left[\!\left[u\right]\!\right]\dot{\overbrace{\underline{a}_n\bullet\nabla w+b_n w}}\\ \mathcal{K}(w,u) &\equiv \left[\!\left[\underline{a}_n\bullet\nabla w+b_n w\right]\!\right]\dot{u}-\left[\!\left[w\right]\!\right]\dot{\overbrace{\underline{a}_n\bullet\nabla u}}\end{aligned} \tag{3.7}$$

Here, $\underline{a}_n \equiv \underline{\underline{a}}\underline{n}$ and $b_n \equiv \underline{b}\bullet\underline{n}$. Thus, Green-Herrera formula is:

$$\left.\begin{aligned}&\int_\Omega w\mathcal{L}u\,dx-\int_\Gamma\left(\dot{w}\left[\!\left[\underline{a}_n\bullet\nabla u\right]\!\right]-\left[\!\left[u\right]\!\right]\dot{\overbrace{\underline{a}_n\bullet\nabla w+b_n w}}\right)dx=\\ &\int_\Omega u\mathcal{L}^{*}w\,dx-\int_\Gamma\left(\left[\!\left[\underline{a}_n\bullet\nabla w+b_n w\right]\!\right]\dot{u}-\left[\!\left[w\right]\!\right]\dot{\overbrace{\underline{a}_n\bullet\nabla u}}\right)dx\end{aligned}\right\}\forall u\in\hat{D}_0(\Omega)\ and\ \forall w\in\hat{D}_0^{*}(\Omega) \tag{3.8}$$

The *'boundary-value problem with prescribed jumps'* is: *"Find* $u\in\hat{D}_0(\Omega)$ *such that*

$$\begin{aligned}&\mathcal{L}u=f_\Omega,\ in\ \Omega\\ &\left[\!\left[u\right]\!\right]=j_\Gamma^0\ and\ \left[\!\left[\underline{a}_n\bullet\nabla u\right]\!\right]=j_\Gamma^1,\ in\ \Gamma\end{aligned} \tag{3.9}'$$

The functions f_Ω, j_Γ^0 and j_Γ^1 are the data of this problem; each one of them is assumed to be in the range of the corresponding operator. The problem of Eq.(3.9) is said to be a Neumann problem when $j_\Gamma^0=0$, and it is said to be a Dirichlet problem when $j_\Gamma^1=0$.

Applying *Green-Herrera formula* a variational formulation of this problem is obtained [5, 6]; in view of Eqs.(3.5), (3.7) and (3.9), it is

$$\left.\begin{aligned}&\int_\Omega w\mathcal{L}u\,dx-\int_\Gamma\left(\dot{w}\left[\!\left[\underline{a}_n\bullet\nabla u\right]\!\right]-\left[\!\left[u\right]\!\right]\dot{\overbrace{\underline{a}_n\bullet\nabla w+b_n w}}\right)dx=\\ &\int_\Omega wf_\Omega\,dx-\int_\Gamma\left(\dot{w}\,j_\Gamma^1-j_\Gamma^0\,\dot{\overbrace{\underline{a}_n\bullet\nabla w+b_n w}}\right)dx\end{aligned}\right\},\ \forall w\in\hat{D}_0^{*}(\Omega) \tag{3.10}$$

4. THE 'DERIVED VECTOR-SPACE'

When dealing with domain decomposition methods, a shortcoming of standard domain partitions (i.e., of standard non-overlapping domain decompositions) is that such a partition does not yield non-overlapping subsets of *original nodes* of the discretized problem. Indeed, nodes are usually set in the intersection of the subdomains boundaries and therefore each one of such internal boundary nodes are shared by more that one subdomain. In our approach, such a property is only satisfied when *derived nodes* are considered [9, 10]. In our notation, the abstract representation of an *original node* is natural number, while that of a *derived node* is a number pair: the number of the node followed by the subdomain to which it is associated. Thus, the set of *original nodes* is a set of natural numbers, while the set of *derived nodes* is a set of number pairs..

A conspicuous feature of the approach we are discussing is that it is an axiomatic one; It starts with the matrix that is obtained after the problem has been discretized (referred to as the *original matrix*) and its application does not require any information about the partial differential equation from which it originated, it only requires that the basic axioms be fulfilled by the matrix considered. The vector space we work on is the *derived vector-space*, which is constituted by the functions defined in the set of *derived nodes*; this, of course, is a finite set. Clearly, the *original matrix* is not defined on it, since such a matrix is defined on the vector-space whose members correspond to functions defined in the original set of nodes. Therefore, another matrix has to be introduced, which as it is usual in DDM procedures is never constructed, in terms of which the problem is formulated in the *derived vector-space*. From there on, in our theory, all the work is done in the *derived vector-space*. This is a novel approach, to our knowledge this is the first time that non-overlapping domain decomposition methods have been formulated and treated in the *derived vector-space*. So, we describe the procedure with some detail.

Consider the set of nodes of the "non-overlapping" domain decomposition shown in Fig.1, assuming a left-to-right nodes numbering as well as a downwards rows

ordering. The total set of *"original nodes"*, to be denoted as Ω, can be taken to be $\{1,...,25\}$. Then, the set of *original nodes* $\Omega = \{1,...,25\}$ corresponding to such a

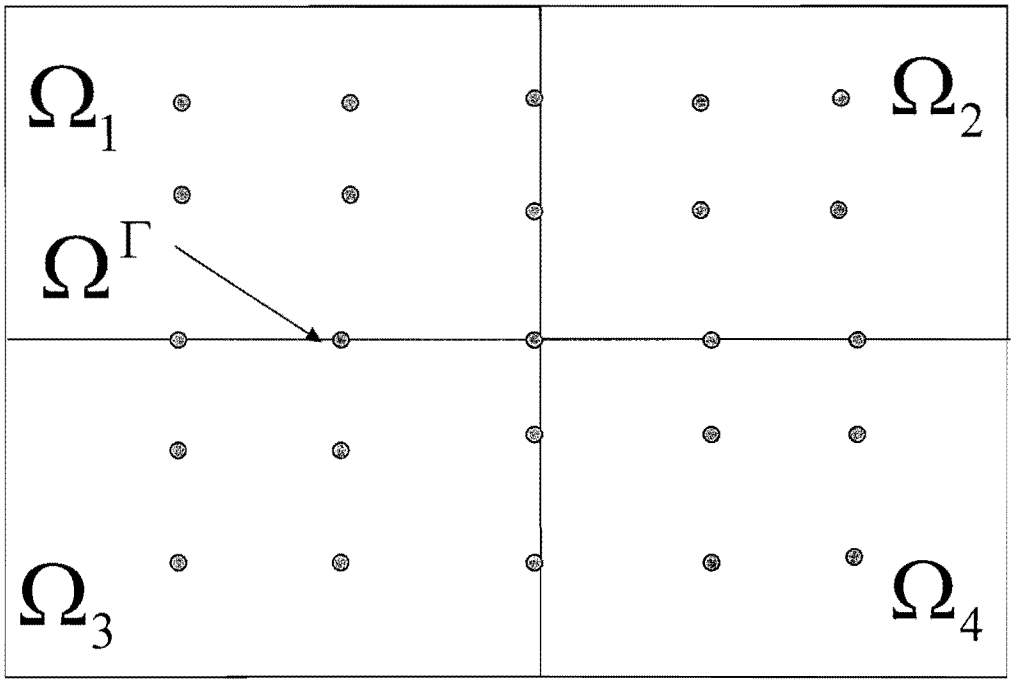

Figure 1. The *'original nodes'*

"non-overlapping" domain decomposition is actually overlapping, since the family of subsets

$$\begin{aligned} \Omega_1 &\equiv \{1,2,3,6,7,8,11,12,13\} & \Omega_2 &\equiv \{3,4,5,8,9,10,13,14,15\} \\ \Omega_3 &\equiv \{11,12,13,16,17,18,21,22,23\} & \Omega_4 &\equiv \{13,14,15,18,19,20,23,24,25\} \end{aligned} \tag{4.1}$$

is not disjoint. Indeed, for example:

$$\Omega_1 \cap \Omega_2 = \{3,8,13\} \tag{4.2}$$

However, it is advantageous to work with a "truly" non-overlapping decomposition of the set of nodes and, to this end, we replace the set Ω of *original nodes* by another set, the set of *"derived nodes"* (see Fig.2), which will be denoted by $\bar{\Omega}$. Each *derived node* is defined to be a pair of numbers, (p,α), such that p is an

original node, while $\alpha \in \{1,2,3,4\}$ is such that $p \in \Omega_\alpha$. Then the set of *derived nodes* is given by

$$\bar{\Omega} \equiv \{(p,\alpha) | \, p \in \Omega_\alpha\} \tag{4.3}$$

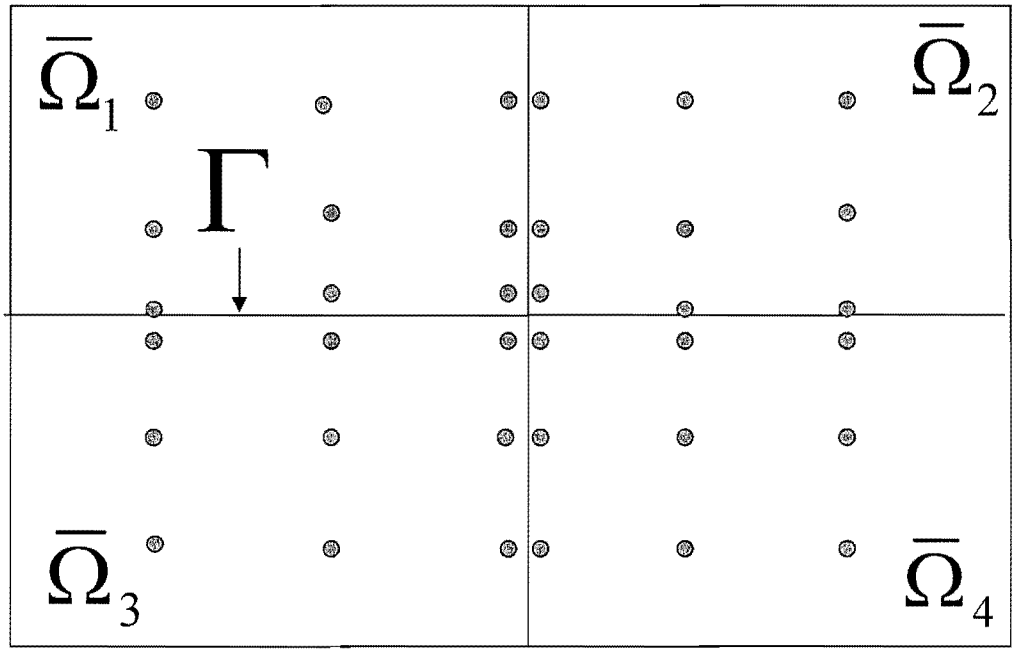

Figure 2. The *'derived nodes'*

If, for each $\alpha = 1,2,3,4$, we define the subsets $\bar{\Omega}_\alpha \subset \bar{\Omega}$ by

$$\bar{\Omega}_\alpha \equiv \{(p,\alpha) | \, p \in \Omega_\alpha\}, \alpha = 1,2,3,4 \tag{4.4}$$

Then the family of subsets $\{\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3, \bar{\Omega}_4\}$ is a truly non-overlapping decomposition of the set $\bar{\Omega}$, in the sense that

$$\bar{\Omega} = \bigcup_{\alpha=1}^{4} \bar{\Omega}_\alpha \; while \; \bar{\Omega}_\alpha \cap \bar{\Omega}_\beta = \varnothing \; when \; \alpha \neq \beta \tag{4.5}$$

Motivated by the above discussion, in the general theory we are discussing [9, 10], the set of *'original nodes'* is defined to be a set of *natural numbers* $\Omega \equiv \{1,...,d\}$, where d is the total number of nodes that occur in the discretized version of the differential equation (or system of equations) that is being treated (generally, nodes

that lie on the domain's external boundary are not included). The *'domain decomposition'* is a collection $\{\Omega_1,...,\Omega_N\}$ of subsets of Ω, usually not disjoint, such that

$$\Omega = \bigcup_{\alpha=1}^{N} \Omega_\alpha \tag{4.6}$$

The *original internal boundary* is, by definition, the subset Ω^Γ of *original nodes* that occur in more than one subdomain of the *domain decomposition*; in particular, we observe that in the example of Fig.1 the cardinality of Ω^Γ is 9. The set of "derived nodes" is defined by Eq.(4.3), and observe that the cardinality of $\bar{\Omega}$ is always greater or equal than that of Ω. In the particular case of Fig.1 these cardinalities are 36 and 25, respectively (see Fig.2). For the sake of brevity, in what follows, the word *node* will be used as a synonymous of *derived node*.

The family of subsets of $\{\bar{\Omega}_1,...,\bar{\Omega}_N\}$ is:

$$\bar{\Omega}_\alpha \equiv \{(p,\alpha) \,|\, p \in \Omega_\alpha\}, \alpha = 1,...,N \tag{4.7}$$

It constitutes a truly non-overlapping partition of the set $\bar{\Omega}$, since

$$\bar{\Omega} = \bigcup_{\alpha=1}^{N} \bar{\Omega}_\alpha \; and \; \bar{\Omega}_\alpha \cap \bar{\Omega}_\beta = \varnothing \; when \; \alpha \neq \beta \tag{4.8}$$

When applying domain decomposition techniques this property is very important. As it was mentioned previously the main goal of DDM strategy is to process each subdomain in a different processor and the degree of independence of each subdomain is greater when they are fully disjoint.

Given any *original node,* $p \in \Omega$, the set $Z(p) \subset \bar{\Omega}$, is made of the *(derived) nodes* that can be written as (p,α), for some $1 \leq \alpha \leq N$. When $(p,\alpha) \in \bar{\Omega}$ is a *node*, its *multiplicity* is defined to be the cardinality of the set $Z(p) \subset \bar{\Omega}$ and it is denoted by $m(p)$. Such a *node* is said to be an *internal* or *boundary* node depending on

whether its multiplicity is one or greater than one. The subsets $\mathrm{I} \subset \bar{\Omega}$ and $\Gamma \subset \bar{\Omega}$ of *internal* and *boundary nodes*, respectively, decompose $\bar{\Omega}$, in the sense that

$$\bar{\Omega} = \mathrm{I} \cup \Gamma \; and \; \varnothing = \mathrm{I} \cap \Gamma \tag{4.9}$$

The family of subsets $\{\Gamma_1,...,\Gamma_N\}$ decomposes Γ, when they are defined by

$$\Gamma_\alpha \equiv \Gamma \cap \bar{\Omega}_\alpha, \; for \; each \; \alpha = 1,...,N \tag{4.10}$$

Any real-valued[1] function defined in each one of the sets considered thus far defines uniquely a finite-dimensional vector, because they are finite sets. The vector space corresponding to functions defined in Ω will be denoted by $\tilde{D}(\Omega)$, while the vector-space of functions defined in $\bar{\Omega}$ will be $\tilde{D}(\bar{\Omega})$ and this latter space will be referred to as the *'derived vector space'*. The set $\tilde{D}(\bar{\Omega})$ constitutes a finite dimensional *Hilbert-space* with respect to the Euclidean inner product, which is defined by

$$\underline{u} \bullet \underline{w} \equiv \sum_{(p,\alpha)\in\bar{\Omega}} \underline{u}(p,\alpha)\underline{w}(p,\alpha), \forall \underline{u} \in \tilde{D}(\bar{\Omega}) \; and \; \underline{w} \in \tilde{D}(\bar{\Omega}) \tag{4.11}$$

As for notation, when $\Phi \subset \bar{\Omega}$ is a subset of $\bar{\Omega}$, we write $\tilde{D}(\Phi) \subset \tilde{D}(\bar{\Omega})$ for the linear subspace of $\tilde{D}(\bar{\Omega})$ whose functions vanish at every *node* that does not belong to Φ. With this notation we have: when $\Xi \subset \Phi \subset \bar{\Omega}$, $\tilde{D}(\Xi)$ is a subspace of $\tilde{D}(\Phi)$; i.e.,

$$\tilde{D}(\Xi) \subset \tilde{D}(\Phi) \tag{4.12}$$

Furthermore, if $\{\Phi_1,...,\Phi_M\}$ decomposes Φ, in the sense introduced before, then

$$\tilde{D}(\Phi) = \tilde{D}(\Phi_1) \oplus ... \oplus \tilde{D}(\Phi_M) \tag{4.13}$$

Here, the symbol $\oplus$ is used for the *direct sum* of vector spaces; i.e., Eq.(4.13) holds, if and only if,

$$\tilde{D}(\Phi) = \tilde{D}(\Phi_1) + ... + \tilde{D}(\Phi_M) \; and \; \{0\} = \tilde{D}(\Phi_\alpha) \cap \tilde{D}(\Phi_\beta), \; whenever \; \alpha \neq \beta \tag{4.14}$$

[1] When considering systems of equations, as in elasticity problems, such functions are vector-valued.

Then, it is clear that

$$\begin{cases} \tilde{D}(\Gamma) = \tilde{D}(\Gamma_1) \oplus ... \oplus \tilde{D}(\Gamma_N) \\ \tilde{D}(\bar{\Omega}) = \tilde{D}(\mathrm{I}) \oplus \tilde{D}(\Gamma) \\ \tilde{D}(\bar{\Omega}) = \tilde{D}(\bar{\Omega}_1) \oplus ... \oplus \tilde{D}(\bar{\Omega}_N) \end{cases} \tag{4.15}$$

A function (vector) of $\tilde{D}(\bar{\Omega})$ is said to be *'continuous'* when its value at any node, (p,α), depends on p exclusively and it is independent of α. The vectors of $\tilde{D}(\bar{\Omega})$ are generally discontinuous; the subspace of *continuous vectors* will be denoted by $\tilde{D}_{12}(\bar{\Omega})$, while the orthogonal complement of this subspace is $\tilde{D}_{11}(\bar{\Omega})$. The *average* and *jump matrices*, $\underline{\underline{a}}$ and $\underline{\underline{j}}$, are the projections on $\tilde{D}_{12}(\bar{\Omega})$ and on $\tilde{D}_{11}(\bar{\Omega})$, respectively. Therefore, $\underline{\underline{I}} = \underline{\underline{a}} + \underline{\underline{j}}$.

5. GREEN-HERRERA FORMULAS FOR MATRICES

Given a matrix $\underline{\underline{\hat{A}}} : \tilde{D}(\Omega) \to \tilde{D}(\Omega)$, generally non-symmetric, the *'original problem'* consists in searching for a function $\underline{\hat{u}} \in \tilde{D}(\Omega)$ that satisfies

$$\underline{\underline{\hat{A}}}\,\underline{\hat{u}} = \underline{\hat{f}} \tag{5.1}$$

with $\underline{\hat{f}} \in \tilde{D}(\Omega)$ given. As said in Section 4, in order to formulate an equivalent problem in the *derived vector space* another matrix, $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$, is constructed such that Eq.(5.1) is fulfilled if and only if

$$\underline{\underline{a}}\,\underline{\underline{A}}\,\underline{u} = \underline{f} \; and \; \underline{\underline{j}}\,\underline{u} = 0 \tag{5.2}$$

This latter problem is referred to as the *transformed problem*. Details of the construction of $\underline{\underline{A}}$ and $\underline{f}$, when $\underline{\underline{\hat{A}}}$ and $\underline{\hat{f}}$ are given, can be found in [9-11]; when $\underline{\underline{\hat{A}}}$ is non-symmetric, so is $\underline{\underline{A}}$. The manner in which $\underline{\hat{u}}$ and $\underline{u}$ are related is also explained in [9-11].

The matrix $\underline{\underline{A}} : \tilde{D}(\bar{\Omega}) \to \tilde{D}(\bar{\Omega})$ is written as:

$$\underline{\underline{A}} \equiv \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\Gamma} \\ \underline{\underline{A}}_{\Gamma I} & \underline{\underline{A}}_{\Gamma\Gamma} \end{pmatrix} \text{ so that } \underline{\underline{A}}^T \equiv \begin{pmatrix} \underline{\underline{A}}_{II}^T & \underline{\underline{A}}_{\Gamma I}^T \\ \underline{\underline{A}}_{I\Gamma}^T & \underline{\underline{A}}_{\Gamma\Gamma}^T \end{pmatrix} \tag{5.3}$$

Here,

$$\begin{aligned} &\underline{\underline{A}}_{II} : \tilde{D}(\mathrm{I}) \to \tilde{D}(\mathrm{I}), \underline{\underline{A}}_{I\Gamma} : \tilde{D}(\Gamma) \to \tilde{D}(\mathrm{I}) \\ &\underline{\underline{A}}_{\Gamma I} : \tilde{D}(\mathrm{I}) \to \tilde{D}(\Gamma), \underline{\underline{A}}_{\Gamma\Gamma} : \tilde{D}(\Gamma) \to \tilde{D}(\Gamma) \end{aligned} \tag{5.4}$$

We define

$$\begin{aligned} &\underline{\underline{L}} \equiv \begin{pmatrix} \underline{\underline{A}}_{II} & \underline{\underline{A}}_{I\Gamma} \\ 0 & 0 \end{pmatrix} \text{ and } \underline{\underline{R}} \equiv -\begin{pmatrix} 0 & 0 \\ \underline{\underline{A}}_{\Gamma I} & \underline{\underline{A}}_{\Gamma\Gamma} \end{pmatrix} \\ &\underline{\underline{L}}^* \equiv \begin{pmatrix} \underline{\underline{A}}_{II}^T & \underline{\underline{A}}_{\Gamma I}^T \\ 0 & 0 \end{pmatrix} \text{ and } \underline{\underline{R}}^* \equiv -\begin{pmatrix} 0 & 0 \\ \underline{\underline{A}}_{I\Gamma}^T & \underline{\underline{A}}_{\Gamma\Gamma}^T \end{pmatrix} \end{aligned} \tag{5.5}$$

Therefore,

$$\underline{\underline{L}} - \left(\underline{\underline{L}}^*\right)^T = \underline{\underline{R}} - \left(\underline{\underline{R}}^*\right)^T \tag{5.6}$$

This is equivalent to

$$\underline{w} \bullet \underline{\underline{L}}\underline{u} - \underline{u} \bullet \underline{\underline{L}}^* \underline{w} = \underline{w} \bullet \underline{\underline{R}}\underline{u} - \underline{u} \bullet \underline{\underline{R}}^* \underline{w}, \ \forall \underline{u}, \underline{w} \in \tilde{D}\left(\bar{\Omega}\right) \tag{5.7}$$

or

$$\underline{w} \bullet \underline{\underline{L}}\underline{u} - \underline{u} \bullet \underline{\underline{L}}^* \underline{w} = D\left(\underline{u}, \underline{w}\right), \ \forall \underline{u}, \underline{w} \in \tilde{D}\left(\bar{\Omega}\right) \tag{5.8}$$

Here, we define

$$D\left(\underline{u}, \underline{w}\right) \equiv \underline{w} \bullet \underline{\underline{R}}\underline{u} - \underline{u} \bullet \underline{\underline{R}}^* \underline{w} = \left[\!\left[\underline{w}\right]\!\right] \bullet \dot{\widehat{\underline{\underline{R}}\underline{u}}} + \dot{\underline{w}} \bullet \left[\!\left[\underline{\underline{R}}\underline{u}\right]\!\right] - \left[\!\left[\underline{u}\right]\!\right] \bullet \dot{\widehat{\underline{\underline{R}}^* \underline{w}}} - \dot{\underline{u}} \bullet \left[\!\left[\underline{\underline{R}}^* \underline{w}\right]\!\right], \ \forall \underline{u}, \underline{w} \in \tilde{D}\left(\bar{\Omega}\right) \tag{5.9}$$

Where we have used the following notation: whenever $\underline{\underline{P}} = \underline{\underline{R}}$ or $\underline{\underline{P}} = \underline{\underline{R}}^*$, and $\underline{u} \in \tilde{D}\left(\bar{\Omega}\right)$, we write

$$\left[\!\left[\underline{u}\right]\!\right] \equiv \underline{\underline{j}}\underline{u}, \dot{\underline{u}} \equiv \underline{\underline{a}}\underline{u}, \left[\!\left[\underline{\underline{P}}\underline{u}\right]\!\right] \equiv \underline{\underline{\underline{a}}}\underline{\underline{P}}\underline{u} \text{ and } \dot{\widehat{\underline{\underline{P}}\underline{u}}} \equiv \underline{\underline{j}}\underline{\underline{P}}\underline{u} \tag{5.10}$$

From Eq.(5.8); it follows that

$$\underline{w} \bullet \underline{\underline{L}}\underline{u} + \left[\!\left[\underline{u}\right]\!\right] \bullet \dot{\widehat{\underline{\underline{R}}^* \underline{w}}} - \dot{\underline{w}} \bullet \left[\!\left[\underline{\underline{R}}\underline{u}\right]\!\right] = \underline{u} \bullet \underline{\underline{L}}^* \underline{w} + \left[\!\left[\underline{w}\right]\!\right] \bullet \dot{\widehat{\underline{\underline{R}}\underline{u}}} - \dot{\underline{u}} \bullet \left[\!\left[\underline{\underline{R}}^* \underline{w}\right]\!\right], \ \forall \underline{u}, \underline{w} \in \tilde{D}\left(\bar{\Omega}\right) \tag{5.11}$$

This equation is said to be a *'Green-Herrera formula for matrices'* when

$$\underline{w}\bullet\underline{\underline{L}}\,\underline{u}+[\![\underline{u}]\!]\bullet\dot{\widehat{\underline{\underline{R}}^{*}\,\underline{w}}}-\dot{\underline{w}}\bullet[\![\underline{\underline{R}}\,\underline{u}]\!]=0,\ \forall\underline{w}\in\tilde{D}(\bar{\Omega})\Rightarrow\underline{\underline{L}}\,\underline{u}=0,\ [\![\underline{\underline{R}}\,\underline{u}]\!]=0\ and\ [\![\underline{u}]\!]=0 \quad (5.12)$$

In particular, when $\underline{\underline{a}}\,\underline{\underline{A}}\,\underline{\underline{a}}:\tilde{D}_{12}(\bar{\Omega})\rightarrow\tilde{D}_{12}(\bar{\Omega})$ and $\underline{j}\,\underline{\underline{A}}\,\underline{j}:\tilde{D}_{11}(\bar{\Omega})\rightarrow\tilde{D}_{11}(\bar{\Omega})$ are one-to-one this latter condition is granted. We observe, comparing Eqs.(5.11) and (3.8), the following correspondences

$$[\![u]\!]\leftrightarrow[\![\underline{u}]\!]\ and\ [\![\underline{a}_{n}\bullet\nabla u]\!]\leftrightarrow[\![\underline{\underline{R}}\,\underline{u}]\!]\equiv\underline{\underline{a}}\,\underline{\underline{R}}\,\underline{u} \quad (5.13)$$

We observe that the second of these correspondences imply that the *Steklov-Poincaré operator*, at the discrete level, should be defined as $\underline{\underline{a}}\,\underline{\underline{R}}$.

The *'problem with prescribed jumps, for matrices'*, is defined to be:

$$\begin{aligned}\underline{\underline{L}}\,\underline{u}&=\underline{f}_{\mathrm{I}},\ in\ \Omega\\ [\![u]\!]&=\underline{j}^{0}\ and\ [\![\underline{\underline{R}}\,\underline{u}]\!]=\underline{j}^{1}\end{aligned} \quad (5.14)$$

The data of this problem satisfies

$$\underline{f}_{\mathrm{I}}\in\tilde{D}(\mathrm{I}),\ \underline{j}^{0}\in\tilde{D}_{11}(\bar{\Omega})\ and\ \underline{j}^{1}\in\tilde{D}_{11}(\bar{\Omega}) \quad (5.15)$$

When $\underline{j}^{0}=0$, we say that the problem is a *'Neumann problem'* and if $\underline{j}^{1}=0$ then the problem is said to be a *'Dirichlet problem'*. We observe that the *'Transformed problem'*, Eq.(5.2), is a Neumann problem. When Eq.(5.11) is a Green-Herrera formula, a variational formulation of this problem is

$$\underline{w}\bullet\underline{\underline{L}}\,\underline{u}+[\![\underline{u}]\!]\bullet\dot{\widehat{\underline{\underline{R}}^{*}\,\underline{w}}}-\dot{\underline{w}}\bullet[\![\underline{\underline{R}}\,\underline{u}]\!]=\underline{w}\bullet\underline{f}_{\mathrm{I}}+\underline{j}^{0}\bullet\dot{\widehat{\underline{\underline{R}}^{*}\,\underline{w}}}-\dot{\underline{w}}\bullet\underline{j}^{1},\ \forall\underline{w}\in\tilde{D}(\bar{\Omega}) \quad (5.16)$$

6.- CONCLUSIONS

As this paper's title indicates, a *'unified theory of differential operators acting on discontinuous functions and operators acting on discontinuous vectors'* has been presented, which is equally applicable to symmetric and non-symmetric matrices. The main motivation for its development has been *domain decomposition methods (DDM)*. In previous articles it has been shown that this theory yields very systematic procedures for applying parallel processing to the algebraic systems that are obtained after the partial differential equations have been discretized.

A fundamental element on which the theory is based are *Green-Herrera formulas* that, for differential operators, are given by Eq.(3.5) and (3.10), while Eq.(5.11) yields them for matrices. These formulas have permitted us to establish very precise correspondences between the concepts of the theory of partial differential equations and those concerning discretized versions of such equations. In this light, in particular, we have introduced an improved definition of the Steklov-Poincaré operator, which is more appropriate than the standard one, which is referred to by many authors working in DDM [2].

The concept of *'problems with prescribed jumps'* has been extended to include matrix equations. In a previous paper [11], we have shown that each *problem with prescribed jumps* can be treated with resource to Lagrange multipliers (the LM procedure) and without resource to them (the MF procedure). Also, each problem can be formulated as a Neumann problem and also as a Dirichlet problem; or, more generally, as a Robin problem. If attention is restricted just to Neumann and Dirichlet formulations, then one obtains four preconditioned algorithms that can be applied to each *problem with prescribed jumps*; two for each one of these formulations.

The most commonly used non-overlapping domain decomposition methods are different versions of FETI and BDD [14-19]. BDD is a direct formulation of a Neumann problem that does not use Lagrange multipliers. Although it is generally believed that FETI formulations correspond to discrete formulations of Dirichlet problems [2] (in particular, see Subsection 1.3.5 of [2]), in the light of our theory it has been shown that it actually corresponds to a Lagrange multipliers formulation of the same Neumann problem as BDD. All this implies that, up to now, no truly Dirichlet preconditioned formulations had been reported in the literature. Therefore, until recently an important gap has existed, since two formulations as significant as FETI or BDD have not received any attention at all; we refer to two formulations of DDM procedures as a Dirichlet problem, one using Lagrange multipliers and the

other one without them. In [11], as a contribution to fill this gap, Herrera et al. have given matrix formulas for the corresponding non-overlapping-preconditioned algorithms.

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